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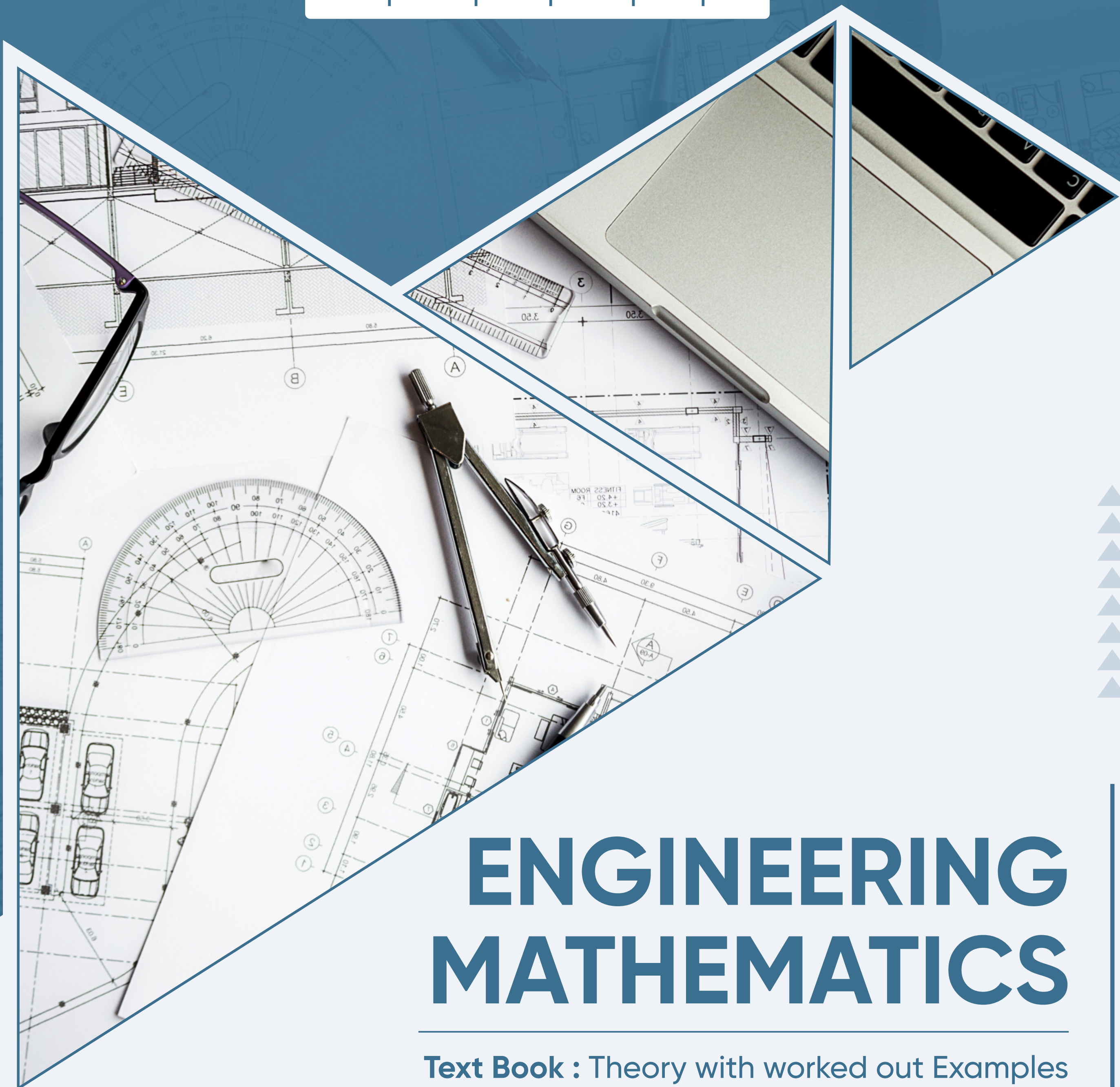


ENGINEERING MATHEMATICS

Text Book : Theory with worked out Examples and Practice Questions

Linear Algebra

(Solutions for Text Book Practice Questions)

01. Ans: 0

Sol: $|A^{2023} - A^{2024}| = |A^{2023}(A-I)|$

$= |A|^{2023}\,|A-I|$

$$|A - I| = \begin{vmatrix} 8 & 6 \\ 8 & 6 \end{vmatrix} = 0$$

$\therefore\ |A^{2023} - A^{2024}| = 0$

02. Ans: 27

Sol: $$|A| = \begin{vmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & -1 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

$C_2 \rightarrow C_2 + C_1$

$C_3 \rightarrow C_3 - C_1$

$C_4 \rightarrow C_4 + C_1$

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & -3 & 3 \\ 3 & 5 & -2 & 2 \\ 4 & 7 & -2 & 5 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 3 & -3 & 3 \\ 5 & -2 & 2 \\ 7 & -2 & 5 \end{vmatrix}$$

$C_2 \rightarrow C_2 + C_1$ & $C_3 \rightarrow C_3 - C_1$

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 5 & 3 & -3 \\ 7 & 5 & -2 \end{vmatrix}$$

$|A| = 3(-6 + 15) = 27$

03. Ans: 0

Sol: Given A & B are symmetric matrices

$\Rightarrow$ AB – BA is a skew – symmetric matrix

$\therefore$ det(AB – BA) = 0

04. Ans: 11

Sol: We know that $|\text{adj}A| = |A|^{n-1}$

$\Rightarrow (12-12) - \alpha(4-6) + 3(4-6) = 4^2$

$\Rightarrow 2\alpha - 6 = 16$

$\Rightarrow 2\alpha = 22$

$\therefore\ \alpha = 11$

05. Ans: (a & d)

Sol: We know that

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ (Property)

$(CBA)^{-1} = A^{-1}B^{-1}C^{-1}$

06. Ans: (a)

Sol: Given $|A| = 2$

$$\left|\text{adj}\,\text{adj}\left(\text{adj}A^{-1}\right)\right| = |A^{-1}|^{(n-1)^3} = |A|^{-(n-1)^3}$$

$$= |A|^{-(3-1)^3} = 2^{-8}$$

$$= \frac{1}{256}$$

07. Ans: (a)

Sol: det (A) = 2 (12 – 2) – (16 – 1) + 4 (8–3)

det (A) = 20 – 15 + 20 = 25

Cofactor of $a_{11} = 2$ is 10

Cofactor of $a_{12} = 1$ is -15

Cofactor of $a_{13} = 4$ is 5

Cofactor of $a_{21} = 4$ is 4

Cofactor of $a_{22} = 3$ is 4

Cofactor of $a_{23} = 1$ is -3

Cofactor of $a_{31} = 1$ is -11

Cofactor of $a_{32} = 2$ is 14

Cofactor of $a_{33} = 4$ is 2

$$\text{Cofactor of } A = \begin{bmatrix} 10 & -15 & 5 \\ 4 & 4 & -3 \\ -11 & 14 & 2 \end{bmatrix}$$

$$\text{adj}\,A = \begin{bmatrix} 10 & 4 & -11 \\ -15 & 4 & 14 \\ 5 & -3 & 2 \end{bmatrix}$$

$$\therefore A^{-1}\frac{\text{adj}A}{|A|} = \frac{1}{25}\begin{bmatrix} 10 & 4 & -11 \\ -15 & 4 & 14 \\ 5 & -3 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2/5 & 4/25 & -11/25 \\ -3/5 & 4/25 & 14/25 \\ 1/5 & -3/25 & 2/25 \end{bmatrix}$$

08. Ans: (c)

Sol: Please Refer ACE Previous maths solution booklet

09. Ans: (b)

Sol: Please Refer ACE Previous maths solution booklet

10. Ans: (a & d)

Sol: Please Refer ACE Previous maths solution booklet

11. Ans: (a)

Sol: If rank $(A_{3\times3}) < 3$ then $|A| = 0$

$(1-x^2) - x(x - x^2) + x(x^2 - x) = 0$

$(1-x^2) - x^2(1-x) - x^2(1-x) = 0$

$(1+x)(1-x) - 2x^2(1-x) = 0$

$\Rightarrow (1-x)[1+x-2x^2] = 0$

$\Rightarrow 1 - x = 0$ & $-2x^2 + x + 1 = 0$

$\Rightarrow x = 1$ & $x = 1, \frac{-1}{2}x$

$\therefore\ x = 1, \frac{-1}{2}$

12. Ans: 5

Sol: For linearly dependent vectors,

det(A) = 0

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ a & 6 & 4 \end{vmatrix} = 0$$

$(12-6) - (8-a) + (12-3a) = 0$

$6 - 8 + a + 12 - 3a = 0$

$-2a + 10 = 0$

$\therefore\ a = 5$

13. Ans: (a, d)

Sol: Let $X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ and $x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$X_1^T X_2 = \frac{1}{2} - \frac{1}{2} = 0$$

Here X_1 and X_2 are orthogonal vectors

Let $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$|A| = \frac{1}{2} + \frac{1}{2} = 1$$

$|A| \neq 0$

X_1 and X_2 are linearly independent vectors.

$\therefore$ Orthogonal vectors are linearly independent.

But linearly independent vectors need not be orthogonal.

For example

Let $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

X_1 and X_2 are linearly independent since one can not be expressed as the other.

But $X_1^T X_2 \neq 0$.

14. Ans: 3

Sol: For infinite solutions of homogeneous system of equations,

We have det(M) = 0

$\alpha(\beta\gamma - 1) - (\gamma - 1) + (1 - \beta) = 0$

$\alpha\beta\gamma - \alpha - \gamma + 1 + 1 - \beta = 0$

$3 - (\alpha+\beta+\gamma) = 0$ ($\because$ $\alpha\beta\gamma = 1$)

$3 - \text{trace}(M) = 0$

$\therefore$ Trace (M) = 3

15. Ans: 4

Sol: Number of linearly independent solutions is given by

$n - r = 1$

$5 - \text{Rank}(P) = 1$

$\text{Rank}(P) = 4$

16. Ans: (b)

Sol: Please Refer ACE Previous maths solution booklet

17. Ans: (a)

Sol: The augmented matrix is given by

$$(A/B) = \left(\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right)$$

$R_2 \to R_2 + 4R_1$

$R_3 \to R_3 + 3R_1$

$$(A/B) = \left(\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2 - 4b_1 \\ 0 & -3 & 12 & b_3 + 3b_1 \end{array}\right)$$

18. Ans: (a)

Sol: Given AX = B

B is a linear combination of columns of A.

$\Rightarrow$ The system of equation is consistent

And A has three linearly independent columns

$\Rightarrow |A| \neq 0$

$\therefore$ The system of equations has unique solution .

19. Ans: (d)

Sol: The augmented matrix is

$$(A/B) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 1 \\ r & s & 0 & s-1 \end{array}\right]$$

$R_2 \to R_2 \to 2R_1$

$R_3 \to R_3 - rR_1$

$$[A\,|\,B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -4 & 3 & -1 \\ 0 & s-2r & 0 & s-r-1 \end{array}\right]$$

$R_3 \to 4R_3 + (s–2r)R_2$

$$[A\,|\,B] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -4 & 3 & - \\ 0 & 0 & 3(s-2r-4) & 13s-2r-4 \end{array}\right]$$

If s = 2r = 2 then,

Rank (A) = rank (A|B) < number of unknowns.

$\therefore$ The system of equations has infinitely many solutions for s = 2r = 2.

20. Ans: (c)

Sol: Trace of matrix = sum of eigen values

Trace of matrix = 9

Sum of eigen values of option (c) = 9

$\therefore$ Option (c) is correct.

21. Ans: 15

Sol: If 2 + i is an eigen value of P then 2 – I is another eigen value of P

det (P) = (2+i)(2–i)(3) = 15 .

22. Ans: 78

Sol: Let λ_1 & λ_2 be eigen values of P

$\lambda_1 + \lambda_2 = 1$ ________ (1)

$\lambda_1\ \lambda_2 = -6$ ________ (2)

From (1) & (2) $\lambda_1 = 3$ & $\lambda_2 = -2$

The eigen value of $P^4 - P^3$ is $\lambda^4 - \lambda^3$

For $\lambda_1 = 3$, $P^4 - P^3 = 54$

For $\lambda_2 = -2$, $P^4 - P^3 = 24$

Trace $(P^4 - P^3) = 54 + 24 = 78$

23. Ans: (a)

Sol: Magnitude of eigen value of orthogonal matrix is 1

Eigen values of skew-symmetric matrix are either purely imaginary (or) zeros by using above properties, Option (A) is correct.

24. Ans: (b & c)

Sol: Please Refer ACE Previous maths solution booklet

25. Ans: (a & d)

Sol: Let $Y = [1 \quad 0]$ $\quad Y^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Y^TY = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let $P = Y^TY$

$P^T = (Y^TY)^T = Y^T(Y^T)^T = Y^TY$

$P^T = P$

$P = Y^TY$ is a symmetric matrix

$\det(Y^TY) = 0$

Y^TY is not invertible

$\text{Rank}(Y^TY) = 1$

$$YY^T = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1+0\end{bmatrix} = 1$$

Eigen values of Y^TY are 1, 0

$\therefore$ Options (a) & (d) are correct.

26. Ans: 1

Sol: Please Refer ACE Previous maths solution booklet

27. Ans: 7

Sol: $(A-\lambda I)X = 0$

$$\begin{bmatrix}8-\lambda & -6 & 2\\ -6 & x-\lambda & -4\\ 2 & -4 & 3-\lambda\end{bmatrix}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$2(8-\lambda) + 12 + 2 = 0$

$-2\lambda + 30 \Rightarrow \lambda = 15$

$-12 - 2(x-\lambda) - 4 = 0$

$-12 - 2x + 2\lambda - 4 = 0$

$-2x + 14 = 0$

$\therefore\ x = 7$

28. Ans: 3

Sol: Given M is a singular matrix one of the eigen value of M is 0. Other two eigen values are 2 & 3 $\lambda^3+2\lambda+1$ is an eigen value of $M^3 + 2M + 1$

For $\lambda = 0$, $\lambda^3 + 2\lambda + 1 = 1$

For $\lambda = 2$, $\lambda^3 + 2\lambda + 1 = 13$

For $\lambda = 3$, $\lambda^3 + 2\lambda + 1 = 34$

$M^3 + 2M + 1$ has three different eigen values.

So, it has three linearly independent eigen vectors.

29. Ans: 2

Sol: Please Refer ACE Previous maths solution booklet

30. Ans: (a)

Sol: Please Refer ACE Previous maths solution booklet

31. Ans: (a)

Sol: $(A-\lambda I)X = 0$

$$\begin{bmatrix}-2-\lambda & 2 & 1\\ 2 & 1-\lambda & 2\\ 1 & 2 & 6-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

For $\lambda = -3$, we have

$$\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2 \to R_2 \to 2R_1$

$R_3 \to R_3 \to R_1$

$$\begin{bmatrix}1 & 2 & 1\\ 0 & 0 & 0\\ 0 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1 + 2x_2 + x_3 = 0$ --------------- (1)

$7x_3 = 0$

$x_3 = 0$

$(1) \Rightarrow x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$$\frac{x_1}{x_2} = \frac{-2}{1} \Rightarrow x_1 = -2 \ \& \ x_2 = 1$$

Option (a) is correct.

32. Ans: 6

Sol: $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\ \{(-2-\lambda)(-3-\lambda)\} = 0$

$\Rightarrow (1-\lambda)\ \{\lambda^2 + 5\lambda + 6\} = 0$

$\Rightarrow -\lambda^3 - 4\lambda^2 - \lambda + 6 = 0$

$\Rightarrow \lambda^3 + 4\lambda^2 + \lambda - 6 = 0$

By cayley - Hamilton theorem we have

$A^3 + 4A^2 + A - 6I = 0$

Multiplying both sides by A^{-1}

$A^{-1}\ (A^3 + 4A^2 + A - 6I) = 0$

$A^2 + 4A + I - 6A^{-1} = 0$

$6A^{-1} = A^2 + 4A + I = aA^2 + bA + CI$

$\Rightarrow a = 1, b = 4, c = 1$

$\therefore\ a + b + c = 6$

33. Ans: (c)

Sol: Eigen values of A are

$$\lambda = 0, 1, \frac{2}{3}$$

The characteristic equation is

$$\lambda(\lambda-1)\left(\lambda - \frac{2}{3}\right) = 0$$

$$\lambda\left(\lambda^2 - \frac{5}{3}\lambda + \frac{2}{3}\right) = 0$$

$$\lambda^3 - \frac{5}{3}\lambda^2 + \frac{2}{3}\lambda = 0$$

By cayley-Hamilton theorem, we have

$$A^3 - \frac{5}{3}A^2 + \frac{2}{3}A = 0$$

$$\Rightarrow A^3 = \frac{5}{3}A^2 - \frac{2}{3}A$$

$$\therefore\ A^3 = \frac{1}{3}\left(5A^2 - 2A\right)$$

34. Ans: (a & c)

Sol: Given $x_1 + x_2 + x_3 = 0$

$\Rightarrow x_1 = -x_2 - x_3$

Putting $x_2 = c_1$ and $x_3 = c_2$

$x_1 = -c_1 - c_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x3 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix} = -c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{Here}$$

$X = -c_1x_1 - c_2x_2$

Where $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Thus

(i) x_1 and x_2 span the given plane

(ii) x_1 and x_2 are linearly independent because they are not scalar multiple of each other

$\therefore$ It forms a basis

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Dimension = 2

35. Ans: (d)

Sol: Please Refer ACE Previous maths solution booklet

Calculus

(Solutions for Text Book Practice Questions)

01. Ans: (A)

Sol: $\lim\limits_{x\to\frac{5}{4}}(x-[x]) = \lim\limits_{x\to\frac{5}{4}} x - \lim\limits_{x\to\frac{5}{4}}[x]$

$$= \frac{5}{4} - 1 = \frac{1}{4}$$

02. Ans: (d)

Sol: Please Refer ACE Previous Maths Solution Booklet

03. Ans: (d)

Sol: $\lim\limits_{x\to 2}\frac{x^2+x-6}{|x-2|} = \lim\limits_{x\to 2}\frac{(x+3)(x-2)}{|x-2|} =$

$$\lim_{x\to 2^-}\frac{(x+3)(x-2)}{|x-2|} = \lim_{x\to 2^-}\frac{(x+3)(x-2)}{-(x-2)} = -5$$

$$\lim_{x\to 2^+}\frac{(x+3)(x-2)}{|x-2|} = \lim_{x\to 2^+}\frac{(x+3)(x-2)}{(x-2)} = 5$$

Since $\lim\limits_{x\to 2^-}\frac{x^2+x-6}{|x-2|} \neq \lim\limits_{x\to 2^+}\frac{x^2+x-6}{|x-2|}$

$\lim\limits_{x\to 2}\frac{x^2+x-6}{|x-2|} =$ does not exist

04. Ans: 2

Sol: $\lim\limits_{x\to 0}\frac{x\sin x}{1-\cos x} = \lim\limits_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)}{\left(\frac{1-\cos x}{x^2}\right)} = \frac{1}{\frac{1}{2}} = 2$

05. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

06. Ans: 0.5

Sol: Please Refer ACE Previous Maths Solution Booklet

07. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

08. Ans: 0.5

Sol: Please Refer ACE Previous Maths Solution Booklet

09. Ans: 0.25

Sol: Please Refer ACE Previous Maths Solution Booklet

10. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

11. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

12. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

13. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

14. Ans: (d)

Sol: Let, f(x) = $x^2 - 2x + 2$ and [a,b] = [1,3]

Then, $f'(x) = 2x - 2$

By a mean value theorem

$\exists\ c \in (1,3) \Rightarrow f'(c) = \dfrac{f(3) - f(1)}{3 - 1}$

$\Rightarrow c - 1 = 1$

$\therefore$ c = 2 (or) x = 2

15. Ans: (d)

Sol: Rolle's Theorem states that in the 3rd condition that there exists a point C in given range such that

$f^1(c) = 0$

$f(x) = x^3 - 4x$

$f^1(x) = 3x^2 - 4$

$f^1(c) = 0$

$3c^2 - 4 = 0 \Rightarrow C = \dfrac{\pm 2}{\sqrt{3}}$

Both $\dfrac{2}{\sqrt{3}}$ & $\dfrac{-2}{\sqrt{3}}$ lies in the given range [–2, 2]

So $C = \dfrac{\pm 2}{\sqrt{3}}$ is the point at which the tangent of f(x) is parallel to X – axis.

