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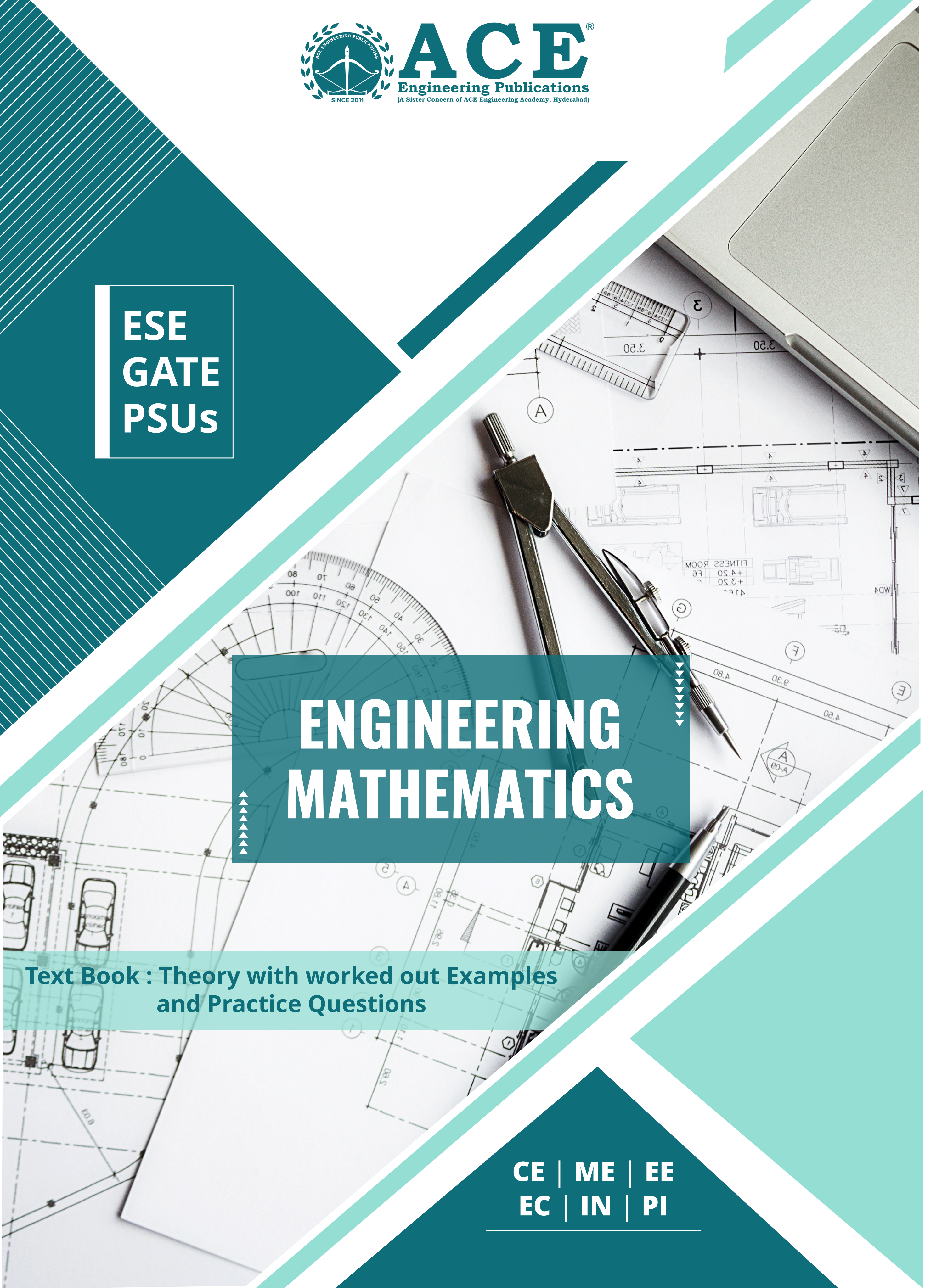
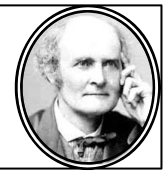


ENGINEERING MATHEMATICS

Text Book : Theory with worked out Examples and Practice Questions

CE | ME | EE
EC | IN | PI

Chapter 1 Linear Algebra


Arthur Cayley
(1821 – 1895)

01. Ans: (c)

Sol: Consider $|A| = \begin{vmatrix} 2 & 2 & 3 & 3 \\ 2 & 1 & 4 & 2 \\ 3 & 2 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{vmatrix}$

$R_2 \to R_2 - R_1$;

$\Rightarrow |A| = \begin{vmatrix} 2 & 2 & 3 & 3 \\ 0 & -1 & 1 & -1 \\ 3 & 2 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{vmatrix}$

$C_2 \to C_2 + C_3$; $C_4 \to C_4 + C_3$

$\Rightarrow |A| = \begin{vmatrix} 2 & 5 & 3 & 6 \\ 0 & 0 & 1 & 0 \\ 3 & 5 & 3 & 5 \\ 3 & 3 & 2 & 6 \end{vmatrix}$

$\Rightarrow |A| \equiv (-1)\begin{vmatrix} 2 & 5 & 6 \\ 3 & 5 & 5 \\ 3 & 3 & 6 \end{vmatrix}$

(expanding along 2nd row)

$\Rightarrow |A| = (-1)[2(30-15) - 5(18-5) + 6(9-15)]$

$\therefore |A| = (-1)[30 - 15 - 36] = 21$

02. Ans: (b)

Sol: Given $A = \begin{bmatrix} 6 & 7 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow |A| = 12 - 14 = -2$