16. Ans: 19

Sol: Please Refer ACE Previous Maths Solution Booklet

17. Ans: 2.64

Sol: Please Refer ACE Previous Maths Solution Booklet

18. Ans: (c)

Sol: $\dfrac{f^1(c)}{g^1(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$f(x) = \ln x,\ f^1(x) = \dfrac{1}{x}$

$f(a) = f(1) = \ln(1) = 0$

$f(b) = f(e) = \ln e = 1$

$f^1(c) = \dfrac{1}{c}$

$g(x) = \dfrac{1}{x}, \qquad g^1(x) = \dfrac{-1}{x^2}$

$g(a) = g(1) = \dfrac{1}{1} = 1$

$g(b) = g(e) = \dfrac{1}{e}$

$g^1(c) = \dfrac{-1}{c^2}$

$\dfrac{1/c}{-1/c^2} = \dfrac{1 - 0}{\frac{1}{e} - 1} = \dfrac{e}{1 - e}$

$-c = \dfrac{e}{1 - e}$

$c = \dfrac{e}{e - 1}$

19. Ans: (b)

Sol: Taylor series expansion for function f(x) around x = 0 is

$$f(x)=f(0)+xf^{1}(0)+\frac{x^2}{2!}f^{11}(0)+\frac{x^3}{3!}f^{111}(0)+----$$

$$f(x)=\frac{-1}{1+x},\qquad f(0)=\frac{-1}{1+0}=-1$$

$$f^{1}(x)=\frac{-1\times-1}{(1+x)^2}=\frac{1}{(1+x)^2},\ f^{1}(0)=1$$

$$f^{11}(x)=\frac{-2}{(1+x)^3},\ f^{11}(0)=-2$$

$$f^{111}(x)=\frac{6}{(1+x)^4},\ f^{111}(0)=6$$

$$f^{iv}(x)=\frac{-24}{(1+x)^5},\ f^{iv}(0)=-24$$

f(x) = –1 + (x × 1) +

$$\frac{x^2}{2!}\times-2+\frac{x^3}{3!}\times6+\frac{x^4}{4!}\times-24+----$$

$$=-1+x-x^2+x^3-x^4+----$$

20. Ans: (a)

Sol: Coefficient of $x^4=\frac{f^{IV}(0)}{4!}$

Given f(x) = log (secx)

$$\Rightarrow f'(x)=\frac{1}{\sec x}\sec x\tan x=\tan x$$

$$\Rightarrow f''(x)=\sec^2x$$

$$\Rightarrow f'''(x)=2\sec^2x\tan x$$

$$\Rightarrow f^{IV}(x)=2[\sec^2x\sec^2x+\tan x.\ 2\sec x.\sec x\tan x]$$

$$\Rightarrow f^{IV}(0)=2$$

$$\therefore \text{Coefficient of } x^4=\frac{f^{IV}(0)}{4!}=\frac{2}{24}=\frac{1}{12}$$

21. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

22. Ans: 4

Sol: Please Refer ACE Previous Maths Solution Booklet

23. Ans: (d)

Sol: Please Refer ACE Previous Maths Solution Booklet

24. Ans: (a)

Sol: Given $u=x^{-2}\tan\left(\frac{y}{x}\right)+3y^3\sin^{-1}\left(\frac{x}{y}\right)$

= f(x, y) + 3 g(x,y)

Where f(x, y) is homogeneous with deg m = –2 and g(x, y) is homogeneous with deg n = 3

$\Rightarrow x^2.\ u_{xx}+2xy\ u_{xy}+y^2\ u_{yy}$

= m(m–1) f(x,y) + n(n–1) g(x,y)

= –2(–2–1) f(x,y) + 3[3(3–1)g(x,y)]

= 6 [f(x,y) + 3 g(x,y)]= 6u

25. Ans: 4

Sol: Please Refer ACE Previous Maths Solution Booklet

26. Ans: 2

Sol: Please Refer ACE Previous Maths Solution Booklet

27. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

28. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

29. Ans: (c)

Sol: $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3(1 - 2) - 2(-1 - 1) - 1(2 + 1)$$

$$= -2$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{-1}{2}$$

30. Ans: (b), (d)

Sol: Please Refer ACE Previous Maths Solution Booklet

31. Ans: (a & d)

Sol: $$f'(x) = 6\left(\frac{4}{3}\right)x^{\frac{1}{3}} - 3\left(\frac{1}{3}\right)x^{-2/3} = \frac{8x-1}{x^{2/3}}$$

Critical points are $x = 0, \frac{1}{8}$

$$f(-1) = 6(-1)^{4/3} - 3(-1)^{1/3} = 9$$

$$f(0) = 0$$

$$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{8}\right)^{4/3} - 3\left(\frac{1}{8}\right)^{1/3} = \frac{-9}{8}$$

$$f(1) = 6 - 3 = 3$$

Clearly from the above values absolute minimum is – 9/8, absolute maximum is 9

32. Ans: 1

Sol: Please Refer ACE Previous Maths Solution Booklet

33. Ans: 2

Sol: Please Refer ACE Previous Maths Solution Booklet

34. Ans: 8

Sol: Please Refer ACE Previous Maths Solution Booklet

35. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

36. Ans: (a)

Sol: Given $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

Consider $f_x = 4x - 4x^3 = 0$

$\Rightarrow x = 0, 1, -1$

Consider $f_y = -4y + 4y^3 = 0$

$\Rightarrow y = 0, 1, -1$

Now, $r = f_{xy} = 4 - 12x^2$, $s = f_{xy} = 0$

and $t = f_{yy} = -4 + 12y^2$

At (0,1), we have $r > 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has minimum at (0,1)

At (–1, 0), we have $r < 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has a maximum at (–1, 0)

37. Ans: (c)

Sol: $f(x, y) = 1 - x^2y^2$

$p = -2xy^2$

$q = -2yx^2$

$r = -2y^2$

$s = -4xy$

$t = -2x^2$

At (0, 0)

$r = 0, \quad s = 0, \quad t = 0$

$rt - s^2 = 0$

f(a+h, b+k)

f(a, b) < f(a+h, b+k)

f(0, 0) < f(h, k)

So point (0, 0) is point of maxima

38. Ans: (a & c)

Sol: Please Refer ACE Previous Maths Solution Booklet

39. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

40. Ans: (d)

Sol: $\int_0^{1.5} x[x^2]\,dx$

$$= \int_0^1 x[x^2]dx + \int_1^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx$$

$$= 0 + \int_0^{\sqrt{2}} x\,dx + \int_{\sqrt{2}}^{1.5} 2x\,dx = \frac{3}{4}$$

41. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

42. Ans: (a)

Sol: Given that, $x\sin(\pi x) = \int_0^{x^2} f(t)dt$

Differentiating both sides, we get

$$x\cos(\pi x).\pi + \sin(\pi x) = f(x).2x$$

Putting x = 4

$$4\pi\cos(4\pi) = f(4).8$$

$$\therefore f(4) = \frac{\pi}{2}$$

43. Ans: (d)

Sol: Please Refer ACE Previous Maths Solution Booklet

44. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

45. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

46. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

47. Ans: 1.7

Sol: Please Refer ACE Previous Maths Solution Booklet

48. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

49. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

50. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

51. Ans: 64

Sol: Please Refer ACE Previous Maths Solution Booklet

52. Ans: (b)

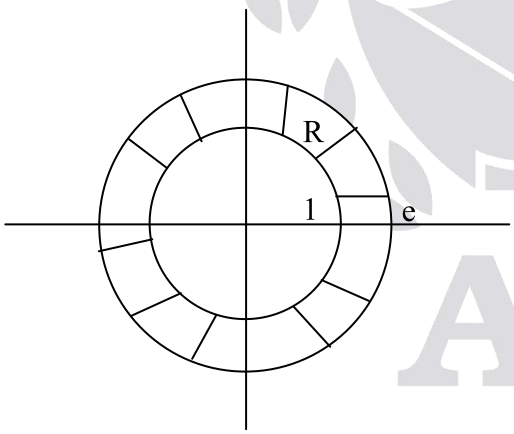
Sol: Please Refer ACE Previous Maths Solution Booklet

53. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

54. Ans: (2π)

Sol:



Let $x = r\cos\theta,\ y = r\sin\theta$

$$\iint_R \frac{\ell n(x^2+y^2)}{x^2+y^2}dxdy = \int_{\theta=0}^{2\pi}\int_{r=1}^{e}\frac{\ell n(r^2)}{r^2}rdrd\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{e}2\frac{\ell n(r)}{r}drd\theta$$

Let $\ell n(r) = t$

$$\frac{1}{r}dr = dt$$

Limits of t are from 0 to 1

$$2\int_{\theta=0}^{2\pi}\int_{t=0}^{1}tdtd\theta = 2\int_{\theta=0}^{2\pi}\left.\frac{t^2}{2}\right|_0^1 d\theta$$

$$= \theta\Big|_0^{2\pi} = 2\pi$$

55. Ans: (2)

Sol: Please Refer ACE Previous Maths Solution Booklet

56. Ans: 0

Sol: Please Refer ACE Previous Maths Solution Booklet

57. Ans: (d)

Sol: Please Refer ACE Previous Maths Solution Booklet

58. Ans: (d)

Sol: $y = \log\sec x$, $\frac{dy}{dx} = \tan x$

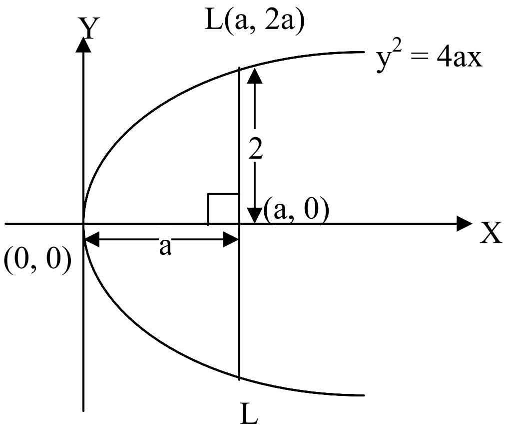
Length of curve $= \int_0^{\pi/4}\sqrt{1+\tan^2 x}\ dx$

$$= \int_0^{\pi/4}\sec x\ dx = \log(\sec x + \tan x)\Big|_0^{\pi/4}$$

$$= \log\left(\sqrt{2}+1\right)$$

59. Ans: (a)

Sol:



$y^2 = 4ax$

The required volume $= \int_{x=0}^{a} \pi y^2\, dx$

$y^2 = 4ax$

$$I = \int_{x=0}^{a} \pi(4ax)dx$$

$$I = \pi 4a \left.\frac{x^2}{2}\right|_0^a$$

$$= \frac{4a\pi}{2} a^2 = 2\pi a^3$$

60. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

61. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

62. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

63. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

64. Ans: (d)

Sol: Please Refer ACE Previous Maths Solution Booklet

65. Ans: 10

Sol: Please Refer ACE Previous Maths Solution Booklet

66. Ans: 5

Sol: Please Refer ACE Previous Maths Solution Booklet

67. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

68. Ans: (d)

Sol: $\bar{u} = y\hat{i} + xy\hat{j}$

$\bar{v} = x^2\hat{i} + xy^2\hat{j}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ y & xy & 0 \\ x^2 & xy^2 & 0 \end{vmatrix}$$

$= i\,(0 - 0) - j(0 - 0) + k\,(xy^3 - yx^3)$

$= k\,(xy^3 - yx^3)$

$$\nabla \times (\vec{u} \times \vec{v}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (xy^3 - x^3 y) \end{vmatrix}$$

$= i\,(3xy^2 - x^3) - j\,(y^3 - 3x^2y) + 0$

$= (3xy^2 - x^3)i - (y^3 - 3x^2y)j$

69. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

70. Ans: 22

Sol: Please Refer ACE Previous Maths Solution Booklet

71. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

72. Ans: 2

Sol: Please Refer ACE Previous Maths Solution Booklet

73. Ans: 9.42

Sol: Please Refer ACE Previous Maths Solution Booklet

74. Ans: 8

Sol: Please Refer ACE Previous Maths Solution Booklet

75. Ans: (d)

Sol: The given surface is a closed surface.

$$\oint\!\!\oint_s \overline{F}\,\overline{N}\,ds = \iiint_V \nabla \bullet \overline{F}\,dv$$

$$\nabla \bullet \overline{F} = 3$$

$$\iiint_V 3dv = 3\iiint dv$$

$= 3$ (Volume of Cylinder)

$= 3\,(\pi\,(4^2)\,(2)) = 96\,\pi$

76. Ans: 264

Sol: Using Gauss–Divergence Theorem,

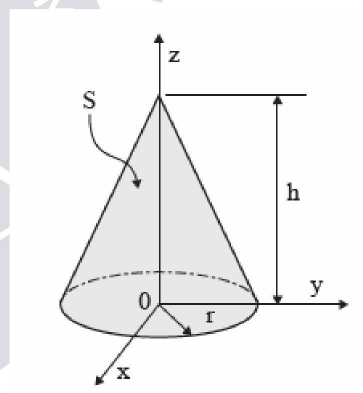
$$\iint_S xy\,dy\,dz + yz\,dzdx + zx\,dx\,dy = \iiint_V \operatorname{div}\overline{F}\,dv$$

$$= \iiint_V (y+z+x)\,dv$$

$$= \int_{x=0}^{4}\int_{y=0}^{3}\int_{z=0}^{4}(x+y+z)\,dzdydx$$

$$= \int_{x=0}^{4}\int_{y=0}^{3}[4x+4y+8]\,dy\,dz$$

$$= \int_{x=0}^{4}[12x+18+24]\,dx = 264$$



77. Ans: 0

Sol: Please Refer ACE Previous Maths Solution Booklet

78. Ans: (b)

Sol: $\oint \vec{F}.d\vec{r} = \oint_C x^3\,dx + (x+y-z)dy + yzdz$

$z = 0,\ dz = 0$

By Using Green's Theorem

$$\oint x^3dx + (2+y)dy$$

$$= \iint 1.dxdy = \pi$$

79. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

80. Ans: –6.58

Sol: Please Refer ACE Previous Maths Solution Booklet

81. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

82. Ans: (c)

Sol: Given function is odd function

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 2\sin nx\,dx$$

$$= \frac{4}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{n}$$

$$b_n = \frac{4}{\pi n}\left(-(-1)^n + 1\right)$$

$$\sum_{n=1}^{\infty}\frac{4}{\pi n}\left(1-(-1)^n\right)\sin(nx)$$

$$\frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x + \frac{8}{5\pi}\sin 5x + ----$$

Let $x = \frac{\pi}{2}$

$$\frac{8}{\pi} + \frac{-8}{3\pi} + \frac{8}{5\pi} + -----$$

$$\frac{8}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-----\right) = 2$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-1----=\frac{\pi}{4}$$

83. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

84. Ans: (c)

Sol: $f(x) = \pi x - x^2$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\pi x - x^2\right)\sin nx\,dx$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\left[\left(\pi x - x^2\right)\sin x\right]dx$$

$$\frac{2}{\pi}\left[\left(\pi x - x^2\right)(-\cos x) - (\pi - 2x)(-\sin x) + (-2)\cos x\right]_0^{\pi} = \frac{8}{\pi}$$

85. Ans: (b)

Sol: $f(x) = (x-1)^2$

The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x-1)^2\cos(n\pi x)\,dx$$

$$\frac{2}{\pi}\left[(x-1)^2.\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1).\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}$$

Answer Key

1	**A**	2	**D**	3	**D**	4	**2**	5	**A**	6	**0.5**	7	**C**	8	**0.5**	9	**0.25**	10	**A**
11	**C**	12	**B**	13	**B**	14	**D**	15	**C**	16	**19**	17	**C**	18	**C**	19	**B**	20	**A**
21	**C**	22	**4**	23	**D**	24	**A**	25	**4**	26	**2**	27	**C**	28	**A**	29	**C**	30	**B, D**
31	**A&D**	32	**1**	33	**2**	34	**8**	35	**C**	36	**A**	37	**C**	38	**A &D**	39	**A**	40	**D**
41	**A**	42	**A**	43	**D**	44	**B**	45	**A**	46	**C**	47	**1.7**	48	**C**	49	**C**	50	**B**
51	**64**	52	**B**	53	**C**	54	**2π**	55	**2**	56	**0**	57	**D**	58	**D**	59	**A**	60	**B**
61	**B**	62	**C**	63	**B**	64	**D**	65	**10**	66	**5**	67	**B**	68	**D**	69	**C**	70	**22**
71	**B**	72	**2**	73	**9.42**	74	**8**	75	**D**	76	**264**	77	**0**	78	**B**	79	**B**	80	**–6.58**
81	**A**	82	**C**	83	**C**	84	**C**	85	**B**										

Probability

(Solutions for Text Book Practice Questions)

01. Ans: (0.6548) (0.653 to 0.655)

Sol: Nine members = 4 Btech + 3 Mtech + 2 PHD

The probability of removing 2 students from the same category & third one from any other

$$= \frac{C_2^4\left(C_1^2 + C_1^3\right) + C_2^3 \times \left(C_1^4 + C_1^2\right) + C_2^2\left(C_1^3 + C_1^4\right)}{C_3^9}$$

$$= \frac{6(2+3) + 3\times(4+2) + 1\times(3+4)}{84}$$

$$= \frac{55}{84} = 0.6548$$

02. Ans: (b)

Sol: Total outcome = 6^6

Possible arrangements of six faces of 6 dices = 6!

The probability of throwing six perfect dice & getting six different faces $= \frac{6!}{6^6}$

03. Ans: (10)

Sol: Please Refer ACE Previous Maths Solution Booklet

04. Ans: (c)

Sol: Given : P(j) α j

$$P(j) = kj,\ j = 1, 2, 3, 4, 5, 6$$

X = j	1	2	3	4	5	6
P(X = j)	k	2k	3k	4k	5k	6k

$$\sum P(X = j) = 1$$

$$21\,k = 1$$

$$k = \frac{1}{21}$$

$\therefore$ P(odd number of dots)

$$= P(X = 1) + P(X = 3) + P(X = 5)$$

$$= k + 3k + 5k = 9\,k$$

$$= \frac{9}{21} = \frac{3}{7}$$

05. Ans: (a)

Sol: Total cases for arranging six boys and six girls in a row = 12!

Assume six girls as one unit.

We have a total of six boys + 1 unit = 7

They can be arranged among themselves in 7! ways and six girls can be arranged among themselves in 6! ways.

Favourable cases = 6! × 7!

$\therefore$ Required probability $= \frac{6!\times 7!}{12!}$

<u>**Conditional Probability**</u>

06. Ans: (b)

Sol:

P(E U F) = 0.8

P (E) = 0.4

P (E/F) = 0.3

$$P(E/F)=\frac{P(E\cap F)}{P(F)}$$

$$P(E/F)=\frac{P(E)+P(F)-P(E\cup F)}{P(F)}$$

$$0.3=\frac{0.4+P(F)-0.8}{P(F)}$$

0.3 P(F) = P(F) – 0.4

0.7 P(F) = 0.4

$$P(F)=\frac{4}{7}$$

07. Ans: (a)

Sol:

Case (i) Let first ball drawn is blue & second ball drawn is also blue

$$P_1=\frac{C_1^b}{C_1^{b+r}}\times\frac{C_1^{b-1}}{C_1^{b+r-1}}=\frac{b(b-1)}{(b+r)(b+r-1)}$$

Case (ii) Let first bal drawn is red & Second ball drawn is blue

$$P_2=\frac{C_1^r}{C_1^{b+r}}\times\frac{C_1^b}{C_1^{r+b-1}}=\frac{b\,r}{(b+r)(r+b-1)}$$

So the probability that the second ball drawn is blue = $P_1 + P_2$

$$=\frac{b(b-1)+br}{(b+r)(b+r-1)}=\frac{b(b-1+r)}{(b+r)(b+r-1)}=\frac{b}{b+r}$$

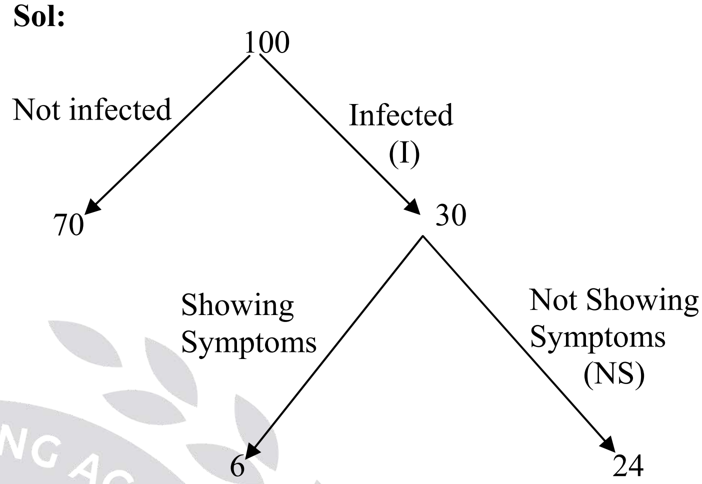
08. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

<u>Baye's Theorem</u>

09. Ans: (0.2553) (0.24 to 0.26)

Sol:



The person selected at random not shown any symptoms of COVID–19 then the probability that the person infected with COVID – 19

$$=P(I/NS)=\frac{P(I\cap NS)}{P(NS)}=\frac{24}{70+24}$$

$$=\frac{24}{94}=0.2553$$

10. Ans: (0.375 to 0.375)

Sol: P(T) = Probability that person speaks truth $=\frac{3}{4}$

P(F) = Probability that person speaks false $=\frac{1}{4}$

P(E/T) = Probability of fair die outcome is 5 given that persons speaks truth $=\frac{1}{6}$

P(E/F) = Probability of fair die outcome is 5 given that person speaks false

$$= 1 - P(E/T)$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

P = The probability that the outcome is really 5 is

$$= \frac{P(T)P(E/T)}{P(T)P(E/T) + P(F)P(E/F)}$$

$$= \frac{\frac{3}{4}\times\frac{1}{6}}{\left(\frac{3}{4}\times\frac{1}{6}\right)+\left(\frac{1}{4}\times\frac{5}{6}\right)}$$

$$= \frac{3}{3+5} = \frac{3}{8} = 0.375$$

Independent Events

11. Ans: (d)

Sol: P(X) = probability that the number greater than four appears $= \frac{2}{6} = \frac{1}{3}$

$P^1(X)$ = Probability that the number less than equal to 4 appears

$$= 1 - P(X) = 1 - \frac{1}{3} = \frac{2}{3}$$

Probability of Number of trials is even

$= P^1(X)P(X) + (P^1(X))^3\,P(X) + (P^1(X))^5\,P(X) + \text{-- --}$

$$= \left(\frac{2}{3}\times\frac{1}{3}\right) + \left(\left(\frac{2}{3}\right)^3\times\frac{1}{3}\right) + \left(\left(\frac{2}{3}\right)^5\times\frac{1}{3}\right) + \text{---}$$

$$= \frac{2}{9} + \frac{8}{81} + \frac{32}{729} + \text{----}$$

$$= \frac{2}{9}\left(1 + \frac{4}{9} + \frac{16}{81} + \text{-----}\right)$$

$$= \frac{2}{9}\times\frac{1}{\left(1-\frac{4}{9}\right)} = \frac{2}{9}\times\frac{9}{5} = \frac{2}{5}$$

12. Ans: (c, d)

Sol: Please Refer ACE Previous Maths Solution Booklet

Random Variables

13. Ans: 0.1875 (Range 0.17 to 0.19)

Sol: Please Refer ACE Previous Maths Solution Booklet

14. Ans: (b)

Sol: P(2 white + 1 Black ball) $= \frac{C_2^5 \times C_1^7}{C_3^{12}} = \frac{7}{22}$

P (2 Black + 1 White ball) $= \frac{C_2^7 \times C_1^5}{C_3^{12}} = \frac{21}{44}$

P (3 White balls) $= \frac{C_3^5}{C_3^{12}} = \frac{1}{22}$

P (3 Black balls) $= \frac{C_3^7}{C_3^{12}} = \frac{7}{44}$

Total expectation =

$$\frac{1}{22}(60) + \frac{7}{44}(30) + (2\times 20 + 10)\frac{7}{22} + (20 + 2\times 10)\times\frac{21}{44}$$

$$= \frac{120 + 210 + 700 + 840}{44} = \frac{1870}{44} = 42.5$$

15. Ans: (0.35)

Sol: Please Refer ACE Previous Maths Solution Booklet

16. Ans: (c)

Sol: E(Area) = E($2x^2$)

$= 2\,E(x^2)$

$$= 2\int_{-\infty}^{\infty} x^2 f(x)dx = 2\int_{x=0}^{2} \frac{x^2}{2}dx$$

$$= \frac{2}{2}\left.\frac{x^3}{3}\right|_0^2 = \frac{1}{3}(8) = \frac{8}{3}$$

Binomial Distribution

17. Ans: 0.275 (Range 0.26 to 0.28)

Sol: p = 0.8,

q = 1 – p = 1 – 0.8 = 0.2

The probability of full occupancy exactly for r = 5 days in a week (n = 7)

$$= C_r^n (p)^2 (q)^{n-r}$$

$$= C_5^7 \times (0.8)^5 (0.2)^2$$

$$= \frac{7!}{5!\times 2!} \times (0.8)^5 \times (0.2)^2$$

$$= \frac{7\times 6}{2} \times (0.8)^5 \times 0.04 = 0.27525$$

18. Ans: 0.1648

Sol: $P(6) = \frac{1}{6}$

Y = P (6 appeared exactly twice)

$$= C_2^3 \times \left(\frac{1}{6}\right)^2 \times \left(1-\frac{1}{6}\right)^1 = C_2^3 \times \frac{1}{36} \times \frac{5}{6}$$

$$= 3 \times \frac{1}{36} \times \frac{5}{6} = \frac{5}{72}$$

X = P (6 appeared at least once)

$$= 1 - \left(C_0^3 \times p^0\ q^3\right)$$

$$= 1 - \left(C_0^3 \times \left(\frac{5}{6}\right)^3\right) = 1 - \frac{125}{216} = \frac{91}{216}$$

$$P(y/x) = \frac{P(y\cap x)}{p(x)} = \frac{P(y)}{P(x)} = \frac{5/72}{91/216}$$

= 0.1648

Poisson Distribution

19. Ans: (0.2593)

Sol: No. of sides coloured Red = 2

No. of sides coloured Blue = 2

No. of sides coloured Green = 2

P = Probability to get red face in one throw

$$= \frac{2}{6} = \frac{1}{3}$$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

The Probability of obtaining red colour on the top face of the die at least twice

$$= \left[C_2^3\left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^1\right] + \left[C_3^3\left(\frac{1}{3}\right)^3 \times \left(\frac{2}{3}\right)^0\right]$$

$$= \left(3\times\frac{1}{9}\times\frac{2}{3}\right) + \left(1\times\frac{1}{27}\right)$$

$$= \frac{6+1}{27} = \frac{7}{27} = 0.2593$$

20. Ans: (12)

Sol: $P(X) = \dfrac{\lambda^x e^{-\lambda}}{x\,!}, \quad x = 0, 1, 2, \ldots\ldots\ldots$

$P(y) = \dfrac{\lambda^y e^{-\lambda}}{y\,!}, \quad y = 0, 1, 2, \ldots\ldots\ldots$

$P(X = 1) = P(X = 2)$

$\lambda e^{-\lambda} = \dfrac{\lambda^2 e^{-\lambda}}{2}$

$\Rightarrow \lambda_x = 2$

$P(Y = 3) = P(Y = 4)$

$\dfrac{\lambda^3 e^{-\lambda}}{3\,!} = \dfrac{\lambda^4 e^{-\lambda}}{4\,!} \Rightarrow \dfrac{1}{6} = \dfrac{\lambda}{24}$

$\lambda_y = 4$

$Var(2X - Y) = (2)^2\, var(X) + (-1)^2\, var(Y)$

$= 4 \times 2 + (1) \times 4 = 12$

21. Ans: (c)

Sol: Please Refer ACE Previous Maths Solution Booklet

22. Ans: (a)

Sol: $E[\cos \pi x] = \sum_{k=0}^{\infty} \dfrac{e^{-\lambda}\lambda^k}{k!} \cos(k\pi)$

$= \sum_{k=0}^{\infty} \dfrac{e^{-\lambda}\lambda^k}{k!} (-1)^k$

In Poisson Theorem→mean= expectation = $\ln 2 = \lambda$

$E[\cos \pi x] = \sum_{k=0}^{\infty} \dfrac{e^{-\ell n2}(\ell n2)^k}{k!}(-1)^k$

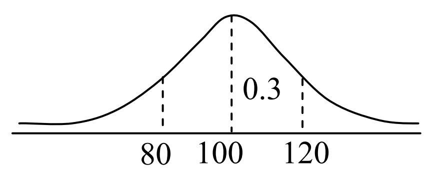
$= e^{-\ell n2}\left[1 - \dfrac{\ell n2}{1!} + \dfrac{(\ell n2)^2}{2!} - \dfrac{(\ell n2)^3}{3!} + - - - - - \right]$

$= e^{-\ln 2}\,.\,e^{-\ln 2} = e^{-2\ell n2} = e^{\ell n\left(\frac{1}{4}\right)} = \dfrac{1}{4}$

Normal Distribution

23. Ans: (0.2)

Sol: The area under normal curve is 1 and the curve is symmetric about mean.



$\therefore P(100 < X < 120) = P(80 < X < 120)$

$= 0.3$

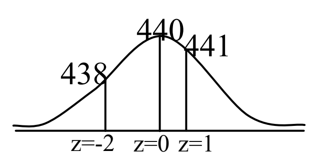
Now, $P(X < 80) = 0.5 - P(80 < X < 120)$

$= 0.5 - 0.3 = 0.2$

24. Ans: (a)

Sol: The standard normal variable Z is given by

$Z = \dfrac{x - \mu}{\sigma}$



When x = 438

$Z = \dfrac{438 - 440}{1} = -2$

When x = 441

$Z = \dfrac{441 - 440}{1} = 1$

The percentage of rods whose lengths lie between 438 mm and 441 mm

$= P(438 < x < 441)$

$= P(-2 < Z < 1)$

$= P(-2 < Z < 0) + P(0 < Z < 1)$

$= \frac{0.9545}{2} + \frac{0.6826}{2} = 0.81855 \approx 81.85\ \%$

25. Ans: (0.804 to 0.806)

Sol: The probability of population has Alzheimer's disease is

$p = 0.04,\ q = 0.96,\ n = 3500$

$\mu = np = (3500)\,(0.04) = 140$

$\sigma^2 = npq = (3500)\,(0.04)\,(0.96)$

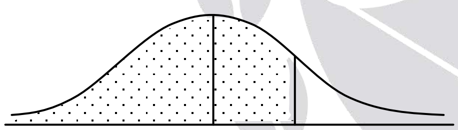
$\sigma^2 = 134.4,\ \sigma \cong 11.59$

Let X = number of people having Alzheimer's disease

$$P(X < 156) = P\left(\frac{X-\mu}{\sigma} < \frac{150-\mu}{\sigma}\right)$$

$$= P\left(Z < \frac{150-140}{11.59}\right)$$

$$= P(Z < 0.86)$$



$= 0.5 +$ Area between $z = 0$ & $z = 0.86$

$= 0.5 + 0.3051 = 0.8051$

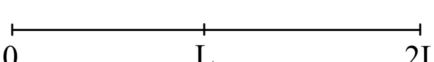
26. Ans: (b)

Sol: Please Refer ACE Previous Maths Solution Booklet

Uniform Distribution

27. Ans: (c)

Sol:



To get a shorter piece, it can be broken anywhere between 0 and L.