Consider $|A^{2004} - 2A^{2003}| = |A^{2003}(A - 2I)|$

$\Rightarrow |A^{2004} - 2A^{2003}| = |A^{2003}|\,|A - 2I|$ ($\because |AB| = |A|\,|B|$)

$\Rightarrow |A^{2004} - 2A^{2003}| = |A|^{2003} \begin{vmatrix} 4 & 7 \\ 2 & 0 \end{vmatrix}$

$\Rightarrow |A^{2004} - 2A^{2003}| = (-2)^{2003}(-14)$

$\Rightarrow |A^{2004} - 2A^{2003}| = (-1)(2)^{2003}(-1)(2)(7)$

$\therefore |A^{2004} - 2A^{2003}| = (2^{2004})(7)$

03. Ans: (b)

Sol: If $A = [a_{11}]_{1\times1}$ then

$|A| = |a_{11}| = a_{11}$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$|A| = a_{11}a_{22} - a_{12}a_{21}$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{31}a_{23} - a_{12}a_{21}a_{33} + a_{13}a_{22}a_{32} - a_{13}a_{31}a_{22}$

$\therefore$ The number of terms in expansion of 1st order determinant is 1!,

The number of terms in expansion of 2nd order determinant is 2!,

The number of terms in expansion of 3rd order determinant is 3!

and so on the number of terms in expansion of n^{th} order determinant is n!

Arthur Cayley was probably the first mathematician to realize the importance of the notion of a matrix and in 1858 published book, showing the basic operations on matrices. He also discovered a number of important results in matrix theory.

04. Ans: (a, b, c)

Sol: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ni a, b, c, d \neq 0$

Then

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

Given that $A^2 = I$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2 + bc = 1$, $ab + bd = 0$, $ac + cd = 0$ and $bc + d^2 = 1$

$\Rightarrow c(a + d) = 0$, $b(a + d) = 0$

$\Rightarrow a + d = 0 \quad (\because b \neq 0 \text{ and } c \neq 0)$

$\Rightarrow tr(A) = 0$

$\therefore$ I is true

Consider $|A| = ad - bc$

$\Rightarrow |A| = a(-a) - bc \quad (\because a + d = 0 \Leftrightarrow d = -a)$

$\Rightarrow |A| = (-1)(a^2 + bc)$

$\Rightarrow |A| = (-1)(1) \quad (\because a^2 + bc = 1)$

$\therefore$ II is false

Hence, option (a), (b), (c) are correct.

05. Ans: 16

Sol: The number of multiplication involved in computing the matrix product (PQ)R is 48 & the matrix product P(QR) is 16.

$\therefore$ The minimum number of multiplication is 16.

06. Ans: (c)

Sol: Now, $B = A^{-1} = \dfrac{adj(A)}{|A|}$

$$\Rightarrow B = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\therefore$ The element in the 2^{nd} row and 3^{rd} column of B is given by

$$\frac{1}{|A|}A_{32} = \frac{1}{|A|}(-1)^{3+2}M_{32}$$

$$= \frac{1}{2}(-1)(1-0) = \frac{-1}{2}$$

07. Ans: 0

Sol: Given that A and B are symmetric matrices.

$\Rightarrow A^T = A$ and $B^T = B$

Consider $(AB - BA)^T = (AB)^T - (BA)^T$ $(\because (A - B)^T = A^T - B^T)$

$\Rightarrow (AB - BA)^T = B^T A^T - A^T B^T$ $(\because (AB)^T = B^T A^T)$

$\Rightarrow (AB - BA)^T = BA - AB$

$\Rightarrow (AB - BA)^T = -(AB - BA)$

$\Rightarrow (AB - BA)$ is a skew symmetric matrix of order (3×3)

$\therefore \ |AB - BA| = 0$

08. Ans: 46

Sol: Here, $|\text{adj } A| = |A|^2$ $(\because |\text{adj}(A_{n\times n})| = |A|^{n-1})$

$\Rightarrow 2116 = |A|^2$

$\Rightarrow |A| = \pm 46$

$\therefore$ Absolute value of $|A|$ is 46

09. Ans: (d)

Sol: $\because \text{adj}(\text{adj}(A_{n\times n})) = |A|^{n-2} A_{n\times n}$

$\Rightarrow \text{adj}(\text{adj}(A_{3\times 3})) = |A|^{3-2} A_{3\times 3}$

$\Rightarrow \text{adj}(\text{adj}(A_{3\times 3})) = [1(0-4) - 2(a-4) + (a - 0)]A$

$\Rightarrow A = (4 - a)A$ $(\because \text{adj}(\text{adj}(A)) = A)$

$\Rightarrow 4 - a = 1$

$\therefore a = 3$

10. Ans: 1

Sol: Given that $B = \text{adj}(A)$ and $C = 5A$

Consider $\dfrac{|\text{adj}(B)|}{|C|} = \dfrac{|\text{adj}(\text{adj}(A))|}{|5A|}$

$$\Rightarrow \frac{|\text{adj}(B)|}{|C|} = \frac{|A|^{(3-1)^2}}{5^3|A|}$$

$$\left(\because |\text{adj}(\text{adj}(A)_{n\times n})| = |A_{n\times n}|^{(n-1)^2} \text{ and } |kA_{n\times n}| = k^n|A_{n\times n}|\right)$$

$$\Rightarrow \frac{|\text{adj}(B)|}{|C|} = \frac{|A|^4}{5^3|A|}$$

$$\Rightarrow \frac{|\text{adj}(B)|}{|C|} = \frac{|A|^3}{5^3}$$

$$\therefore \frac{|\text{adj}(B)|}{|C|} = \frac{5^3}{5^3} = 1 \qquad (\because |A| = 5)$$

11. Ans: (a)

Sol: Each element of the matrix in the principal diagonal and above the diagonal, we can chosen in q ways.

Number of elements in the principal diagonal = n

Number of elements above the principal diagonal = $n\left(\dfrac{n-1}{2}\right)$

By product rule, number of ways we can choose these elements = $q^n . q^{n\left(\frac{n-1}{2}\right)}$

Required number of symmetric matrices $= q^{n\left(\frac{n+1}{2}\right)}$

12. Ans: 4

Sol: Given $A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

$R_4 \to R_4 + R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\Rightarrow A \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_5 \to R_5 + R_4$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$, which is an Echelon form of A

$\therefore$ Rank of A = number of non-zero rows in Echelon form of 'A' = 4

13. Ans: (a)

Sol: Given that $A.adj(A) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$\Rightarrow A.adj(A) = 5I$

$\Rightarrow |A| = 5 \neq 0$,

$(\because A.adj(A) = adj(A)\, A = |A|\, I_n)$

$\therefore \rho(A_{3\times3}) = 3$

14. Ans: (a)

Sol: Here, A^n is a zero matrix.

$\therefore$ rank of $A^n = 0$

15. Ans: (a)

Sol: S1 is true because, any subset of linearly independent set of vectors is always linearly independent set.

S2 is not necessarily true,

for example, $\{X_1, X_2, X_3\}$ can be linearly independent set and X_4 is linear combination of X_1, X_2 and X_3.

16. Ans: (a, b, d)

Sol: The augmented matrix of the given system is

$$(A \mid B) = \begin{pmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_3$

$$\Rightarrow (A \mid B) \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$R_2 \to R_2 - 4R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 - R_1$

$$\Rightarrow (A \mid B) \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\Rightarrow (A \mid B) \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_3 \to R_3 - 4R_2;\ R_4 \to R_4 - 3R_2$

$$\Rightarrow (A \mid B) \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 15 & 21 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\Rightarrow (A \mid B) \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \rho(A) = \rho(A \mid B) = 3$ = no. of variables

$\therefore$ The system AX = B has a unique solution.

Hence, options (a), (b) and (d) are false statements.

17. Ans: (b)

Sol: The condition for many solutions of AX = B is

$\rho(A) = \rho(A|B) \neq n = 3$

$$\text{Consider } (A|B) = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 1 & 1 & -1 & 1 \\ -1 & k & 3 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1;\ R_3 \to R_3 + R_1$

$$\Rightarrow (A|B) \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 2 & -3 & -6 \\ 0 & k-1 & 5 & 7 \end{array}\right]$$

$R_3 \to 2R_3 - (k-1)R_2$

$$\Rightarrow (A|B) \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 7 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 3k+7 & 6k+8 \end{array}\right]$$

If $3k + 7 \neq 0$ (or) $k \neq -7/3$ then $\rho(A) = 3 = \rho(A/B) = n = 3$ and system will have a unique solution.

If $3k + 7 = 0$ (or) $k = -7/3$ then $\rho(A) = 2 \neq \rho(A/B)$ and the system will not have a solution.

Here, there is no real value of k such that $\rho(A) = 2 \neq \rho(A/B) <= n = 3$.

$\therefore$ The system will not have many solutions

Hence, option (b) is correct.

18. Ans: (b)

Sol: The condition for unique solution of AX = B is $\rho(A) = \rho(A \mid B) = n = 3$ (or) $|A| \neq 0$.

$$\text{Given, } A = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix}$$

$\Rightarrow |A| = k(k^2 - 1) - (k - 1) + (1 - k)$

$\Rightarrow |A| = (k - 1)[k^2 + k - 2]$

$\Rightarrow |A| = (k - 1)^2 (k + 2)$

Thus, the system has a unique solution when

$(k - 1)^2 (k + 2) \neq 0$

(or) $k \neq 1$ and $k \neq -2$

19. Ans: (a)

Sol: Given n – r = 1, where r = ρ(A) and n = order of the matrix

$\Rightarrow 3 - r = 1$

$\Rightarrow r = \rho(A) = 2$ = number of non-zero rows in an echelon form

∴ To have rank 2 for matrix A, k must be either –1 or 0.

20. Ans: (a)

Sol: Here Rank of A = Rank of [A|B] = 3

∴ The given system has a unique solution.

21. Ans: (a, b, c)

Sol: Now, $|A| = \begin{vmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{vmatrix}$

$\Rightarrow |A| = 2(3 - 6) - 0 + 1(8 - 0)$

$\Rightarrow |A| = 8 - 6 = 2 \neq 0$

$\Rightarrow \rho(A_{3\times 3}) = 3$

⇒ The system will have a unique solution.

∴ Option (d) is not true and other options are true

22. Ans: 2

Sol: Consider $(A|B) = \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 1 & -3 & 4 & 1 \\ -2 & 4 & -6 & k \end{array}\right]$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 + 2R_1$

$\Rightarrow (A/B) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & k-2 \end{array}\right)$

Here, this system is consistent only when $k - 2 = 0$

∴ For k = 2, the system has infinitely many solutions.

23. Ans: (c)

Sol: Given $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow (1 - \lambda)\{(5 - \lambda)^2 + 36\} = 0$

$\Rightarrow (1 - \lambda)(\lambda^2 - 10\lambda + 61) = 0$

$\therefore \lambda = 1, 5 \pm 6j$

24. Ans: 15

Sol: If $\lambda = 2 + \sqrt{-1} = 2 + i$ is an eigen value of matrix A then 2–i is also an eigen value of matrix A.