Let X be a random variable uniformly distributed in [0, L]

Mean $E(X) = \frac{0+L}{2} = \frac{L}{2}$

$\text{var}(X) = \frac{(L-0)^2}{12} = \frac{L^2}{12}$

28. Ans: (0.39 to 0.41)

Sol: $x \sim$ UNIF $(-5,\ 5)$

$f(x) = \frac{1}{10},\ -5 < x < 5$

$= 0,$ elsewhere

$P[100t^2 + 20tx + 2x+3 = 0$ has complex solutions]

$= P\{[(20x)^2 - 4(100)(2x+3)] \le 0\}$

$= P\{[400x^2 - 800x - 1200] \le 0\}$

$= P\{(x^2 - 2x - 3) \le 0\}$

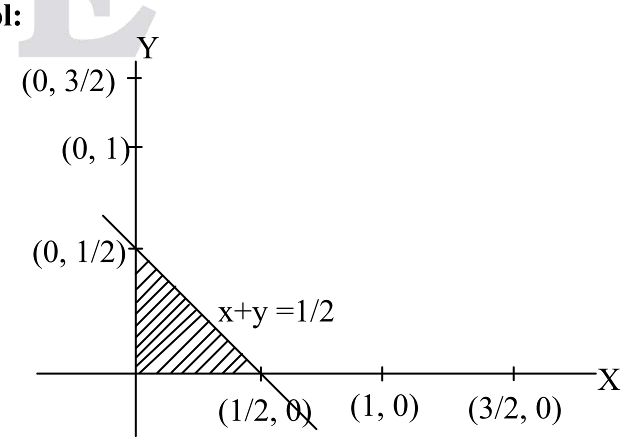
$= P\{[(x-3)\,(x+1) \le 0\}$

$= P(-1 < x < 3)$

$$= \int_{-1}^{3} \frac{1}{10}dx = \frac{1}{10}(x)_{-1}^{3} = \frac{4}{10} = 0.4$$

29. Ans: 0.125

Sol:



Area represents $\left[x+y<\frac{1}{2}\right]$

$P\left(x+y<\frac{1}{2}\right)=$ Area shaded in above figure.

$$=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\ =\frac{1}{8}=0.125$$

Exponential Distribution

30. Ans: 0.367

Sol: For exponential distribution

$$f(x) = \theta e^{-\theta x} \qquad x \ge 0$$
$$= 0 \qquad \text{Otherwise}$$

$$\text{Mean} = E(x)=\frac{1}{\theta}$$

$$p\left(x>\frac{1}{\theta}\right)=\int_{1/\theta}^{\infty} f(x)dx$$

$$=\int_{1/\theta}^{\infty}\theta e^{-\theta x}\,dx$$

$$=\theta\left[\frac{e^{-\theta x}}{-\theta}\right]_{x=\frac{1}{\theta}}^{\infty}$$

$$= -1 \times (0 - e^{-1})$$
$$= e^{-1} = 0.367$$

31. Ans: 0.1 to 0.1

Sol: $\frac{1}{\lambda_1}=1\Rightarrow\lambda_1=1$

$$\frac{1}{\lambda_2}=\frac{1}{2}\Rightarrow\lambda_2=2$$

$$\frac{1}{\lambda_3}=\frac{1}{3}\Rightarrow\lambda_3=3$$

$$\frac{1}{\lambda_4}=\frac{1}{4}\Rightarrow\lambda_4=4$$

The distribution $y = \min(x_1, x_2, x_3, x_4)$ is also an exponential distribution with mean

$$\frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}=0.1$$

32. Ans: 0.0024 to 0.0026

Sol: $\lambda = 2$

$$P(x>3)=\int_3^{\infty}\lambda e^{-\lambda x}dx \ =\lambda\left(\frac{-e^{-\lambda x}}{\lambda}\right)_3^{\infty}$$

$$= (-e^{-\infty}) - (-e^{-3\lambda})$$
$$= e^{-3(2)} = e^{-6} = 0.0025$$

Statistics

33. Ans: 37.2, 39, 43.33

Sol: $$\text{Mean}=\frac{5\times4+15\times5+25\times7+35\times10+45\times12+55\times8+65\times4}{4+5+7+10+12+8+4}=37.2$$

For median

Class	f	Cf	
0 – 10	4	4	
10 – 20	5	9	
20 – 30	7	16	
$\boxed{30 – 40}$	$\boxed{10}$	$\boxed{26}$	→Median class
40 – 50	12	38	
50 – 60	8	46	
60 – 70	4	50	

Here : $N = \sum f = 50,\ \frac{N+1}{2}=25.5$

l = lower limit of the class interval of the median class = 30

m = Cumulative frequency preceding the median class = 16

f = frequency of the median class = 10

C = size of the class = 10

$$\text{median} = \ell + \left\{ \frac{\frac{N}{2} - m}{f} \right\} C$$

$$= 30 + \left\{ \frac{\frac{50}{2} - 16}{10} \right\} 10 = 39$$

For Mode :

Class	Freq
0 – 10	4
10 – 20	5
20 – 30	7
30 – 40	10
40 – 50	12
50 – 60	8
60 – 70	4

→Modal class (40 – 50)

l = lower limit of the modal class = 40

f = frequency of modal class = 12

f_{-1} = frequency preceding the modal class

f_1 = frequency succeeding the modal class

C = size of the class = 10

$f_{-1} = 10$, $f_1 = 8$

$\Delta_1 = f - f_{-1} = 12 - 10 = 2$

$\Delta_2 = f - f_1 = 12 - 8 = 4$

$$\text{mode} = \ell + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2} \right) C$$

$$= 40 + \left(\frac{2}{2+4} \right) 10 = 43.33$$

34. Ans: 4

Sol: Please Refer ACE Previous Maths Solution Booklet

Correlation and Regression Analysis

(Only for EE, CE, IN)

35. Ans: (a)

Sol: Please Refer ACE Previous Maths Solution Booklet

36. Ans: 1

Sol: Please Refer ACE Previous Maths Solution Booklet

37. Ans: 0.18

Sol: Given: $b_{yx} = 1.6$ and $b_{xy} = 0.4$

$$r = \sqrt{b_{yx}\, b_{xy}}$$

$$r = \sqrt{1.6 \times 0.4}$$

$$r = 0.8$$

Now, $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

$$1.6 = 0.8\frac{\sigma_y}{\sigma_x}$$

$$\frac{\sigma_y}{\sigma_x} = \frac{1.6}{1.8} = \frac{2}{1}$$

$\Rightarrow \sigma_x = 1$ and $\sigma_y = 2$

The angle between two regression lines is

$$\tan\theta = \left(\frac{1 - r^2}{r} \right) \left(\frac{\sigma_x\ \sigma_y}{\sigma_x{}^2 + \sigma_y{}^2} \right)$$

$$= \left\{\frac{1-(0.8)^2}{0.8}\right\}\left\{\frac{(1)(2)}{(1)^2+(2)^2}\right\} = 0.18$$

38. Ans: (a, b, c, d)

Sol: Y = 5X – 15 ------ (i) (given regression line)

The regression coefficient of y on x is $b_{yx} = 5$

so option (b) is correct.

Y = 10X – 35 ------ (ii) (given regression line)

$$x = \frac{y}{10} + \frac{35}{10} = 0.1y + 3.5$$

The regression coefficient of x on y is

$b_{xy} = 0.1$ so option (a) is also correct.

Correlation coefficient $r = \sqrt{b_{xy} b_{yx}}$

$$= \sqrt{0.1 \times 5}$$

$$r = \sqrt{\frac{5}{10}} = \frac{1}{\sqrt{2}}$$

So option (c) is also correct.

Equate both equation (i) & (ii) line

$5x - 15 = 10x - 35 \rightarrow \bar{x} = 4, \ \bar{y} = 5$

So means of $\bar{x} = 4$ & $\bar{y} = 5$

So option (d) is also correct.

Joint Distribution (only For EC)

39. Ans: (0.32 to 0.34)

Sol: Please Refer ACE Previous Maths Solution Booklet

Differential Equation

(Solutions for Text Book Practice Questions)

First Order Differential Equations

01. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

02. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

03. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

04. Ans: (d)

Sol: Please Refer ACE previous maths solution booklet

05. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

06. Ans: 0.4462

Sol: Please Refer ACE previous maths solution booklet

07. Ans: (a)

Sol: $\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$

Let $t = \frac{y}{x}$

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = 1 + t + t^2$$

$$x\frac{dt}{dx} = 1 + t^2$$

$$\int \frac{dt}{1+t^2} = \int \frac{dx}{x}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \log x + C$$

08. Ans: (a)

Sol: Let $t = \frac{y}{x}$

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = t + \cot t$$

$$\frac{dt}{\cot(t)} = \frac{dx}{x}$$

$$\int \tan(t)dt = \int \frac{dx}{x} + C$$

log sec t = log x + logC

$$\sec\left(\frac{y}{x}\right) = xC$$

09. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

10. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

11. Ans: (d)

Sol: Given that $(y+x^2)\,dx + (ax+by^3)\,dy = 0$ is an exact differential equation.

Comparing the given differential equation with general differential equation of the form $M(x,y)dx+N(x,y)dy = 0$,

we get $M = x^2 + y$ and $N(x,y) = ax + by^3$.

$Mdx+Ndy = 0$ is exact Differential Equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Consider $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow 1 = a$

$\therefore$ The given differential equation is exact for a = 1 and for all values of b.

12. Ans: (b)

Sol: Given that

$(2y)dx + (2x\ \log x - xy)dy = 0$ ______(1)

$(\because\ M\,dx + N\,dy = 0)$

$\Rightarrow M = 2y$ and $N = 2x\log x - xy$

Here, $\frac{\partial M}{\partial y} = 2 \neq \frac{\partial N}{\partial x} = 2[x.\frac{1}{x} + \log x] - y$

$= 2 + 2\log x - y$

$\Rightarrow$ The given D.E (1) is not an exact D.E.

Consider $\frac{M_y - N_x}{N} = \frac{2-[2+2\log x - y]}{2x\log x - xy}$

$= \frac{-(2\log x - y)}{x(2\log x - y)} = \frac{-1}{x}$

$\Rightarrow \text{I.F} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

Now,

$\left(\frac{1}{x}\right)[(2y)dx + (2x\log x - xy)dy] = \left(\frac{1}{x}\right)(0)$

$\Rightarrow \left(\frac{2y}{x}\right)dx + (2\log x - y)dy = 0$

$\Rightarrow \int\left(\frac{2y}{x}\right)dx + \int(0-y)dy = c$

$\therefore 2y\log x - \frac{y^2}{2} = c$ is a general solution of (1)

13. Ans: (d)

Sol: Given that

$(y+xy^2)\,dx + (x-x^2y)dy = 0$ …….. (1)

$(\because\ M\,dx + N dy = 0)$

$\Rightarrow M = y + xy^2$ and $N = x - x^2y$

Here, $\frac{\partial M}{\partial y} = 1 + 2xy \neq \frac{\partial N}{\partial x} = 1 - 2xy$

$\Rightarrow$The given D.E (1) is not an exact D.E

Consider I.F =

$\frac{1}{M.x - N.y} = \frac{1}{(xy + x^2y^2) - (xy - x^2y^2)} = \frac{1}{2x^2y^2}$

Now, $\left(\frac{1}{2x^2y^2}\right)[(y+xy^2)dx + (x - x^2y)dy] = \left(\frac{1}{2x^2y^2}\right)(0)$

$$\Rightarrow \left[\frac{1}{2x^2y}+\frac{1}{2x}\right]dx+\left[\frac{1}{2xy^2}-\frac{1}{2y}\right]dy=0$$

$$\Rightarrow \int\left[\frac{1}{2x^2y}+\frac{1}{2x}\right]dx+\int\left[0-\frac{1}{2y}\right]dy=c$$

$$\Rightarrow \frac{-1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$\therefore \log\left(\frac{x}{y}\right)-\frac{1}{xy}=c$ is a required general solution of (1).

14. Ans: (a)

Sol: Given $(x^2y)dx+[-x^3-y^3]\,dy=0$ …… (1)

($\because$ Mdx + Ndy = 0)

$\Rightarrow M = x^2y$ and $N = -x^3-y^3$

Here $\frac{\partial M}{\partial y}=x^2\neq\frac{\partial N}{\partial x}=-3x^2$

$\Rightarrow$ The given D.E (1) is not an Exact D.E

Consider I.F=

$$\frac{1}{M.x+Ny}=\frac{1}{(x^2y)x+[-x^3-y^3]y}=\frac{-1}{y^4}$$

Now,

$$\left(\frac{-1}{y^4}\right)\left[(x^2y)dx+(-x^3-y^3)dy\right]=\left(-\frac{1}{y^4}\right)(0)$$

$$\Rightarrow\left(\frac{-x^2}{y^3}\right)dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0$$

$$\Rightarrow\int\frac{-x^2}{y^3}dx+\int\left(0+\frac{1}{y}\right)dy=c$$

$\therefore \frac{-x^3}{3y^3}+\log(y)=c$ is a general solution of (1)

15. Ans: (b)

Sol: Given D.E is

$(xe^x+e^x)\,dx+(ye^y-xe^x)\,dy=0$ …… (1)

($\because$ Mdx + Mdy = 0)

$\Rightarrow M = xe^x+e^x$ and $N = ye^y-xe^x$

Here, $M_y = 0 \neq N_x = -e^x-xe^x$

$\Rightarrow$The given D.E (1) is not exact D.E

Consider

$$\frac{N_x-M_y}{M}=\frac{(-e^x-xe^x)-0}{xe^x+e^x}=\frac{-(xe^x+e^x)}{xe^x+e^x}=-1$$

$\Rightarrow$ I.F $= e^{\int f(y)dy} = e^{\int -dx}=e^{-y}$

Now,

$(e^{-y})\,[(xe^x+e^x)\,dx+(ye^y-xe^x)dy]=(e^{-y})\,(0)$

$\Rightarrow\int[xe^xe^{-y}+e^xe^{-y}]dx+\int[y-0]\,dy=c$

$\Rightarrow (xe^x-e^x)e^{-y}+e^xe^{-y}+\frac{y^2}{2}=c$

$\therefore xe^{x-y}+\frac{y^2}{2}=c$ is a solution of (1)

16. Ans: (c)

Sol: Given that

$x\,dx+ydy+2(x^2+y^2)\,dx=0$ …… (1)

$$\Rightarrow \frac{xdx+ydy}{x^2+y^2}+2dx=0$$

$$\Rightarrow \frac{1}{2}\left[\frac{2xdx+2ydy}{x^2+y^2}\right]+2dx=0$$

$$\Rightarrow \frac{1}{2}\left[\frac{d(x^2+y^2)}{x^2+y^2}\right]+2dx=0$$

$$\Rightarrow \frac{1}{2}\int\frac{1}{(x^2+y^2)}d(x^2+y^2)+\int 2dx=c$$

$$\therefore \frac{1}{2}\log(x^2+y^2)+2x=c$$

17. Ans: (b)

Sol: Given $x\,dy - y\,dx + 2x^3\,dx = 0$ ……. (1)

$$\Rightarrow \frac{xdy-ydx}{x^2}+2xdx=0$$

$$\Rightarrow d\left(\frac{y}{x}\right)+2xdx=0$$

$$\Rightarrow \int d\left(\frac{y}{x}\right)+2\int x\,dx=0$$

$\therefore \frac{y}{x}+x^2=c$ is a general solution of (1).

18. Ans: (–2)

Sol: Given

$(y-2x^3)dx+(x^2y-x)dy=0$…………. (1)

If x^n is an I.F then after multiplication with x^n, the given equation (1) becomes exact.

Now, $x^n[(y-2x^3)dx+(x^2y-x)dy]=(x^n)(0)$

$\Rightarrow (x^ny-2x^{3+n})dx+(x^{n+2}y-x^{n+1})dy=0$

$\Rightarrow M = x^ny-2x^{3+n}$ & $N = x^{n+2}y-x^{n+1}$ for Exact D.E $M\,dx+N\,dy=0$.

Consider $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$x^n = y(n+2)x^{n+1}-(n+1)x^n$

$\Rightarrow 1=-(n+1)$ ($\because$ coefficient of x^n)

$\therefore n=-2$

19. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

20. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

21. Ans: (a) & (d)

Sol: Given that $\frac{dy}{dx}=\frac{x^3y^2-2y}{x}$

$$\Rightarrow \frac{dy}{dx}+\left(\frac{2}{x}\right)y=x^2y^2 \quad \ldots\ldots \quad (1)$$

$\left(\because \frac{dy}{dx}+P(x).y=Q(x).y^n\right)$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx}+\left(\frac{2}{x}\right)\left(\frac{1}{y}\right)=x^2 \ldots\ldots (2)$$

Let $\frac{1}{y}=z$……. (3)

Then $\frac{-1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}$ …… (4)

Using (3) & (4), (2) becomes

$$\frac{-dz}{dx}+\left(\frac{2}{x}\right)z=x^2$$

$$\Rightarrow \frac{dz}{dx}+\left(\frac{-2}{x}\right)z=\left(-x^2\right)\ldots\ldots. \quad (5)$$

$\left(\because \frac{dz}{dx}+P(x).z=Q(x)\right)$

$$\Rightarrow \text{I.F} = e^{\int\frac{-2}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=\frac{1}{x^2}$$

Now, the general solution of (5) is given by $z.\frac{1}{x^2}=\int\left(-x^2\right)\left(\frac{1}{x^2}\right)dx+c$

$$\Rightarrow \frac{z}{x^2}=-x+c$$

$\therefore \frac{1}{y}=-x^3+cx^2$ is a required solution of (1)

22. Ans: (c)

Sol: Given $x\frac{dy}{dx}+y\log y=xy\,e^x$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}\log y=ye^x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}+(\log y)\left(\frac{1}{x}\right)= e^x\ldots\ldots\ldots \quad (1)$$

$\left(\because f'(y)\frac{dy}{dx}+f(y)P(x)=Q(x)\right)$

Let log y = z …… (2)

Then $\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}$ …. (3)

Using (2) & (3). (1) becomes

$$\frac{dz}{dx}+\left(\frac{1}{x}\right)z=e^x \ldots\ldots (4)$$

$\left(\because \frac{dy}{dx}+P(x)\,z=Q(x)\right)$

$$\Rightarrow \text{I.F} = e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

Now, the general solution of (1) is given by

$z.x=\int e^x.x\,dx+c$

$\Rightarrow(\log y)(x)=xe^x-e^x+c$

$\therefore$ x.log(y) $=e^x(x-1)+c$ is a required solution of (1)

Higher order Differential Equations

23. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

24. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

25. Ans: 0.368

Sol: Please Refer ACE previous maths solution booklet

26. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

27. Ans: (d)

Sol: Please Refer ACE previous maths solution booklet

28. Ans: (4.54)

Sol: Please Refer ACE previous maths solution booklet

29. Ans: (a)

Sol: Given that f(D)y = 0 ….. (1)

where $f(D) = D^4 + 8D^2 + 16$

$\Rightarrow$ A.E is $m^4 + 8m^2 + 16 = 0$

$\Rightarrow (m^2 + 4)^2 = 0$

$\Rightarrow m = 0 \pm 2i, 0 \pm 2i$

$\therefore$ The general solution (G.S) of (1) is

$y = e^{0x}[(c_1+c_2x)\cos(2x) + (c_3 + c_4x)\sin(2x)]$

30. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

31. Ans: (b)

Sol: Given that f(D) y = Q(x) …… (1)

where $f(D) = D^2 - 3D + 2$ & $Q(x) = e^{3x}$

C.F:

A.E is $m^2 - 3m + 2 = 0$

$\Rightarrow m = 1,2$

$\therefore y_c = c_1e^x + c_2e^{2x}$

P.I

Given $Q(x) = e^{3x}$ $(\because Q(x) = ke^{ax})$

$\Rightarrow$ Given $f(D) = f(a) = f(3) = (3)^2 - 3(3) + 2 = 2 \neq 0$

Now, $y_p = \dfrac{1}{f(a)}Q(x)$

$\therefore y_p = \dfrac{e^{3x}}{2}$

Hence, the general solution (G.S) of (1) is

$y = y_c + y_p = (c_1e^x + c_2e^{2x}) + \left(\dfrac{e^{3x}}{2}\right)$

32. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

33. Ans: (d)

Sol: Given f(D)y = Q(x) …… (1),

where $f(D) = D^2 + 6D + 9$ & $Q(x) = 5e^{-3x}$

$(\because Q(x) = ke^{ax})$

Here, $f(D) = f(a) = f(-3) = 9 - 18 + 9 = 0$

$\Rightarrow f'(D) = f'(a) = f'(-3) = 2(-3) + 6 = 0$

$(\because f'(D) = 2D + 6)$

$\Rightarrow f'(D) = f''(a) = f''(-3) = 2 \neq 0$

$(\because f''(D) = 2)$

Now, $y_p = x^2\left[\dfrac{1}{f''(a)}Q(x)\right]$

$\therefore y_p = x^2\left[\dfrac{5e^{-3x}}{2}\right] = \dfrac{5}{2}x^2e^{-3x}$

34. Ans: (a)

Sol: Given f(D)y = Q(x) ……. (1)

where $f(D) = D^2 + 2D + 2$

& Q(x) =

$\sin h(x) = \frac{e^x - e^{-x}}{2} = \left(\frac{e^x}{2}\right) + \left(\frac{-e^{-x}}{2}\right) = Q_1(x) + Q_2(x)$