∴ |P| = product of eigen values of P

$= (2 + i)(2 - i)\,3$

$= (4 + 1)3 = 15$

25. Ans: (a)

Sol: Given $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$

⇒ The characteristic equation of a given matrix $A_{2\times 2}$ is given by $|A - \lambda I| = 0$.

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \lambda = 2, -1$

If $\lambda_1 = 2$ and $\lambda_2 = -1$ are the eigen values of a matrix $A_{2\times2}$ then the eigen values of A^{1000} are $\lambda_1^{1000} = 2^{1000}$ and $\lambda_2^{1000} = (-1)^{1000} = 1$

$\therefore$ tr $(A^{1000}) = 2^{1000} + 1$

26. Ans: –6

Sol: If λ is an Eigen values of A, then

$\lambda^3-3\lambda^2$ is an Eigen value of A^3-3A^2

Putting $\lambda = 1, -1,$ and 3 in $\lambda^3 - 3\lambda^2$,

we get the eigen values of $A^3 - 3A^2$ are -2, -4, 0

$\therefore$ Trace of $(A^3-3A^2) = -6$

27. Ans: (a)

Sol: Since, A is singular, $\lambda = 0$ is an eigen value.

Also, rank of A = 1.

The root $\lambda = 0$ is repeated $n - 1$ times.

trace of $A = n = 0 + 0 + ... + \lambda_n$.

$\Rightarrow \lambda_n = n$

$\therefore$ The distinct eigen values are 0 and n.

28. Ans: 5

Sol: The characteristic equation of M is

$\lambda^3 - 4\lambda^2 + a\lambda + 30 = 0$ ……………(1)

Substituting $\lambda = 2$ in (1), we get $a = -11$

Now, the characteristic equation is

$\lambda^3 - 4\lambda^2 - 11\lambda + 30 = 0$

$\Rightarrow (\lambda-2)(\lambda^2 - 2\lambda - 15) = 0$

$\Rightarrow \lambda = 2, -3, 5$

$\therefore$ The largest among the absolute values of the eigen values of M = 5

29. Ans: (c)

Sol: The given matrix is upper triangular. The eigen values are same as the diagonal elements 1, 2, –1 and 0.

The smallest eigen value is $\lambda = -1$. The eigen vectors for $\lambda = -1$ is given by

$(A - \lambda I)X = 0$

$\Rightarrow (A + I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$\Rightarrow w = 0, y = 0, 2x - z = 0$

$\therefore X = k[1 \ \ 0 \ \ 2 \ \ 0]^T$

30. Ans: 2

Sol: Consider $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ $\lambda = 2$ is an eigen value of a given matrix A.

31. Ans: 7

Sol: Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}$

and eigen vector $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

We know that AX = λX

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 30 \\ -16-2x \\ 15 \end{bmatrix} = \begin{bmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{bmatrix}$$

$\Rightarrow \lambda = 15$ and $-16 - 2x = -30$

$\Rightarrow -2x = -14$

$\therefore\ x = 7$

32. Ans: (2)

Sol: $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\Rightarrow \lambda = 2,2,3$

For the repeated eigen value $\lambda = 2$,

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of $(A - \lambda I) = r = 2$, number of variables = n = 3

Number of linearly independent eigen vectors = n – r = 3 – 2 = 1

The number of linearly independent eigenvectors corresponding to the eigen value λ = 2 is one. The number of linearly independent eigen vectors corresponding to an eigen value λ = 3 is one

($\because$ λ = 3 is a distinct eigen value)

$\therefore$ The number of linearly independent eigen vectors of A is 2.

33. Ans: (b)

Sol: Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

⇒ The characteristic equation of a given matrix $A_{3\times3}$ is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\Rightarrow \lambda^3 = \lambda^2 + \lambda - 1$

$\Rightarrow A^3 = A^2 + A - I$ ———(1)

(by Cayley-Hamilton's theory)

Here, for n = 3 only the option (b) gives equation (1)

$\therefore A^n = A^{n-2} + A^2 - I$.

34. Ans: 1

Sol: Given $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\Rightarrow$ Characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

By Calay-Hamilton's theorem,

$A^3 - A^2 - 4A + 4I = O$ .... (1)

Given that $A^3 - A^2 - 4A + 5I = B$ .... (2)

From (1) and (2), we get B = I

$\therefore |B| = 1$

Basis(Only for ECE)

35. Ans: (a)

Sol: Given $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & -5 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$\Rightarrow \rho(A) = 3$

= number of linearly independent rows

$\therefore$ The set of vectors is linear set and it forms a basis of R^3

36. $k \neq 0$

Sol: If the given vectors form a basis, then they are linearly independent

$$\Rightarrow \begin{vmatrix} k & 1 & 1 \\ 0 & 1 & 1 \\ k & 0 & k \end{vmatrix} \neq 0$$

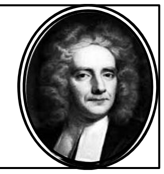
$\Rightarrow k^2 + k - k \neq 0$

$\therefore k \neq 0$

Chapter 2 Calculus

(With Vector Calculus & Fourier Series)

Sir Isaac Newton
(1643 – 1727)



G. W. Von Leibniz
(1646 – 1716)

01. Ans: (a)

Sol: $\lim\limits_{x\to\frac{5}{4}}(x-[x]) = \lim\limits_{x\to\frac{5}{4}} x - \lim\limits_{x\to\frac{5}{4}}[x]$

$$= \frac{5}{4} - 1 = \frac{1}{4}$$

02. Ans: (d)

Sol: $\lim\limits_{x\to 2}\frac{|x-2|}{x-2}$

Left Limit = $\lim\limits_{x\to 2^-}\frac{-(x-2)}{x-2} = -1$

Right Limit = $\lim\limits_{x\to 2^+}\frac{x-2}{x-2} = 1$

$\therefore$ Left Limit $\neq$ Right Limit

$\Rightarrow$ Limit does not exist

03. Ans: (d)

Sol: $\lim\limits_{x\to 2}\frac{x^2+x-6}{|x-2|} = \lim\limits_{x\to 2}\frac{(x+3)(x-2)}{|x-2|} =$

$\lim\limits_{x\to 2^-}\frac{(x+3)(x-2)}{|x-2|} = \lim\limits_{x\to 2^-}\frac{(x+3)(x-2)}{-(x-2)} = -5$

$\lim\limits_{x\to 2^+}\frac{(x+3)(x-2)}{|x-2|} = \lim\limits_{x\to 2^+}\frac{(x+3)(x-2)}{(x-2)} = 5$

Since $\lim\limits_{x\to 2^-}\frac{x^2+x-6}{|x-2|} \neq \lim\limits_{x\to 2^+}\frac{x^2+x-6}{|x-2|}$

$\lim\limits_{x\to 2}\frac{x^2+x-6}{|x-2|}$ = does not exist

04. Ans: 2

Sol: $\lim\limits_{x\to 0}\frac{x\sin x}{1-\cos x} = \lim\limits_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)}{\left(\frac{1-\cos x}{x^2}\right)} = \frac{1}{\frac{1}{2}} = 2$

05. Ans: (3)

Sol: $\lim\limits_{x\to 2}\frac{\tan(x-2)\left(x^2+(k-2)x-2k\right)}{(x-2)\ (x-2)} = 5$

$\lim\limits_{x\to 2}\frac{\tan(x-2)}{x-2}\cdot\lim\limits_{x\to 2}\frac{x^2+(k-2)x-2k}{x-2} = 5$

$1.\ \lim\limits_{x\to 2}\frac{x^2+(k-2)x-2k}{x-2} = 5$

In the above equation LHS is of the form 0/0, so applying L' Hospital rule, we get

$\lim\limits_{x\to 2}\frac{2x+(k-2)1}{1} = 5$

4 + k–2 = 5

k = 3

06. Ans: (0.5)

Sol: $\lim\limits_{x\to 1}\left(\frac{1}{\ell n} - \frac{1}{x-1}\right) = \frac{1}{0} - \frac{1}{0} = \infty - \infty$

(Indeterminate form)

$\lim\limits_{x\to 1}\left(\frac{1}{\ell n} - \frac{1}{x-1}\right) = \lim\limits_{x\to 1}\frac{x-1-\ell n\ x}{(\ell n\ x)(x-1)} = \frac{0}{0}$

(Indeterminate form)

Using L' Hospital rule, we get

$\lim\limits_{x\to 1}\frac{1-0-\frac{1}{x}}{\ell n\ x + \frac{x-1}{x}} = \frac{0}{0}$

Issac Newton(most influential scientist) and Leibniz (universal genius) independently developed calculus which leads to the development of differential and integral equations of mathematical physics

Applying L' Hospital rule again, we get

$$\lim_{x\to 1}\frac{1/x^2}{\frac{1}{x}+\frac{x(1)-(x-1)}{x^2}}=\frac{1}{1+\frac{1-0}{1}}=\frac{1}{2}$$

Correct answer 0.5

07. Ans: (c)

Sol: $\lim\limits_{x\to\infty}\frac{x\ell n(x)}{1+x^2}=\frac{\infty}{\infty}\to$ Indeterminate form

using L' Hospital rule. We get

$$\lim_{x\to\infty}\frac{(x)\frac{1}{x}+\ell n(x)}{2x}=\lim_{x\to\infty}\frac{1+\ell n\,x}{2x}=\frac{\infty}{\infty}$$

Apply L' Hospital rule again, we get

$$\lim_{x\to\infty}\frac{1/x}{2}=0$$

$$\therefore \lim_{x\to\infty}\frac{x\,\ell n\,x}{1+x^2}=0$$

08. Ans: (c)

Sol: Let $y=\lim\limits_{x\to\infty}x^{\frac{1}{x}}$

Taking logarithm on both sides, we get

$$\log\ y=\lim_{x\to\infty}\left\{\log\left[x^{\frac{1}{x}}\right]\right\}$$

$$\Rightarrow \log y=\lim_{x\to\infty}\left\{\frac{1}{x}\log x\right\}\quad\left(\frac{\infty}{\infty}\text{ form}\right)$$

Applying L 'Hospital rule, we get

$$\log y=\lim_{x\to\infty}\left(\frac{1}{x}\right)$$

$$\Rightarrow \log y=0$$

$$\therefore\quad y=\operatorname{Lim}_{x\to\infty}x^{\frac{1}{x}}=e^0=1$$

09. Ans: (0.25)

Sol: $\lim\limits_{x\to -3}\frac{\sqrt{2x+22}-4}{x+3}=\frac{0}{0}$ (Indeterminate form)

$$\lim_{x\to -3}\frac{\left(\sqrt{2x+22}-4\right)\left(\sqrt{2x+22}+4\right)}{(x+3)\left(\sqrt{2x+22}+4\right)}$$

$$=\lim_{x\to -3}\frac{2x+22-16}{(x+3)\left(\sqrt{2x+22}+4\right)}$$

$$\lim_{x\to -3}\frac{2}{\sqrt{2x+22}+4}=0.25$$

10. Ans: (a)

Sol: If $\operatorname{Lt}_{x\to a^-}f(x)=\operatorname{Lt}_{x\to a^+}f(x)=f(a)$ then

f(x) is continuous at x = a

Here $\operatorname{Lt}_{x\to 3^-}f(x)=\operatorname{Lt}_{x\to 3}\frac{x+3}{3}=2$