For $Q_1(x)$:

Here, $Q_1(x) = \frac{1}{2}e^x$ $(\because Q_1(x) = ke^{ax})$

$\Rightarrow f(D) = f(a) = f(1) = 1 + 2(1) + 2 = 5 \neq 0$

$$\therefore\ y_{p_1} = \frac{1}{f(a)}Q_1(x) = \frac{1}{5}\left(\frac{1}{2}e^x\right) = \frac{e^x}{10}$$

For $Q_2(x)$:

Here, $Q_2(x) = \left(\frac{-1}{2}\right)e^{-x}$ $(\because Q_2(x) = ke^{ax})$

$\Rightarrow\ f(D) = f(a) = f(-1) = 1 - 2 + 2 = 1 \neq 0$

$$\therefore y_{p_2} = \frac{1}{f(a)}Q_2(x) = \frac{1}{1}\left(\frac{-1}{2}e^{-x}\right) = \left(\frac{-1}{2}\right)e^{x}$$

Hence, $y_p = y_{p_1} + y_{p_2} = \frac{e^x}{10} - \frac{1}{2}e^{-x}$

35. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

36. Ans: (b)

Sol: Given f(D) y = Q(x) ….. (1),

where $f(D) = D^2 + 4$

& $Q(x) = \sin(2x)$ $(\because Q(x) = k\sin(ax))$

Here, $f(D) = \phi(D^2) = \phi(-a^2) = \phi(-4) = -4 + 4 = 0$

$\Rightarrow f'(D) = \phi'(D^2) = \phi'(-a^2) = \phi'(-4) = 2D \neq 0$

$$\text{Now, } y_p = x\left[\frac{1}{\phi'(-a^2)}Q(x)\right]$$

$$\Rightarrow y_p = x\left[\frac{1}{2D}\sin(2x)\right]$$

$$\therefore y_p = \frac{-x}{4}\cos(2x)$$

37. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

38. Ans: (a)

Sol: Given f(D) y = Q(x)….. (1)

where $f(D) = D^2 + 1$ and $Q(x) = x$

with $y(0) = 1$ …… (2)

& $y'(0) = 1$ …… (3)

C.F:

A.E is $m^2 + 1 = 0$

$\Rightarrow m = 0 \pm i$

$\therefore\ y_c = c_1\cos(x) + c_2\sin(x)$

P.I:

$$\text{Now, } y_p = \frac{1}{f(D)}Q(x) = \frac{1}{D^2+1}(x)$$

$$\Rightarrow y_p = \frac{1}{[1+D^2]}(x) = \left[1+(D^2)\right]^{-1}(x)$$

$\Rightarrow y_p = [1-(D^2) + (D^2)^2 - (D^2)^3 + \ldots\ldots]$ (x)

$\Rightarrow y_p = x - D^2(x) + D^4(x) - D^6(x) + \ldots\ldots$

$\Rightarrow y_p = x$

$\therefore$ The G.S of (1) is $y = y_c + y_y = c_1\cos(x) + c_2\sin(x) + x$…….. (4)

$\Rightarrow y' = -c_1\sin(x) + c_2\cos(x) + 1$……. (5)

Using (2), (4) becomes

$1 = c_1 + 0 + 0$ ……. (6)

Using (3), (5) becomes

$1 = 0 + c_2 + 1$ (or) $c_2 = 0$……. (7)

$\therefore$ The solution of (1) from (4),(6)&(7) is

$y = y(x) = x + \cos x$

39. Ans: (c)

Sol: Given $f(D)\,y = Q(x)$ …. (1),

where $f(D) = D^4 + 4\lambda^4$ & $Q(x) = 1 + x + x^2$

Now,

$$y_p = \frac{1}{f(D)}Q(x) = \frac{1}{D^4 + 4\lambda^4}(1 + x + x^2)$$

$$\Rightarrow y_p = \frac{1}{4\lambda^4\left[1 + \frac{D^4}{4\lambda^4}\right]}(1 + x + x^2)$$

$$\Rightarrow y_p = \frac{1}{4\lambda^4}\left[1 + \left(\frac{D^4}{4\lambda^4}\right)\right]^{-1}(1 + x + x^2)$$

$$\Rightarrow y_p = \frac{1}{4\lambda^4}\left[1 - \left(\frac{D^4}{4\lambda^4}\right) + \left(\frac{D^4}{4\lambda^4}\right)^2 \ldots\ldots\right](1 + x + x^2)$$

$$\Rightarrow y_p = \frac{1}{4\lambda^4}[(1 + x + x^2) - (0) + (0)\ldots\ldots]$$

$$\therefore y_p = \frac{1 + x + x^2}{4\lambda^4}$$

40. Ans: (a),(b) & (d)

Sol: Given that $f(D)y = Q(x)$ …… (1)

where $f(D) = D^2 - 2D - 1$,

$Q(x) = e^x \cos(2x)$

C.F:

A.E is $m^2 - 2m - 1 = 0$

$\Rightarrow m = 1 \pm \sqrt{2}$

$\therefore y_c = [c_1 \cosh\left(\sqrt{2}\right)x + c_2 \sinh(\sqrt{2})\,x]e^x$

P.I:

Given $Q(x) = e^x \cos(x)$ ($\because Q(x) = e^{bx}\,v(x)$)

Now, $y_p = \frac{1}{f(D)}[e^x.\cos(x)]$

$$\Rightarrow y_p = e^x\left[\frac{1}{f(D+1)}\text{cox}(x)\right]$$

$$\Rightarrow y_p = e^x\left[\frac{1}{(D+1)^2 - 2(D+1) - 1}\text{cox}(x)\right]$$

$$\Rightarrow y_p = e^x\left[\frac{1}{D^2 - 2}\text{cox}(x)\right]$$

$$\Rightarrow y_p = e^x\left[\frac{1}{(-1) - 2}\text{cox}(x)\right]$$

($\because D^2 = -a^2 = -1^2 = -1$)

$\therefore$ The general solution of (1) is given by y = y_c + y_p = [c_1 cosh $\left(\sqrt{2}\right)$x + $c_2\sinh\left(\sqrt{2}\right)x]e^x - \dfrac{e^x\cos(x)}{3}$

41. Ans: (a)

Sol: Given that f(D)y = Q(x) ……. (1),

where f(D) = $D^2 - 4D + 4$ & Q(x) = $e^{2x}\,x^3$

($\because Q(x) = e^{bx}.v(x)$)

$$\text{Now, } y_p = \frac{1}{f(D)}[e^{2x}x^3]$$

$$\Rightarrow y_p = e^{2x}\left[\frac{1}{f(D+2)}x^3\right]$$

$$\Rightarrow y_p = e^{2x}\left[\frac{1}{(D+2)^2-4(D+2)+4}(x^3)\right]$$

$$\Rightarrow y_p = e^{2x}\left[\frac{1}{D^2}(x^3)\right]$$

$$\therefore y_p = \frac{e^{2x}.x^5}{20}$$

42. Ans: (a)

Sol: Given f(D)y = Q(x)……. (1)

where f(D) = $D^2 - 2D + 1$ & Q(x) = x. sin(x)

Now,

$$y_p = \frac{1}{f(D)}[x.V(x)]$$

$$= x\left[\frac{1}{f(D)}V(x)\right]-\left[\frac{f'(D)}{(f(D))^2}V(x)\right]$$

$$\Rightarrow y_p = x\left[\frac{1}{D^2-2D+1}\sin(x)\right] - \left[\frac{2D-2}{(D^2-2D+1)^2}\sin(x)\right]$$

$$\Rightarrow y_p = x\left[\frac{1}{-1-2D+1}\sin(x)\right] - \left[\frac{2D-2}{1}\left\{\frac{1}{(-1-2D+1)^2}\sin(x)\right\}\right]$$

$$\Rightarrow y_p = x\left[\frac{1}{-2D}\sin(x)\right]-\left[\frac{2D-2}{1}\left\{\frac{1}{4D^2}\sin x\right\}\right]$$

$$\Rightarrow y_p = x\left[\frac{1}{2}\cos(x)\right]-\left[\frac{2D-2}{1}\left\{\frac{1}{4(-1)}\sin(x)\right\}\right]$$

$$\Rightarrow y_p = \frac{x\cos(x)}{2}+\frac{1}{4}[2\cos x - 2\sin(x)]$$

$$\therefore y_p = \frac{x\cos(x)}{2}+\frac{1}{2}\cos(x)-\frac{1}{2}\sin(x)$$

Method of Variation of Parameters

43. Ans: (a)

Sol: Given that f(D)y = Q(x) ….. (1)

where f(D) = $D^2 + 1$ & Q(x) = cosec(x)

C.F:

A.E is $m^2 + 1 = 0$

$\Rightarrow m = 0 \pm i$

$\therefore y_c = c_1\cos(x) + c_2\sin(x)$

Let $y_c = c_1\,u(x) + c_2v(x) = c_1\cos(x) + c_2\sin(x)$

Then W(u,v) =

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1$$

P.I:

Let $y_p = c_1(x).\ u(x) + c_2(x).v(x)$ be the required particular integral of (1)

Then $c_1(x) =$

$$-\int \frac{Q(x).V(x)}{W} \ \&\ c_2(x) = \int \frac{Q(x).u(x)}{W} dx$$

$$\Rightarrow c_1(x) = -\int \frac{\operatorname{cosec}(x).\sin(x)}{1} dx = -x$$

and $c_2(x) = \int \frac{\operatorname{cosec}(x).\cos(x)}{1} dx = \int \cot(x) dx$

$= \log(\sin x)$

$\therefore\ y_p = (-x).\cos(x) + [\log(\sin x)]\sin(x)$

Hence, the general solution of (1) is

$y = y_c + y_p$

44. Ans: (b)

Sol: Given that $f(D)y = Q(x)$ ……. (1)

where $f(D) = D^2 - 6D + 9\ \&\ Q(x) = \frac{e^{3x}}{x^2}$

C.F:

A.E is $m^2 - 6m + 9 = 0$

$\Rightarrow m = 3, 3$

$\Rightarrow y_c = (c_1 + c_2x)e^{3x}$

Let $y_c = c_1\ .u(x) + c_2v(x) = c_1.\ e^{3x} + c_2xe^{3x}$

Then $W = W(u,v) =$

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (3xe^{3x} + e^{3x}) \end{vmatrix}$$

$\Rightarrow W = 3xe^{6x} + e^{6x} - 3xe^{6x} = e^{6x}$

P.I:

Let $y_p = c_1(x)\ .u(x) + c_2(x)\ v(x)$ be the P.I of (1)

Then

$$c_1(x) = -\int \frac{Q(x).v(x)}{W} dx \ \ \&\ \ c_2(x) = \int \frac{Q(x)u(x)}{W} dx$$

$$\Rightarrow c_1(x) = -\int \frac{e^{3x} x^{-2} xe^{3x}}{e^{6x}} dx = -\log x \text{ and}$$

$$c_2(x) = \int \frac{e^{3x}.x^{-2}.e^{3x}}{e^{6x}} dx = \int \frac{1}{x^2} dx = \frac{-1}{x}$$

$\therefore\ y_p = (-\log x)(e^{3x}) + \left(\frac{-1}{x}\right)(x\,e^{3x})$

Hence, the G.S of (1) is given by

$y = y_c + y_p = (c_1 + c_2x)\,e^{3x} + [(\log x)e^{3x} + e^{3x}]$

Euler – Cauchy's Form

45. Ans: 09

Sol: Please Refer ACE previous maths solution booklet

46. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

Partial Differential Equations

47. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

48. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

49. Ans: (a)

Sol: A partial differential equation of the form (A)

$$\frac{\partial^2 u}{\partial x^2} + (B)\frac{\partial^2 u}{\partial x \partial y} + (C)\frac{\partial^2 u}{\partial y^2} + (D)\frac{\partial u}{\partial x}$$

$$+ (E)\frac{\partial u}{\partial y} + (F)u = Q ----(1)$$

is said to be

(i) elliptic type if $B^2 - 4AC < 0$

(ii) parabolic type if $B^2 - 4AC = 0$

(iii) hyperbolic type $B^2 - 4AC > 0$

For the given differential equation, we have $A = y^2$, $B = 2xy$ and $C = x^2$

Consider $B^2 - 4AC = (2xy)^2 - 4(y^2)(x^2)$

$\Rightarrow B^2 - 4AC = 0$

$\therefore$ The given P.D.E is parabolic type.

50. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

Method of Separation of Variables

51. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

52. Ans: (a)

Sol: Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ …….. (1)

with $u(x,0) = 6e^{-3x}$……… (2)

Let $u(x, t) = X(x).T(t)$ ……. (3) be the solution of (1)

Then $\frac{\partial u}{\partial t} = X.T'$ & $\frac{\partial u}{\partial x} = X'T$ ….. (4),

where $T' = \frac{dT}{dt}$ & $X' = \frac{dX}{dx}$

Put (4) in (1), we get

$$X'T = 2XT' + XT$$

$$\Rightarrow \quad \frac{X'}{X} = \frac{2T'}{T} + 1$$

Let $\frac{X'}{X} = \frac{2T'}{T} + 1 = k$, where k is a constant

Then $\frac{X'}{X} = k$ and $\frac{2T'}{T} + 1 = k$

$$\Rightarrow \frac{1}{X}\frac{dX}{dx} = k \quad \text{and} \quad \frac{1}{T}\frac{dT}{dt} = \frac{(k-1)}{2}$$

$$\Rightarrow \int \frac{1}{X}dX = k\int dx + c' \ \& \int \frac{1}{T}dT = \left(\frac{k-1}{2}\right)\int dt + c''$$

$\Rightarrow \log X = kx + c' \ \& \ \log T = \frac{(k-1)}{2}k + c''$

$\Rightarrow$ X= $e^{kx+c'}$ & T = $e^{\frac{(k-1)t}{2}+c''}$ …… (5)

Now the solution of (1) from (3) & (5) is given by

$u(x,t) = (e^{kx+c'})\,(e^{\left(\frac{k-1}{2}\right)t} + c'')$

$\Rightarrow u(x,t) = e^{[kx+\frac{(k-1)}{2}t]+(c'+c'')}$ ……. (6)

$\because u(x,0) = 6e^{-3x}$

$\Rightarrow e^{kx+(c+c')} = 6e^{-3x}$

$\Rightarrow k = -3 \ \& \ e^{(c+c')} = 6$…….. (7)

$\therefore$ The solution of (1) from (6) & (7) is given by $u(x,t) = 6e^{-3x-2t}$

Lagranges Linear Equation and Standard Types (Only for EC, EE and IN Branches)

53. Ans: (d)

Sol: Given (tanx) p +(tany) q = (tanz) –(1)

$(\because Pp + Qq = R)$

$\Rightarrow$ P = tanx, Q = tany & R = tan(z).

Consider Lagrange's auxiliary equation for (1)

$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$ ……. (2)

Taking the first two fractions of (2), we get (3)

$\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$\Rightarrow \int \cot(x)dx = \int \cot(y)dy + \log(c_1)$

$\Rightarrow \log(\sin x) = \log(\sin y) + \log(c_1)$

$\therefore \frac{\sin x}{\sin(y)} = c_1$ ….. (3)

Taking the last two fractions of (2), we get

$\frac{dy}{\tan y} = \frac{dz}{\tan z}$

$\Rightarrow \int \cot(y)dy = \int \cot(z)dz + \log(c_2)$

$\Rightarrow \log(\sin y) = \log(\sin z) + \log c_2$

$\therefore \frac{\sin(y)}{\sin(z)} = c_2$ …… (4)

Hence the required general solution of (1) is

$\frac{\sin(x)}{\sin(y)} = \phi\left(\frac{\sin y}{\sin z}\right)$

54. Ans: (a)

Sol: Given that

$[x(y-z)]p+[y(z-x)]q=z(x-y)$…… (1)

$\Rightarrow P= x(y-z)$, $Q = y(z-x)$ & $R = z(x-y)$

Consider

$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$ ….. (2)

Let $\ell = \frac{1}{x}, m = \frac{1}{y} \ \& \ n = \frac{1}{z}$

Then $\ell P + mQ + nR = 0$

$\Rightarrow \int (\ell dx + mdy + ndz) = \log c_1$

$\Rightarrow \int \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = \log c_1$

$\Rightarrow \log x + \log y + \log z = \log c_1$

$\therefore \boxed{xyz = c_1}$ …… (3)

Let $\ell = 1$, m =1 and n =1

Then $\ell P + mQ + nR = 0$

$\Rightarrow \int (\ell dx + mdy + ndz) = c_2$

$\Rightarrow \int[dx+dy+dz] = c_2$

$\therefore x + y + z = c_2$ ……. (4)

Hence, the general solution of (1) from (3) & (4) is $\phi(xyz, x + y + z) = 0$

55. Ans: (c)

Sol: Give that pq = k …… (1)

Let p = a …… (2), where 'a' is an arbitrary constant

Put (2) in (1), we get

aq = k

$\Rightarrow q = \frac{k}{a}$ ……. (3)

Consider dz = pdx + qdy

$\Rightarrow dz = a\,dx + \frac{k}{a}dy$

$\Rightarrow \int dz = \int a\,dx + \int \frac{k}{a}dy + c$

$\therefore$ $z = ax + \left(\frac{k}{a}\right)y + c$ is a required solution of (1)

56. Ans: (d)

Sol: Given that $p^2 = qz$ …… (1)

Let q = ap ….. (2)

Put (2) in (1), we get

$p^2 = apz$

$\Rightarrow p^2 - apz = 0$

$\Rightarrow p(p - az) = 0$

Consider p – az = 0

$\Rightarrow p = az$

$\Rightarrow q = ap = a(az) = a^2z$

Consider dz = p dx + q dy

$\Rightarrow dz = az\,dx + a^2\,zdy$

$\Rightarrow \int \frac{1}{z}dz = \int a dx + \int a^2 dy + c$

$\therefore \log(z) = ax + a^2y + c$ is a solution of (1)

57. Ans: (a)

Sol: Given that x(1+y)p=y(1+x)q….. (1)

$\Rightarrow \frac{x}{1+x}p = \frac{y}{1+y}q$

Let $\frac{x}{1+x}p = \frac{y}{1+y}q = a$, where 'a' is an arbitrary constant.