$\operatorname{Lt}_{x\to 3^+}f(x)=\operatorname{Lt}_{x\to 3}x-1=2$ and $f(3)=2$

$\therefore$ Option (a) is correct

11. Ans: (d)

Sol: $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^+}f(x)$

$$a+b=1^2+3(1)+3$$

$$\Rightarrow a+b=7\quad\ldots(1)$$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\lim_{x\to 1^-}\frac{ax+\cancel{b}-a-\cancel{b}}{x-1}=\lim_{x\to 1^+}\frac{x^2+3x+3-a-b}{x-1}$$

$$\lim_{x\to 1^-}\frac{a(x-1)}{(x-1)}=\lim_{x\to 1^+}\frac{x^2+3x+3-7}{x-1}$$

$$a=\lim_{x\to 1^+}\frac{x^2+4x-x-4}{x-1}$$

$$a = \lim_{x\to 1^+} \frac{(x-1)(x+4)}{(x-1)} \Rightarrow a = 5 \Rightarrow b = 2$$

$\therefore f(x)$ is differentiable for unique values of a and b

Option (d) is correct.

12. Ans: (b)

Sol: Given $f(x) = \begin{cases} e^x, & x < 1 \\ \log x + ax^2 + bx, & x \geq 1 \end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} e^x, & x < 1 \\ \frac{1}{x} + 2ax + b, & x \geq 1 \end{cases}$$

If f(x) is differentiable at x = 1, then $f'(1^+) = f'(1^-)$

$\Rightarrow 1 + 2a + b = e$

$\Rightarrow 2a + b = e - 1$ …………(1)

If f(x) is differentiable at x = 1 then f(x) is continuous at x = 1.

$\Rightarrow \underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^+}{Lt} f(x)$

$\Rightarrow a + b = e$ ……….(2)

Solving (1) and (2)

$a = -1$

$b = e + 1$

$\therefore$ f(x) is differentiable at x = 1, for unique values of 'a' and 'b'.

13. Ans: (a)

Sol: Since, f is differentiable at x = 2,

$f'(2^-) = f'(2^+)$

$\Rightarrow (2x)_{x=2} = m$

$\therefore m = 4$

Since, f is continuous at x = 2

i.e., $(x^2)_{x=2} = (mx + b)_{x=2}$

$\Rightarrow 4 = 2m + b$

$\therefore b = -4$

Hence, option (A) is correct.

14. Ans: (d)

Sol: Let, $f(x) = x^2 - 2x + 2$ and $[a,b] = [1,3]$

Then, $f'(x) = 2x - 2$

By a mean value theorem

$\exists c \in (1,3) \Rightarrow f'(c) = \frac{f(3) - f(1)}{3-1}$

$\Rightarrow c - 1 = 1$

$\therefore c = 2$ (or) $x = 2$

15. And: (b)

Sol: Let

$f'(x) = \sin(x) + 2.\sin(2x) + 3\sin(3x) - \frac{8}{\pi} = 0$

be the given equation.

Then,

$f(x) = -\cos(x) - \cos(2x) - \cos(3x) - \frac{8}{\pi}(x) + k$

Here, if the function f(x) satisfies the all the three conditions of the Rolle's theorem in [a, b], then the equation $f'(x) = 0$ has at least one real root in (a, b).

As cos(ax) is continuous & differentiable function and $a_0 + a_1x$ is continuous & differentiable function for all x, the function f(x) is continuous and differentiable for all x.

Here, (i) f(x) is continuous on $\left[0, \frac{\pi}{2}\right]$

(ii) f(x) is differentiable on $\left(0, \frac{\pi}{2}\right)$

(iii) f(0) = – 3 + k = $f\left(\frac{\pi}{2}\right)$

$\therefore$ By a Rolle's theorem, the given equation has at least one root in $\left(0, \frac{\pi}{2}\right)$.

Hence, option (B) is correct.

16. Ans: 19

Sol: Applying Lagrange mean value theorem we get

$$f'(c) = \frac{f(3)-f(-3)}{3-(-3)} = \frac{f(3)-7}{6}$$

$$\frac{f(3)-7}{6} \le 2$$

$$f(3)-7 \le 12$$

$$f(3) \le 19$$

Correct answer 19.

17. Ans: (b)

Sol: Let f(x) be defined in [0, 1].

By Lagrange's Mean Value Theorem,

$\exists\, c \in (0,1)$ such that

$$f'(c) = \frac{f(1)-f(0)}{1-0}$$

$$\Rightarrow \frac{1}{5-c^2} = \frac{f(1)-2}{1}$$

$$\because \underset{0<x<1}{\text{Min}}\{f'(x)\} < f'(c) < \underset{0<x<1}{\text{Max}}\{f'(x)\}$$

$$\Rightarrow \frac{1}{5} < f(1) - 2 < \frac{1}{4}$$

$\therefore$ 2.2 < f(1) < 2.25

18. Ans: 2.5 range 2.49 to 2.51

Sol: By Cauchy's mean value theorem,

$$\frac{f'(c)}{g'(c)} = \frac{f(3)-f(2)}{g(3)-g(2)}$$

$$\Rightarrow -e^{2c} = \frac{e^3 - e^2}{e^{-3} - e^{-2}}$$

$\therefore$ c = 2.5 $\in$ (2, 3)

19. Ans: (a)

Sol: $f(x) = e^{\sin x} \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^{\sin x}.\cos x \Rightarrow f'(0) = 1$

$f''(x) = e^{\sin x}.\cos^2 x + e^{\sin x}(-\sin x) \Rightarrow f''(0) = 1 - 0 = 1$

Taylor's Series for f(x) about x = 0 is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots\ldots\ldots$$

$$= 1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots$$

20. Ans: (a)

Sol: Coefficient of $x^4 = \frac{f^{IV}(0)}{4!}$

Given f(x) = log (secx)

$$\Rightarrow f'(x) = \frac{1}{\sec x}\sec x \tan x = \tan x$$

$$\Rightarrow f''(x) = \sec^2 x$$

$$\Rightarrow f'''(x) = 2\sec^2 x \tan x$$

$\Rightarrow f^{IV}(x) = 2[\sec^2x \sec^2x +$

$\tan x.\ 2\sec x.\sec x \tan x]$

$\Rightarrow f^{IV}(0) = 2$

$\therefore$ Coefficient of $x^4 = \frac{f^{IV}(0)}{4!} = \frac{2}{24} = \frac{1}{12}$

21. Ans: (a)

Sol: Let f(x) = 3 sin x + 2 cos x

$$= 3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}.....\right) + 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}.....\right)$$

$$\therefore f(x) = 2 + 3x - x^2 - \frac{x^3}{2} + ...$$

22. Ans: (c)

Sol: $e^{x+x^2} = 1 + \frac{(x+x^2)}{1!} + \frac{(x+x^2)^2}{2!} + \frac{(x+x^2)^3}{3!} + ...$

$$\left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right)$$

$$\therefore e^{x+x^2} = 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6}$$

23. Ans: (a)

Sol: Given

$$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right) \Rightarrow \text{Sin}u = \frac{x^2+y^2}{x+y}$$

$\Rightarrow$ f(u) = sinu is homogeneous with deg,

n = 1

By Euler's theorem

$$x.u_x + y.u_y = n\frac{f(u)}{f'(u)} = 1\frac{\sin u}{\cos u} = \tan\ u$$

24. Ans: (a)

Sol: Given $u = x^{-2}\tan\left(\frac{y}{x}\right) + 3y^3\sin^{-1}\left(\frac{x}{y}\right)$

= f(x, y) + 3 g(x,y)

Where f(x, y) is homogeneous with deg m = –2

and g(x, y) is homogeneous with deg n = 3

$\Rightarrow x^2.\ u_{xx} + 2xy\ u_{xy} + y^2\ u_{yy}$

= m(m–1) f(x,y) + n(n–1) g(x,y)

= –2(–2–1) f(x,y) + 3[3(3–1)g(x,y)]

= 6 [f(x,y) + 3 g(x,y)]

= 6u

25. Ans: (a)

Sol: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$

$= (3x^2 + z^2 + yz)\ e^t + (3y^2 + xz)\ (–\text{Sint}) + (2xz + xy)3t^2$

At t = 0,

$$\frac{du}{dt} = (3(1)+0+0)(1) + [3(1)+0](0) + [0+1](0)$$

= 3

26. Ans: (a)

Sol: $y^2 - x^2 = e^{2u}\left(\sec^2 v - \tan^2 v\right) = e^{2u} = f(u)$

$$\frac{x}{y} = \frac{e^u \tan v}{e^u \sec v} = \sin\ v = g(v)$$

f(u) and g(v) are homogenous functions of degree 2 and 0 respectively

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial v} = 2f \Rightarrow 2e^{2u}x\frac{\partial u}{\partial x} + 2e^{2u}y\frac{\partial u}{\partial y} = 2e^{2u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Similarly

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial r} = 0.g$$

$$\cos v\ x\frac{\partial v}{\partial x} + \cos v\ y\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0$$

$$\therefore \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$$

27. Ans: (a)

Sol: u = x log(xy)

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \left[x.\frac{1}{xy}(y) + \log(xy)\right](1) + \left[x.\frac{1}{xy}.(x).\frac{dy}{dx}\right]$$

Given

$$\underbrace{x^3 + y^3 + 3xy}_{f(x,y)} = 1 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\left[\frac{3x^2 + 3y}{3y^2 + 3x}\right]$$

$$\therefore \frac{du}{dx} = [1 + \log xy] - \frac{x}{y}\left[\frac{x^2 + y}{y^2 + x}\right]$$

28. Ans: (b)

Sol: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 - \frac{y^2}{x} & \frac{2y}{x} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

$$= \frac{2y}{x}\left[1 - \frac{y^2}{x^2} - \left(-\frac{y^2}{x^2}\right)\right]$$

$$= \frac{2y}{x}$$

29. Ans: (c)

Sol: $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3(1 - 2) - 2(-1 - 1) - 1(2 + 1)$$