Then $\frac{x}{1+x}p = a$ and $\frac{y}{1+y}q = a$

$\Rightarrow P = \frac{a(1+x)}{x}$ and $q = \frac{a(1+y)}{y}$

Consider dx = p dx + q dy

$$\Rightarrow \int dz = \int \frac{a(1+x)}{x}dx + \int \frac{a(1+y)}{y}dy + c$$

$\therefore$ z = a[x + log x] + a[y + logy]+ c is a solution of (1)

58. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

Solution of Wave, Heat and Laplace Equations

(Only for CE, ME and PI branches)

59. Ans: (b)

Sol: Given that $\frac{\partial u}{\partial t} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}$ ……. (1)

$$(\because \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2})$$

with B.C's : u(0,t) = 0 $\quad$ ($\because$ u(0,t)= 0)

u(1,t) = 0 $\quad$ ($\because$ u(l,t)= 0)

and I.C: u(x,0) = sin (πx)

($\because$ u(x,0)= f(x))

$$\text{Now, } a_n = \frac{2}{\ell}\int_0^{\ell} f(x).\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow a_4 = \frac{2}{1}\int_0^1 \sin(\pi x).\sin(4\pi x)dx$$

$$\Rightarrow a_4 = \int_0^1 2\sin(4\pi x).\sin(\pi x)dx$$

$$\Rightarrow a_4 = \int_0^1 \cos(4\pi x - \pi x) - \cos(4\pi x + \pi x)dx$$

$$\Rightarrow a_4 = \int_0^1 [\cos(3\pi x) - \cos(5\pi x)]dx$$

$$\Rightarrow a_4 = \left[\frac{\sin(3\pi x)}{3\pi} - \frac{\sin(5\pi x)}{5\pi}\right]_0^1$$

$$\Rightarrow a_4 = \left[\frac{\sin(3\pi)}{3\pi} - \frac{\sin(5\pi)}{5\pi}\right] - [0 \quad -0]$$

$\therefore a_4 = 0$

60. Ans: 0.395

Sol: Please Refer ACE previous maths solution booklet

61. Ans: (a)

Sol: Given $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ …… (1)

$$\left(\because \frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2}\right)$$

with B.C′s: u(0,t) = 0 $\quad$ ($\because$ u(0,t) = 0)

u(π,t) = 0 $\quad$ ($\because$ u(π,t) = 0)

& I.C : u(x,0) = sin (x) $\quad$ ($\because$ u(x,0) = f(x))

$$\frac{\partial}{\partial t}u(x,0) = 0$$

$$\left(\because \frac{\partial}{\partial t}u(x,0) = 0\right)$$

Now, the formula of general solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n.\sin\left(\frac{n\pi x}{\ell}\right).\cos\left(\frac{n\pi ct}{\ell}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n.\sin(nx).\cos(2nt) \ldots\ldots. (2)$$

Put t = 0 in above, we get

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$\Rightarrow \sin(x) = a_1.\ \sin(x) + a_2 \sin(2x) + a_3\sin(3x) + \ldots.$

$\Rightarrow a_1 = 1,\ a_2 = 0,\ a_3 = 0, \ldots\ldots\ldots (3)$

$\therefore$ The solution of (1) from (2) & (3) is

$u(x,t) = a_1.\sin(x).\cos(2t) + a_2\sin(2x).\cos(4t) + \ldots\ldots..$

$= \sin(x).\cos(2t)$

Hence,

$$u\left(\frac{3\pi}{2},\frac{\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right).\cos\left(\frac{2\pi}{2}\right) = (-1)(-1) = 1$$

62. Ans: (a)

Sol: Given $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ldots\ldots (1)$

$$\left(\because \frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2}\right)$$

with I.C′s: $u(x,0) = 0$

$(\because\ u(x,0) = f(x))$

$$\frac{\partial}{\partial t}u(x,0) = 4xe^{-x^2}$$

$$\left(\because \frac{\partial}{\partial t}u(x,0) = g(x)\right)$$

$\Rightarrow C = 1,\ f(x) = 0\ \&\ g(x) = 4x\,e^{-x^2}$

Now, the formula of general solution of (1) is

$u(x,t) =$

$$\frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

$$\Rightarrow u(x,t) = \frac{1}{2}[0+0]+\frac{1}{2}\int_{x-t}^{x+t}4s.e^{-s^2}ds$$

$$\Rightarrow u(x,t) = (-1)\int_{x-t}^{x+t}(-2s)e^{-s^2}ds$$

$$\Rightarrow u(x,t) = (-1)\left(e^{-s^2}\right)_{x-t}^{x+t}$$

$$\Rightarrow u(x,t) = e^{-(x-t)^2} - e^{-(x+t)^2}$$

$\therefore u(1,1) =$

$$e^{-(1-1)^2} - e^{-(1+1)^2} = e^0 - e^{-2^2} = 1 - e^{-4}$$

63. Ans: (b)

Sol: Given that $u_{tt} = u_{xx} \ldots\ldots (1)$

$(\because\ u_{tt} = c^2 u_{xx})$

with B.C′s: $u(0,t) = 0$

$(\because u(0,t) = 0)$

$u(\pi,t) = 0 \qquad (\because u(l,t) = 0)$

& I.C′s: $u(x,0) = 0 \qquad (\because u(x,0) = 0)$

$$\frac{\partial u(x,0)}{\partial t} = 2\sin(x)$$

$$\left(\because \frac{\partial u(x,0)}{\partial t} = f(x)\right)$$

Then the solution of (1) is given by

$$u(x,t)=\sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{\ell}\right).\sin\left(\frac{n\pi ct}{\ell}\right)$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty} b_n.\sin(nx).\sin(nt) \ldots\ldots\ (2)$$

$$\Rightarrow \frac{\partial u(x,t)}{\partial t}=\sum_{n=1}^{\infty} b_n.\sin(nx).\cos(nt).(n)$$

$$\Rightarrow \frac{\partial}{\partial t}u(x,0)=\sum_{n=1}^{\infty} n\, b_n.\sin(nx)$$

$\Rightarrow 2\sin x = b_1 \sin(x) + 2b_2 \sin(2x) + 3b_3 \sin(3x) + \ldots\ldots$

$\Rightarrow b_1 = 2, b_2 = 0, b_3 = 0, \ldots\ldots$ (3)

$\therefore$ The solution of (1) from (2) & (3) is given by

$u(x,t)=b_1 \sin(x).\sin(t) + b_2 \sin(2x).\cos(2t) + \ldots\ldots = 2\sin(x)\sin(t)$

64. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

65. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

66.

Sol: $$L\{f(t)\}=\int_0^5 e^{-st}2dt+\int_5^{10} e^{-st}0dt+\int_{10}^{\infty} e^{-st}e^{4t}dt$$

$$=2\left(\frac{e^{-st}}{-s}\right)_0^5+\left(\frac{e^{(4-s)t}}{4-s}\right)_{10}^{\infty}$$

$$=\frac{-2}{s}\left(e^{-5s}-1\right)+\left(0-\frac{e^{10(4-s)}}{4-s}\right)$$

$$=\frac{2}{s}+\frac{-2e^{-5s}}{s}+\frac{e^{-10(s-4)}}{s-4},\quad s>4$$

67. Ans: (d)

Sol: Let $f(t) = \cos t$

$$L\{f(t)\}=\frac{s}{s^2+1}=f(s)$$

By first shifting theorem

$L\{e^{-at} f(t)\} = f(s + a)$

$$\text{Now } L\{e^{-4t}\cos t\}=f(s+4)=\frac{s+4}{(s+4)^2+1}$$

68. Ans: (c)

Sol: Let $f(t) = \sin 6t$

$$L\{f(t)\}=\frac{6}{s^2+36}=f(s)$$

$$f^1(s)=\frac{-6}{(s^2+36)^2}(2s)=\frac{-12s}{(s^2+36)^2}$$

We know that $L\{t^n f(t)\}=(-1)^n f^n(s)$

Now $L\{t.\sin 6t\}=(-1)^1 f^1(s)$

$$=-\frac{(-12s)}{(s^2+36)^2}=\frac{12s}{(s^2+36)^2}$$

69.

Sol: we have $L\{t\sin 6t\}=\dfrac{12s}{(s^2+36)^2}=f(s)$

By first shifting theorem

$$L\{e^{-t}(t\sin 6t)\} = f(s+1) = \frac{12(s+1)}{\left[(s+1)^2+36\right]^2}$$

70. Ans: (a)

Sol: given $f(t) = u(t-4)(t-4)^2 + 4u(t-4)$

Let $g(t) = t^2$

$$L\{g(t)\} = \frac{2!}{s^3} = g(s)$$

Now $L\{f(t)\} = e^{-4s}g(s) + 4\frac{e^{-4s}}{s}$

($\because$ By second shifting theorem)

$$\Rightarrow L\{f(t)\} = e^{-4s}\frac{2!}{s^3} + 4\frac{e^{-4s}}{s} = e^{-4s}\left(\frac{2}{s^3}+\frac{4}{s}\right)$$

71. Ans: (a)

Sol: Let $f(t) = e^{at} - e^{-bt}$

$$L\{f(t)\} = L\{e^{at} - e^{-bt}\} = \frac{1}{s-a} - \frac{1}{s+b} = f(s)$$

By Laplace transform of division with 't'

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\,ds$$

$$\therefore L\left\{\frac{e^{at}+e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{s+b}\right)ds$$

$$= \left[\log(s-a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{s+b}\right)\right]_s^\infty = \left[\log\left(\frac{1-a/s}{1+b/s}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{1-a/s}{1+b/s}\right)$$

$$= -\log\left(\frac{s-a}{s+b}\right) = \log\left(\frac{s+b}{s-a}\right)$$

72.

Sol: let $f(t) = \cos\omega t$

$$L\{f(t)\} = \frac{s}{s^2+\omega^2} = f(s)$$

By first shifting theorem $L\{e^{-3t} f(t)\} =$

$$f(s+a) = \frac{s+3}{(s+3)^2+\omega^2} = g(s)$$

Now by Laplace transform of integrals

$$L\left\{\int_0^\infty e^{-3t}\cos\omega t\right\} = \frac{g(s)}{s} = \frac{s+3}{s\left[(s+3)^2+\omega^2\right]}$$

73. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

74.

Sol: $\int_0^\infty e^{-2t}\cos 3t\,dt = L\{\cos 3t\}$

$$= \frac{s}{s^2+9} = \frac{2}{\left((2)^2+9\right)} = \frac{2}{13} \qquad (\because s = 2)$$

75. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

76. Ans: (a)

Sol: $L^{-1}\left\{\frac{s+5}{(s+1)(s+3)}\right\} = L^{-1}\left\{\frac{-1}{s+3}+\frac{2}{s+1}\right\}$

(By partial fraction)

$= -e^{-3t} + 2e^{-t} = 2e^{-t} - e^{-3t}$

77. Ans: (a)

Sol: Let $f(s) = \frac{1}{s^2(s+1)} = \frac{-1}{s}+\frac{1}{s^2}+\frac{1}{s+1}$

$L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = L^{-1}\left\{\frac{-1}{s}+\frac{1}{s^2}+\frac{1}{s+1}\right\}$

$= -1 + t + e^{-t} = t-1+e^{-t}$

78. Ans: (a)

Sol:

$L^{-1}\left\{\frac{s+4}{s^2+9}+\frac{3}{s^2}+\frac{4}{s-3}\right\} = L^{-1}\left\{\frac{s+4}{s^2+9}\right\}+L^{-1}\left\{\frac{3}{s^2}\right\}$

$+L^{-1}\left\{\frac{4}{s-3}\right\}$

$= L^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{1}{12}L^{-1}\left\{\frac{3}{s^2+9}\right\}$

$+L^{-1}\left\{\frac{3}{s^2}\right\}+4L^{-1}\left\{\frac{1}{s-3}\right\}$

$= \cos 3t + \frac{1}{12}\sin 3t + 3t + 4e^{3t}$

79.

Sol: Unit step function is

$$u(t-a) = H(t-a) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0 \end{cases}$$

We have $L\{u(t-a)\} = \frac{e^{-as}}{s}(s>0)$

$\therefore L^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a)$ or $H(t-a)$

80. Ans: (d)

Sol: Let $f(s) = \frac{1}{s^2} \Rightarrow L^{-1}\{f(s)\} = t$

By first shifting theorem

$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$

By second shifting theorem

$L^{-1}\{e^{-as}f(s)\} = f(t-a)\,H(t-a)$

$\therefore$

$4L^{-1}\left\{e^{-2s}.\frac{e^{-2s}}{(s+1)^2}\right\} = 4e^{-(t-2)}.(t-2)H(t-2)$

81. Ans: (b)

Sol: Let $f(s) = \log\left(\frac{s-a}{s-b}\right)$ =log(s–a)–log(s– b)

$f^1(s) = \frac{1}{s-a} - \frac{1}{s-b}$

$L^{-1}\{f^1(s)\} = e^{at} - e^{bt}$

$\Rightarrow (-1)t\, f(t) = e^{at} - e^{bt}$

$\Rightarrow f(t) = -\frac{1}{t}\left(e^{at}-e^{bt}\right)$

$\therefore f(t) = \frac{e^{bt}-e^{at}}{t}$

82. Ans: (a)

Sol: Given D.E. is $f'(t) + 5f(t) = 1$ for $t > 0$

($\because$ $u(t) = 1$ for $t \geq 0$)

$\Rightarrow L\{ f'(t)\ 5f(t)\} = L\{1\}$

$\Rightarrow L\{f'(t)\} + 5f(t)\} = \frac{1}{s}$

$\Rightarrow sf(s) - f(0) + 5L\{f(t)\} = \frac{1}{s}$

$\Rightarrow sL\{(t+1)\} - 1 + 5L\{f(t)\} = \frac{1}{s}$

$\Rightarrow L\{f(t)\}(s+5) - 1 = \frac{1}{s}$

$\Rightarrow L\{f(t)\} = \frac{s+1}{s(s+5)}$

$\Rightarrow$

$$f(t) = L^{-1}\left\{\frac{s+1}{s(s+5)}\right\} = L^{-1}\left\{\frac{1}{5s} + \frac{-4}{-5(s+5)}\right\}$$

$$= \frac{1}{5}(1) + \frac{4}{5}e^{-5t}$$

$$= \frac{1}{5}\left(1 + 4e^{-5t}\right)$$

$\therefore\ f(t) = 0.2 + 0.8e^{-5t}$

83. Ans: (a)

Sol: Given,

D.E is $f^{11}(t) - f(t) = 1$

$\Rightarrow L\{f^{11}(t) - f(t)\} = L\{1\}$

$\Rightarrow \{s^2 f(s) - f(0) - f^1(0)\} - L\{f(t)\} = \frac{1}{s}$

$\Rightarrow L\{f(t)\}(s^2 - 1) = \frac{1}{s}$

$\Rightarrow L\{f(t)\} = \frac{1}{s(s^2-1)}$

$\therefore L\{f(t)\} = \frac{1}{s(s+1)(s-1)}$

Complex Variables

(Solutions for Text Book Practice Questions)

Analytic Function

01. Ans: (a, b, c)

Sol: Let $u+iv = w = f(z) = (x + e^{-x} \sin y - 4) + i(y+e^{-x} \cos y)$

Then $u = x + e^{-x} \sin (y) - 4$ and $v = y + e^{-x} \cos(y)$

$\Rightarrow u_x = 1 - e^{-x} \sin y$, $u_y = e^{-x} \cos (y)$,

$\Rightarrow v_x = - e^{-x} \cos(y)$ and $v_y = 1 + e^{-x} (-\sin y)$

Here, $u_x = v_y$ and $v_x = -u_y$ at every point and also u, v, u_x, u_y, v_x, v_y are continuous at every point

$\Rightarrow w = f(z) = u+ iv$ is differentiable at every point.

$\Rightarrow w = f(z) = u +iv$ is analytic at every point.

$\therefore$ Options (a), (b) and (c) are correct.

02. Ans: (a)

Sol: Let $u + iv = f(z) = xz$

Then $u + iv = f(z) = x(x + iy) = x^2 + ixy$

$\Rightarrow u = x^2$ and $v = xy$

$\Rightarrow u_x = 2x$, $u_y = 0$, $v_x = y$, $v_y = x$

Here, $u_x = v_y$ and $v_x = - u_y$ satisfy only at origin and u, v, u_x, u_y, v_x, v_y are also continuous at origin.

$f(z) = u + iv$ is differentiable only at origin.

$\therefore$ Option (a) is correct.

03. Ans: (d)

Sol: Let $w = f(z) = \bar{z}$

Then $u + iv = f(z) = x - iv$

$u = x$ and $v = - y$

$u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$

Here, $u_x = v_y$ will not satisfy at any point

$\Rightarrow$ $f(z) = u + iv$ is not differentiable at any point

$\therefore$ $f(z) = u + iv$ is not analytic at any point.

04. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

05. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

06. Ans: (b)

Sol: Let $u + iv = f(z) = (x^2 + c_1 y^2 - 2xy) + i (c_2 x^2 - y^2 + 2xy)$ be analytic. Then the C–R equations $u_x = v_y$ and $v_x = - u_y$ will satisfy.

$\Rightarrow$ $2x-2y = -2y + 2x$

and $2c_2x + 2y = -(2c_1 y - 2x)$

$\therefore$ $c_1 = -1$ and $c_2 = 1$

07. Ans: (2)

Sol: Please Refer ACE previous maths solution booklet

08. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

09. Ans: (2)

Sol: Please Refer ACE previous maths solution booklet

10. Ans: (c)

Sol: Given that $u = u(r, \theta) = r^2 \cos(2\theta)$

$\Rightarrow u_r = 2r\cos(2\theta)$ and $u_\theta = -2r^2 \sin(2\theta)$.

Consider $dv = \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial \theta}d\theta$ for $v(r, \theta)$

$$\Rightarrow dv = \left(\frac{-1}{r}u_\theta\right)dr + (ru_r)d\theta$$

$$\left(\because u_r = \frac{1}{r}v_\theta \ \& \ v_r = \frac{-1}{r}u_\theta\right)$$

$$\Rightarrow dv = \left[\frac{-1}{r}\left(-2r^2\sin(2\theta)\right)\right]dr + \left[r\left(2r\cos(2\theta)\right)\right]d\theta$$

$\Rightarrow dv = [2r\sin(2\theta)]dr + [2r^2\cos(2\theta)]d\theta$

$\Rightarrow dv = d[r^2 \sin(2\theta)]$

$$\Rightarrow \int dv = \int d\left[r^2\sin(2\theta)\right] + k$$

$\therefore v = v(r, \theta) = r^2.\sin(2\theta) + k$ is a required harmonic conjugate function of $u(r, \theta)$.

11. Ans: (a, c)

Sol: Given that $v = y^3 + 2y - 3x^2 y$

$\Rightarrow v_x = -6xy,\ v_y = 3y^2 + 2 - 3x^2$

Consider $du = u_x dx + u_y dy$

$\Rightarrow du = (v_y)dx + (-v_x)dy$

$(\because u_x = v_y \ \& \ v_x = -u_y)$

$\Rightarrow du = (3y^2 + 2 - 3x^2)\,dx + (6xy)\,dy$

$$\Rightarrow \int du = \int\left(2dx - 3x^2dx\right) + \int d\left(3xy^2\right) + k$$

$\therefore u = 2x - x^3 + 3xy^2 + k$ is a real part of $f(z)$.

Consider $f(z) = u + iv$

$\Rightarrow f(z) = (2x - x^3 + 3xy^2 + k) + i\,(y^3 + 2y - 3x^2y)$

$\therefore f(z) = 2z - z^3 + (k + i\,0) = 2z - z^3 + c$, where $c = k + i\,0$, is a required analytic function.

12. Ans: (c)

Sol: Given $u = (x-1)^3 - 3xy^2 + 3y^2$

$\Rightarrow u_x = 3(x-1)^2 - 3y^2$ and $u_y = -6xy + 6y$

Consider $f^1(z) = u_x - iu_y$

$\Rightarrow f'(z) = [3(x-1)^2 - 3y^2] + i\,[6xy - 6y]$

$\Rightarrow f'(z) = [3(z-1)^2 - 0] + i(0-0)$

$(\because x \text{ by } z \ \& \ y \text{ by } 0)$

$$\Rightarrow \int f'(z)dz = \int 3(z-1)^2 dz + c$$

$\therefore f(z) = (z-1)^3 + c$ is a required analytic function.

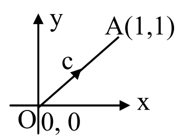
Complex Integration

13. Ans: (a)

Sol: Let $I = \int_C f(z)dz$

Then $I = \int_{z=(0,0)}^{(1,1)} \left(y - x - 3ix^2\right)(dx + i\,dy)$

The equation of line segment from a point (0, 0) to (1, 1) is given by $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$



$\Rightarrow \dfrac{y-0}{1-0} = \dfrac{x-0}{1-0}$

$\Rightarrow y = x$

$\Rightarrow dy = dx$

Now, $I = \int_{x=0}^{1} \left(x - x - 3i\,x^2\right)(dx + i\,dx)$

$\Rightarrow I = (1+i) \int_{x=0}^{1} \left(-3ix^2\right)dx$

$\Rightarrow I = (1+i) \left(-3i\dfrac{x^3}{3}\right)_0^1$

$\Rightarrow I = -i(1+i) = -\left(i + i^2\right)$

$\therefore I = 1 - i$

14. Ans: (b)

Sol: Let $I = \int_c f(z)dz$, where $f(z) = z^2$

Then $I = \int_{z=0}^{1+i} z^2 dz$

$\Rightarrow I = \left(\dfrac{z^3}{3}\right)_0^{1+i}$ ($\because$ f(z) is analytic function)

$\Rightarrow I = \dfrac{(1+i)^3}{3}$

$\Rightarrow I = \dfrac{1^3 + 3(1)^2(i) + 3(1)(i)^2 + (i)^3}{3} = \dfrac{1 + 3i - 3 - i}{3}$

$\therefore I = \dfrac{-2 + 2i}{3}$

15. Ans: (0)

Sol: Let $f(z) = e^{-z^2}$

Here, z^2 is an analytic function every where

$\Rightarrow -z^2$ is an analytic function every where

$\Rightarrow e^{-z^2}$ is an analytic function every where

$\Rightarrow f(z) = e^{-z^2}$ is an analytic function in the closed region R bounded by closed curve 'c' in the complex plane

$\therefore$ By Cauchy's theorem, we have $\oint f(z)dz = 0$

16. Ans: (0)

Sol: Let $f(z) = \dfrac{z^2 + \cos(z)}{(z-4)^3(z+2)}$

Them the singular points of f(z) are given by equating the denominator to zero i.e $(z - 4)^3 (z + 2) = 0$

$\Rightarrow z = 4, z = -2$ are singular points.

$\Rightarrow$ Both singular points lie out side the given region R bounded by c : $|z-2| = 1$.

$\therefore$ By Cauchy's theorem, we have

$\int_c f(z)dz = 0$

17. Ans: (c)

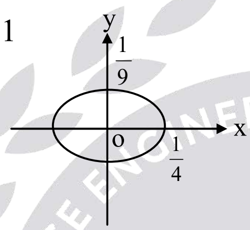
Sol: Please Refer ACE previous maths solution booklet

18. Ans: (a)

Sol: Let f(z) = $\frac{1}{z^2 e^z}$

Then the singular point of f(z) is z = 0

$$\frac{x^2}{\left(\frac{1}{9}\right)^2}+\frac{y^2}{\left(\frac{1}{4}\right)^2}=1$$



⇒ The singular point z =0 lies inside the given region bounded by c : $9x^2 + 4y^2 = 1$

So, consider f(z) = $\frac{\phi(z)}{[z-z_o]^n}=\frac{e^{-z}}{[z-0]^2}$

Now, by cauchy's integral formula, we have

$$\oint_c f(z)dz=\frac{2\pi i}{(2-1)!}\left[\frac{d}{dz}\left(e^{-z}\right)\right]_{z=0}$$

$$\therefore \oint_c f(z)\,dz = 2\pi i\left(-e^{-z}\right)=2\pi i$$

19. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

20. Ans: (0.5)

Sol: Please Refer ACE previous maths solution booklet

21. Ans: (c)

Sol: Let f(z) = $\frac{z}{(z+1)(z+2)}$

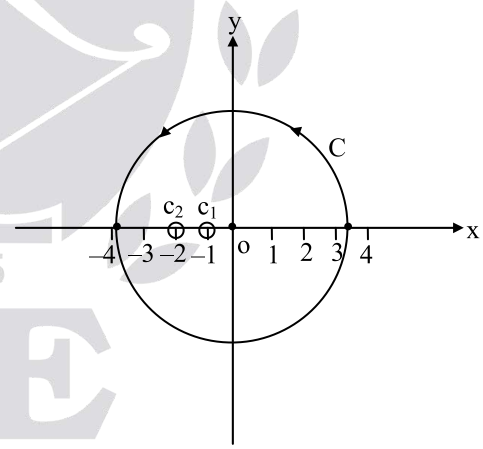
Then the singular points of f(z) are z = –1 and z = –2

⇒ Both singular points z = –1 and z = –2 lie inside the circle |z| = 4

$$\text{So, consider } f(z)=\frac{\left(\frac{z}{z+2}\right)}{[z-(-1)]}+\frac{\left(\frac{z}{z+1}\right)}{[z-(-2)]}$$

$$\Rightarrow \oint_C f(z)dz=\oint_{C_1}\frac{\left(\frac{z}{z+2}\right)}{[z-(-1)]}dz+\oint_{C_2}\frac{\left(\frac{z}{z+1}\right)}{[z-(-2)]}dz,$$

where c_1 & c_2 are circles $|z-(-1)|=r_1$ and $|z-(-2)|=r_2$ respectively.



$$\Rightarrow \oint_C f(z)dz=2\pi i\left[\frac{z}{z+2}\right]_{z=-1}+2\pi i\left[\frac{z}{z+1}\right]_{z=-2}$$

$$\Rightarrow \oint_C f(z)dz=2\pi i\left(\frac{-1}{-1+2}\right)+2\pi i\left(\frac{-2}{-2+1}\right)$$

$$\therefore \oint_C f(z)dz=2\pi i[-1+2]=2\pi i$$

22. Ans: (d)

Sol: Please Refer ACE previous maths solution booklet

Taylor and Laurent Series

23. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

24. Ans: (a, b, d)

Sol: Given that $f(z) = z^3\ e^{\frac{1}{z}}$

$\Rightarrow$f(z) has a singular point at $z = 0$

Now, $f(z) = z^3\ e^{\frac{1}{z}}$

$$\Rightarrow f(z) = z^3\left[1+\frac{\left(\frac{1}{z}\right)^1}{1!}+\frac{\left(\frac{1}{z}\right)^2}{2!}+\frac{\left(\frac{1}{z}\right)^3}{3!}+\frac{\left(\frac{1}{z}\right)^4}{4!}+\frac{\left(\frac{1}{z}\right)^5}{5!}+....\right]$$

$$\Rightarrow f(z) = \left[z^3+z^2+\frac{1}{2}z\right]+\left[\frac{1}{3!}+\frac{1}{4!}\frac{1}{z}+\frac{1}{5!}\frac{1}{z^2}+.....\right]$$

which is a Laurent series expansion of f(z) about $z = 0$

The coefficient of $\frac{1}{z^2}$ and $\frac{1}{z}$ are $\frac{1}{5!}$ & $\frac{1}{4!}$ respectively.

$\therefore$ Options (a), (b) and (d) are correct.

25. Ans: (a)

Sol: Let $f(z) = \log(z)$ and $z_0 = 1$

Then the Taylor series expansion of f(z) about a point $z = z_0$ in the given region $|z-z_0|<r$ is given by

$$f(z) = f(z_0)+ (z-z_0)f'(z_0)+ \frac{(z-z_0)^2}{2!}f''(z_0)+\frac{(z-z_0)}{3!}f'''(z_0)+...(1)$$

Consider $f(z) = \log z$ and

$f(z_0) = f(1) = \log(1) = 0$

$\Rightarrow f'(z) = \frac{1}{z}$ and $f'(z_0) = f'(1) = 1$

$\Rightarrow f''(z) = \frac{-1}{z^2}$ and $f''(z_0) = f''(1) = -1$

$\Rightarrow f'''(z) = \frac{2}{z^3}$ and $f'''(z_0) = f'''(1) = 2$

Substituting above all in (1), we get

$$f(z) = (0) + (z-1)(1)+ \frac{(z-1)^2}{2!}(1)+\frac{(z-1)^3}{3}(2)+...$$

$$\therefore \log(z) = (z-1) - \frac{(z-1)^2}{2}+\frac{(z-1)^3}{3}-\frac{(z-1)^4}{4}+...$$

which is a Taylor series expansion of f(z) about $z = 1$ in the region $|z-1|<1$

26. Ans: (d)

Sol: Please Refer ACE previous maths solution booklet

Residues

27. Ans: (d)

Sol: Please Refer ACE previous maths solution booklet

28. Ans: (a, b, c)

Sol: Given that $f(z) = \frac{\alpha z^2+\beta}{(z-1)^2(z+2)}$

$\Rightarrow$ z = 1, and z = –2 are the singular points of f(z) and z = 1 & z = – 2 are poles of order 2 and one respectively.

Also given that Res(f(z): z = 1) = $\frac{5}{9}$

$$\Rightarrow \frac{1}{(2-1)}\underset{z\to 1}{\text{Lt}}\left[\frac{d}{dz}\left\{(z-1)^2.\frac{\alpha z^2+ß}{(z-1)^2(z+2)}\right\}\right]=\frac{5}{9}$$

$$\Rightarrow \underset{z\to 1}{\text{Lt}}\left[\frac{d}{dz}\left(\frac{\alpha z^2+ß}{z+2}\right)\right]=\frac{5}{9}$$

$$\Rightarrow \underset{z\to 1}{\text{Lt}}\left[\frac{(z+2)(2\alpha z)-(\alpha z^2+\beta)(1)}{(z+2)^2}\right]=\frac{5}{9}$$

$$\Rightarrow \frac{6\alpha-(\alpha+\beta)}{9}=\frac{5}{9}$$

$$\Rightarrow \frac{5\alpha-\beta}{9}=\frac{5(1)-(0)}{9}$$

$\therefore$ α = 1 & ß = 0

Hence, options (a), (b) & (c) are true.

29. Ans: (b)

Sol: Given $f(z)=\frac{z+\cos(z)}{\left(z-\frac{\pi}{2}\right)}$ $\left(\because f(z)=\frac{\phi(z)}{z-z_0}\right)$

$\Rightarrow$ f(z) has a singular point at z = $\frac{\pi}{2}$

$\Rightarrow$ z = $\frac{\pi}{2}$ is a pole of order one

Now, R, = Res(f(z) : z = $\frac{\pi}{2}$) = $\phi\left(\frac{\pi}{2}\right)$

$$\therefore R_1=\frac{\pi}{2}+\cos\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$$

30. Ans: (1)

Sol: Given that f(z)=$\frac{e^z}{\sin(z)}$ and singular point z = 0

$\Rightarrow$ The singular point z = 0 is a pole of order one.

$\Rightarrow$ Now, R_1 = Res(f(z) : z = 0) = $\frac{\phi(0)}{\psi'(0)}$

$$\left(\because f(z)=\frac{\phi(z)}{\psi(z)}\right)$$

$$\Rightarrow R_1=\frac{e^0}{\cos(0)}=\frac{1}{1}$$

$\therefore$ R_1 = 1

31. Ans: (b)

Sol: Please Refer ACE previous maths solution booklet

32. Ans: (0)

Sol: Given f(z) = $\frac{\sin(z)}{z}$

$\Rightarrow$ The singular point of f(z) is z = 0

$$\text{Now, } f(z)=\frac{1}{z}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+....\right]$$

$$\Rightarrow f(z)=1-\frac{1}{3!}z^2+\frac{1}{5!}z^4-\frac{1}{7!}-z^6+....$$

$\therefore$ The singular point z = 0 is a removable singular point of f(z) and the residue of f(z) at z = 0 is zero.

33. Ans: (c)

Sol: Given that $f(z) = z.e^{\frac{7}{z}}$

$\Rightarrow$ The singular point of f(z) is z = 0

Now, $f(z) = z.\,e^{\frac{7}{z}}$

$$\Rightarrow f(z) = z.\left[1 + \frac{\left(\frac{7}{z}\right)}{1} + \frac{\left(\frac{7}{z}\right)^2}{2!} + \frac{\left(\frac{7}{z}\right)^3}{3!} + \ldots\right]$$

$$\Rightarrow f(z) = z + (7) + \left(\frac{7^2}{2}\right)\frac{1}{z} + \left(\frac{7^3}{3!}\right)\frac{1}{z^2} + \ldots.$$

$\therefore$ z = 0 is an essential singular point of f(z) and the residue of f(z) at z = 0 is $\frac{49}{2}$

34. Ans: (a, b, c)

Sol: **(a)** Let $f(z) = \frac{z}{z^2 - 1}$ and singular point be z = 1

Then the singular point z = 1 is a pole of order one.

$$\Rightarrow R_1 = \text{res}(f(z)\,z = 1) = \underset{z \to 1}{\text{Lt}}\left[(z-1).\frac{z}{(z-1)(z+1)}\right]$$

$$\Rightarrow R_1 = \frac{1}{1+1} = \frac{1}{2}$$

$\therefore$ Option (a) is a true statement.

(b) Let $f(z) = z^2$ & c : |z| = 1

Then f(z) is an analytic function everywhere

$\Rightarrow$ f(z) is analytic inside & on 'C'

$\therefore$ By cauchy's theorem, we have

$$\oint_C f(z)\,dz = 0$$

Hence, option (b) is a true statement

(c) Let $f(z) = \frac{1}{z} = \frac{1}{(z-1)}\left(\because f(z) = \frac{\phi(z)}{(z - z_0)}\right)$

Then the singular point of f(z) is z = 0

$\Rightarrow$ The singular point lies inside 'c'.

Now, by cauchy's integral formula, we have

$$\oint_c f(z)dz = 2\pi i\,\phi(0)$$

$$\Rightarrow \oint_c f(z)dz = 2\pi i(1)_{z=0} = 2\pi i$$

$$\Rightarrow \frac{1}{2\pi i}\oint_c f(z)dz = \frac{1}{2\pi i}(2\pi i) = 1$$

$\therefore$ Option (c) is a true statement.

(d) Let $f(z) = \bar{z}$

Then $f(z) = \bar{z}$ is not differentiable at any point

$\Rightarrow f(z) = \bar{z}$ is not analytic at any point.

$\therefore$ Option (d) is not a true statement.

Hence, option (a), (b) & (c) are correct answer.

35. Ans: (c)

Sol: Please Refer ACE previous maths solution booklet

Sequence and Series (Only for EC)

36. Ans: (a, b)

Sol : $\text{Let} \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3.4^n}$

Then $u_n = \dfrac{(z+2)^{n-1}}{(n+1)^3.4^n}$ & $u_{n+1} = \dfrac{(z+2)^n}{(n+2)^3.4^{n+1}}$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{(z+2)^n}{(n+2)^3.4^{n+1}} \times \frac{(n+1)^3.4^n}{(z+2)^{n-1}}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{n^3\left(1+\frac{1}{n}\right)^3}{n^3\left(1+\frac{2}{n}\right)^3}.\frac{(z+2)}{4}$$

Now, $l = \underset{n\to\infty}{\text{Lt}}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}}\left|\frac{\left(1+\frac{1}{n}\right)^3}{\left(1+\frac{2}{n}\right)^3}.\frac{z+2}{4}\right|$

$$= \left|\frac{z+2}{4}\right|$$

$\Rightarrow$ The given series is convergent if $l < 1$ (by Ratio test)

$$\Rightarrow \left|\frac{z+2}{4}\right| < 1$$

$$\Rightarrow |z+2| < 4,$$

$\therefore$ The region of convergence (R.O.C) is $|z+2| < 4$ radius of convergence is R = 4 and centre of the circle of convergence is $z_0 = -2$.

37. Ans: (0.2)

Sol: Let $\Sigma u_n = \Sigma(3+4i)^n.z^n$

Then $u_n = (3+4i)^n.z^n$

Now, $l = \underset{n\to\infty}{\text{Lt}}|u_n|^{\frac{1}{n}} = \underset{n\to\infty}{\text{Lt}}\left|(3+4i)^n z^n\right|^{\frac{1}{n}}$

$\Rightarrow l = \underset{n\to\infty}{\text{Lt}}|(3+4i)z|$

$\Rightarrow l = |3+4i|\ |z| = 5\ |z|$

$\Rightarrow$ The given series is convergent if $l < 1$

$\Rightarrow 5\ |z| < 1$

$\Rightarrow |z| < \frac{1}{5}$ of $|z-0| < \frac{1}{5}$

$\therefore$ The R.O.C of the given series is $|z-0| < \frac{1}{5}$

and radius of convergence is $R = \frac{1}{5}$

38. Ans: (b, d)

Sol: Please Refer ACE previous maths solution bookle

NUMERICAL METHODS

SOLUTIONS OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

(Solutions for Text Book Practice Questions)

01. Ans: (a)

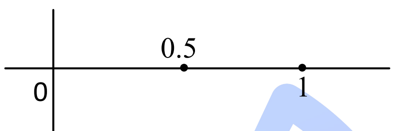
Sol: Please Refer ACE previous maths solution booklet

02. Ans: (b)

Sol: Let $f(x) = xe^x - 1$

$f(0) = -1 < 0$

$f(1) = e - 1 > 1$



1^{st} approx $x_1 = \frac{0+1}{2} = 0.5$

Now $f(x_1) = f(0.5) = (0.5)e^{0.5} - 1$

$= -0.1756 < 0$

Here f(0.5), f(1) have opposite signs

$\therefore$ The root lies between 0.5 and 1

2^{nd} approx $x_2 = \frac{0.5+1}{2} = 0.75$

03.

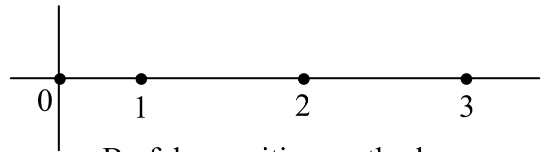
Sol: Please Refer ACE previous maths solution booklet

04. Ans: (a)

Sol: Let $f(x) = x^3 - 2x - 5$

$f(2) = -1 < 0$

$f(3) = 16 > 0$



By false position method

1^{st} approx $x_1 = \frac{2\times f(3) - 3\times f(2)}{f(3) - f(2)}$

$\Rightarrow$

$x_1 = \frac{2\times 16 - 3\times(-1)}{16+1} = \frac{35}{17} = 2.058$

05.

Sol: Please Refer ACE previous maths solution booklet

06. Ans: (d)

Sol: Let $f(x) = x^2 - 2$

$f^1(x) = 2x$

Given $x_0 = -1$

By N–R method

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$\Rightarrow x_1 = -1 - \frac{f(-1)}{f^1(1)}$$

$$\Rightarrow x_1 = -1 - \frac{(-1)}{-2}$$

$\therefore x_1 = -1.5$

<u>2^{nd} Approximation:</u>

$$x_2 = x_1 - \frac{f(x)}{f^1(x_1)}$$

$$\Rightarrow x_2 = -1.5 - \frac{f(-1.5)}{f^1(-1.5)}$$

$\Rightarrow x_2 = -1.416$

If we continue like this the root converges to $-\sqrt{2}$

07. Ans: (a)

Sol: Please Refer ACE previous maths solution booklet

08.

Sol: Please Refer ACE previous maths solution booklet

09. Ans: (a)

Sol: Given N–R iterative formula is

$$x_{n+1} = \frac{x_n^2 + b}{2x_n} \text{ ------------ (1)}$$

If 'x' is a root at nth approximation, then

$x_n = x$ then $x_{n+1} = x$: $x_{n+2} = x$, ……….

$\therefore$ From (1)

$$x = \frac{x^2 + b}{2x} \Rightarrow 2x^2 = x^2 + b$$

$$\therefore x^2 - b = 0$$

10. Ans: (b)

Sol: Let $f(x) = x^2 - 4x + 4$

Given initial values are $x_0 = 3$, $x_1 = 2.5$

By secant method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{3 \times f(2.5) - 2.5 \times f(3)}{f(2.5) - f(3)}$$

$$= \frac{3 \times (0.25) - 2.5 \times (1)}{0.25 - 1} = \frac{0.75 - 2.5}{-0.75} = 2.33$$

NUMERICAL INTEGRATION

11. Ans: (c)

Sol: Given $y = 2^x$

Number of sub intervals n = 4 : h = 1

x	–1	0	1	2	3
y	2^{-1}	2^0	2^1	2^2	2^3
	0.5 y_0	1 y_1	2 y_2	4 y_3	8 y_4

By Trapezoidal rule

$$\int_{-1}^{3} y\,dx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{2}[(0.5 + 8) + 2(1 + 2 + 4)] = \frac{22.5}{2} = 11.25$$

12.

Sol: Please Refer ACE previous maths solution booklet

13. Ans: (b)

Sol: Let $f(x) = xe^x$

$f^1(x) = xe^x + e^x = e^x (x+1)$

$f^{11}(x) = e^x(1) + (x+1) e^x$

$= e^x (x + 2)$

Here (a, b) = (1, 2)

$$h = \frac{b-a}{n} = \frac{1}{n}$$

$\text{Max } f^{11}(c) = f^{11}(2)$

$1 \le C \le 2 \quad = 4e^2$

We know that absolute error in Trapezoidal

rule $\le \left|\frac{h^2}{12}(b-a)\times \max f^{11}(c) 1 \le C \le 2\right|$

Given $\frac{h^2}{12}\times(b-a)\times \max f^{11}(c) = \frac{1}{3}\times 10^{-6}$

$1 \le C \le 2$

$\Rightarrow \frac{1}{12n^2}\times 1\times 4e^2 = \frac{1}{3}\times 10^{-6}$

$\Rightarrow \frac{e^2}{n^2} = 10^{-6}$

$\Rightarrow 10^6 e^2 = n^2$

$\Rightarrow (10^3 e)^2 = n^2$

$\therefore n = 1000e$

14. Ans: (d)

Sol: Please Refer ACE previous maths solution booklet

15.

Sol: Please Refer ACE previous maths solution booklet

16. Ans: (d)

Sol: Please Refer ACE previous maths solution booklet

17. Ans: (b)

Sol: Fly wheel energy $= \int_0^T T d\theta \quad (\theta \text{ in radions})$

$$= \frac{h}{3}\left[(T_0 + T_6) + 4(T_1 + T_3 + T_5) + 2(T_2 + T_4)\right]$$

(By Simpson's $\frac{1}{3}$ rule)

$$\therefore \text{Energy } = \frac{\pi/3}{3}\left[\begin{array}{r}(0+0)+4(1066+0-355)\\ +2(-323+323)\end{array}\right]$$

$$= 992.74 \simeq 993$$

Angle (degree)	0	60	120	180	240	300	360
Torque (Nm)	0	1066	–323	0	323	–355	0

NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

18.

Sol: given D.E is $\frac{dy}{dx} = xy + x^2$

$y(0) = 0.1 \Rightarrow x_0 = 0 : y_0 = 0.1$

$y(0.2) = ? \Rightarrow x_1 = 0.2 : y_1 = ?$

$h = 0.2$

we have

$y^1 = xy + x^2 \Rightarrow y_0^1 = x_0 y_0 + x_0^2 = 0 + 0 = 0$

$y^{11} = (xy^1 + y) + 2x$

$\Rightarrow y_0^{11} = x_0 y_0^1 + y_0 + 2x_0$

$= 0 + 0.1 + 0 = 0.1$

By Taylor series up to h^2 term

$$y_1 = y(x_1) = y_0 + \frac{h}{1!}y_0^1 + \frac{h^2}{2!}y_0^{11}$$

$$\Rightarrow y(0.2) = 0.1 + \frac{0.2}{1!}(0) + \frac{(0.2)^2}{2!}(0.1)$$

$\Rightarrow y(0.2) = 0.1 + 0.2 + 0.002$

$= 0.302$

19.

Sol: Given D.E is $y^1 = x - y$ with $y(0) = 0$

$h = 0.1$

Here $f(x, y) = x - y$:

$x_0 = 0 : y_0 = 0$

$x_1 = 0.1 : y_1 = ?$

By forward Euler's method

$y_1 = y_0 + hf(x_0, y_0)$

$\Rightarrow y_1 = y_0 + h(x_0 - y_0)$

$\Rightarrow y(0.1) = 0 + (0.1)(0 - 0)$

$= 0$

$\therefore y(0.1) = 0$

20.

Sol: Please Refer ACE previous maths solution booklet

21.

Sol: Please Refer ACE previous maths solution booklet

22. Ans: (0.96)

Sol: Given D.E is $\frac{dy}{dx} = -2xy^2$

$y_1^* = y_0 + h\left(-2x_0y_0^2\right) = 1$

$y_1 = y_0 + \frac{1}{2}0.2\left(-2x_0y_0^2 - 2x_1y_1^*\right)$

$= 1 - 0.04 = 0.96$

23. Ans: (0.00067)

Sol: Given D.E. is $y^1 = y + 2x - x^2$

Here $f(x, y) = y + 2x - x^2$

Given $y(0) = 1 \Rightarrow x_0 = 0 : y_0 = 1$

$h = 0.1$

By modified Euler's method

$$y_1 = y_0 + \frac{1}{2}\left(K_1 + K_2\right) \text{ -------------- (1)}$$

$K_1 = hf(x_0, y_0)$

$\Rightarrow K_1 = h(y_0 + 2x_0 - x^2) = 0.1(1) = 0.1$

$K_2 = hf(x_0 + h, y_0 + K_1)$

$\Rightarrow K_2 = h[(y_0 + K_1) + 2(x_0 + h) - (x_0 + h)^2]$

$= (0.1)[(1+0.1)+2(0.1)-(0.1)^2] = 0.129$

$\therefore$ From (1) $y_1 = 1 + \frac{1}{2}\left(0.1 + 0.129\right)$

$\Rightarrow y(0.1) = 1.1145$

Exact solution is $y(x) = x^2 + e^x$

$y(0.1) = (0.1)^2 + e^{0.1} = 1.1151$

Absolute error $= |1.1145 - 1.1151| = 0.00067$

24. Ans: (0.1103)

Sol: Given D.E. is $\frac{dy}{dx} = x + y$

Given $y(0) = 1$ and $h = 0.1$

Here $f(x, y) = x + y$

$x_0 = 0 : y_0 = 1$

$x_1 = 0.1$

By 4^{th} order R-K method

$$y_1 = y_0 + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) \text{ ----------- (1)}$$

$K_1 = hf(x_0, y_0) \Rightarrow K_1 = (0.1)(0+1)$

$\Rightarrow K_1 = 0.1$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{K_1}{2}\right)\right]$$

$$\Rightarrow K_2 = (0.1)\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)\right]$$

$\Rightarrow K_2 = 0.11$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{K_2}{2}\right)\right]$$

$$\Rightarrow K_3 = (0.1)\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.11}{2}\right)\right]$$

$K_3 = 0.1 \times 1105 = 0.1105$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= h[(x_0 + h) + (y_0 + K_3)]$

$\Rightarrow K_4 = (0.1)[(0+0.1)+(1+0.1105)]$

$\therefore K_4 = 0.12105$

From (1)

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + K_3 + K_4)$$

$\therefore y_1 = y(0.1) = 0.1103$

<u>INTERPOLATION (For CE only)</u>

25.

Sol: (i) <u>Lagranges interpolation</u>

$x_0 = 1$: $x_1 = 2$: $x_2 = 4$

$y_0 = 3$: $y_1 = 4$: $y_2 = 6$

By Lagranges interpolation

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$\Rightarrow f(x) = \frac{(x-2)(x-4)}{(-1)(-3)}(3) + \frac{(x-1)(x-4)}{(1)(-2)}(4) + \frac{(x-1)(x-2)}{(3)(2)}(6)$$

$$\Rightarrow f(x) = \frac{6(x^2-6x+8) - 12(x^2-5x+4) + 6(x^2-3x+2)}{6}$$

$$\Rightarrow f(x) = \frac{6x+12}{6} = x+2$$

$\therefore f(x) = x + 2$

$f^1(x) = 1$

Now $f(3) = 5 : f^1(3) = 1$

<u>(ii) Newton's divided difference interpolation</u>

x	y	Δy	
x_0 (1)	(3) y_0		
		(1)	
x_1 2	4 y_1		0
		1	
x_2 4	6 y_2		

By Newton's divided interpolation

$f(x) = y_0 + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - 1)\, f[x_0, x_1, x_2]$

$\Rightarrow f(x) = 3 + (x - 1)(1) + (x - 1)(x - 2)0$

$f(x) = x + 2$

$f^1(x) = 1$

Now $f(3) = 5$

$f^1(3) = 1$

(iii) Newton's backward interpolation

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0 4	1 y_0			
		$2 = \nabla y_1$		
x_1 6	3 y_1		$3 = \nabla^2 y_2$	
		$5 = \nabla^2 y_3$		$(0) = \nabla^3 y_3$
x_2 8	8 y_2		$3 = \nabla^2 y_3$	
x_3 10	16 y_3	$8 = \nabla y_3$		

Here $h = 2$

$$P = \frac{x - x_n}{h} = \frac{x - x_3}{h} = \frac{x - 10}{2}$$

By Newton's backward interpolation

$$f(x) = y_3 + P\nabla y_3 + \frac{P(P+1)}{2!}\nabla^2 y_3 + \frac{P(P+1)(P+2)}{3!}\nabla^2 y_3$$

$$\Rightarrow f(x) = 16 + \left(\frac{x-10}{2}\right)(8) + \left(\frac{x-10}{2}\right)\left(\frac{x-8}{2}\right) + (3) + 0$$

$$\Rightarrow f(x) = 16 + \frac{8x - 80}{2} + \frac{x^2 - 18x + 80}{8}(3)$$

$$\Rightarrow f(x) = \frac{128 + (32x - 320) + 3x^2 - 54x + 240}{8}$$

$$\Rightarrow f(x) = \frac{3x^2 - 22x + 48}{8}$$

$$f^1(x) = \frac{6x - 22}{8}$$

Now $f(1) = \frac{29}{8}$

$f^1(1) = -2$

26.

Sol: (i) Newton's forward interpolation

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0 4	1 y_0			
		2 Δy_0		
6	3		3	
		5	$\Delta^2 y_0$	0
8	8			$\Delta^3 y_0$
10	16	8	3	

Here $h = 2$;

$$P = \frac{x - x_0}{h} = \frac{x - 4}{2}$$

By Newton's Forward interpolation

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$\Rightarrow f(x) = 1 + \left(\frac{x-y}{2}\right)(2) + \frac{(x-4)(x-6)}{4\times 2!}\times(3) + 0$$

$$\Rightarrow f(x) = 1 + (x-4) + \frac{3}{8}(x^2 - 10x + 24)$$

$$\Rightarrow f(x) = \frac{8 + 8x - 32 + 3x^2 - 30x + 72}{8}$$

$$\Rightarrow f(x) = \frac{3x^2 - 22x + 48}{8}$$

$$f^1(x) = \frac{6x - 22}{8} = \frac{3x - 11}{4}$$

Now $f(1) = \frac{29}{8}$

$f^1(1) = -2$

FOR ESE ONLY

27. Ans: (a)

Sol: Given equations can be written as

$$x_1 = \frac{3-2x_2-x_3}{3} \qquad \text{(From equation 3)}$$

$$x_2 = \frac{1-2x_1-x_3}{3} \qquad \text{(From equation 2)}$$

$$x_3 = \frac{5-x_1-2x_2}{3} \qquad \text{(From equation 1)}$$

Initial values are $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

1st approximations

$$x_1^{(1)} = \frac{3-2x_2^{(0)}-x_3^{(0)}}{3} = 1$$

$$x_2^{(1)} = \frac{1-2x_1^{(1)}-x_3^{(0)}}{3} = \frac{1-2(1)-0}{3} = -\frac{1}{3}$$

$$x_3^{(1)} = \frac{5-x_1^{(1)}-2x_2^{(1)}}{3} = \frac{5-1+2\left(\frac{1}{3}\right)}{3}$$

$$= \frac{4+\frac{2}{3}}{3} = \frac{14}{9} = 1.55$$

28. Ans: (b)

Sol: Given equation is x = cosx

This equation is in a form $x = \phi(x)$

Let $\phi(x) = \cos x$

$\phi^1(x) = -\sin x$

Clearly $|\phi^1(x)| < 1 \ \forall\ x$

Given $x_0 = \frac{\pi}{4}$

1st approximation $x_1 = \phi(x_0)$

$\Rightarrow x_1 = \cos x_0$

$\Rightarrow x_1 = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$\therefore x_1 = \frac{1}{\sqrt{2}} = 0.7072$

29. Ans: (a)

Sol: Since 5 data points given $\Delta^4 y_0 = 0$

$\Rightarrow (E-1)^4\, y_0 = 0$

$\Rightarrow [(E^2-2E+1)(E^2-2E+1)]y_0 = 0$

$\Rightarrow E^4y_0 - 4E^3y_0 + 6E^2y_0 - 4Ey_0 + y_0 = 0$

$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$\Rightarrow 81 - 4y_3 + 6\times 9 - 4\times 3 + 1 = 0$

$\Rightarrow 4y_3 = 124$

$\therefore y_3 = 31$

30. Ans: (d)

Sol: $\Delta e^x = e^{x+h} - e^x = e^x\,(e^h - 1)$

$\Delta^2 e^x = e^{x+h}\,(e^h-1) - e^x\,(e^h-1)$

$= (e^h-1)\,[e^{x+4} - e^x]$

$= (e^h-1)(e^h-1)e^x = (e^h-1)^2e^x$

Continue like this

$\Delta^n e^x = (e^h-1)^n\, e^x$

From the options put h = 1

$\therefore \Delta^n e^x = (e-1)^n\, e^x$