$$= -2$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{-1}{2}$$

30. Ans: (a, b, c)

Sol: $f(x) = x^3 - 3x^2 - 24x + 100$

$f'(x) = 3x^2 - 6x - 24$

Equating $f'(x)$ to zero for obtaining stationary points

$3x^2 - 6x - 24 = 0 \Rightarrow x^2 - 2x - 8 = 0$

$x = -2, 4$

$f(-3) = (-3)^3 - 3(-3)^2 - 24(-3) + 100 = 118$

$f(-2) = (-2)^3 - 2(-2)^2 - 24(-2) + 100 = 132$

$f(3) = (3)^3 - 3(3)^2 - 24(3) + 100 = 28$

minimum at x = 3

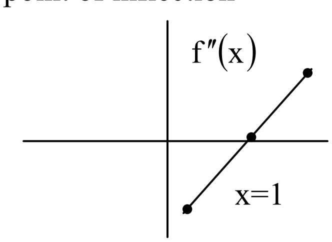
maximum at x = –2

Let us check for point of inflection

$f''(x) = 6x - 6$

$6x - 6 = 0$

$\Rightarrow x = 1$



Clearly $f''(x)$ is changing it's sign about x = 1 ⇒ x = 1 is point of inflection

a, b, c, are correct

31. Ans: (a, d)

Sol: $f'(x) = 6\left(\frac{4}{3}\right)x^{\frac{1}{3}} - 3\left(\frac{1}{3}\right)x^{-2/3}$

$= \frac{8x-1}{x^{2/3}}$

Critical points are $x = 0, \frac{1}{8}$

$f(-1) = 6(-1)^{4/3} - 3(-1)^{1/3} = 9$

$f(0) = 0$

$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{8}\right)^{4/3} - 3\left(\frac{1}{8}\right)^{1/3} = \frac{-9}{8}$

$f(1) = 6 - 3 = 3$

Clearly from the above values absolute minimum is – 9/8, absolute maximum is 9

32. Ans: (c)

Sol: Given $f(x) = x^3 - 9x^2 + 24x + 5$ in [1, 6]

$\Rightarrow f'(x) = 3x^2 - 18x + 24, f''(x) = 6x - 18$

Consider $f'(x) = 0$

$\Rightarrow 3x^2 - 18x + 24 = 0$

$\Rightarrow x = 2, 4$ are the stationary points

At $x = 2$, $f''(2) = -6 < 0$ and

at $x = 4$, $f''(4) = 6 > 0$

$\Rightarrow$ f(x) has a maximum at x = 2 and a minimum at x = 4.

$\therefore$ The maximum value of f(x) in [1, 6] = max{f(1), f(6), f(2)} = max{21, 41, 25}= 41

33. Ans: (c)

Sol: Given $f(x) = (k^2 - 4)x^2 + 6x^3 + 8x^4$

$\Rightarrow f'(x) = 32x^3 + 18x^2 + 2(k^2-4)x$

and $f''(x) = 96x^2 + 36x + 2(k^2-4)$

f(x) has local maxima at x = 0

$\Rightarrow f''(0) < 0$

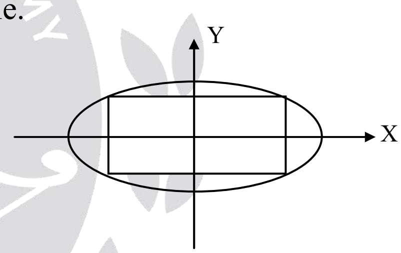
$\Rightarrow 2(k^2-4) < 0$

$\Rightarrow k^2 - 4 < 0$ (or) $(k-2)(k+2) < 0$

$\therefore -2 < k < 2$

34. Ans: 1

Sol: Let 2x & 2y be the length & breadth of the rectangle.



Let A = 2x×2y = 4xy be the area of the rectangle.

Then $A^2 = 4x^2y^2 = x^2(1-x^2) = x^2 - x^4$

Let $f(x) = x^2 - x^4$

Then $f'(x) = 2x - 4x^3$ and $f''(x) = 2 - 12x^2$

For maximum, we have

$$f'(x) = 0$$

$$\Rightarrow 2x(1-2x^2) = 0$$

$$\Rightarrow x = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

Here $f''(0) > 0$, $f''\left(\frac{1}{\sqrt{2}}\right) < 0$

$\therefore$ Area $A = 4xy = 4x \times \frac{\sqrt{1-x^2}}{2}$

$= 2x\sqrt{1-x^2}$

$= 2 \times \frac{1}{\sqrt{2}} \times \sqrt{1-\frac{1}{2}} = 1$

35. Ans: 49

Sol: Let $A = \begin{pmatrix} x & y \\ y & 14-x \end{pmatrix}$

$|A| = x(14-x) - y^2$

For maximum value of $|A|$, $y = 0$

Now, $A = \begin{pmatrix} x & 0 \\ 0 & 14-x \end{pmatrix}$

$\Rightarrow |A| = x(14-x) = 14x - x^2$

Let $f(x) = 14x - x^2$

$\Rightarrow f'(x) = 14 - 2x$ and $f''(x) = -2$

Consider, $f'(x) = 0 \Rightarrow x = 7$

At $x = 7$, $f''(x) = -2 < 0$

$\therefore$ At $x = 7$, the function $f(x)$ has a maximum and is equal to 49.

36. Ans: (a)

Sol: Given $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

Consider $f_x = 4x - 4x^3 = 0$

$\Rightarrow x = 0, 1, -1$

Consider $f_y = -4y + 4y^3 = 0$

$\Rightarrow y = 0, 1, -1$

Now, $r = f_{xy} = 4 - 12x^2$, $s = f_{xy} = 0$

and $t = f_{yy} = -4 + 12y^2$

At (0,1), we have $r > 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has minimum at (0,1)

At (–1, 0), we have $r < 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has a maximum at (–1, 0)

37. Ans: (b and c)

Sol: $f(x, y) = x^2 - y^2$

$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = -2y,\ \frac{\partial^2 f}{\partial x^2} = 2$

$\frac{\partial^2 f}{\partial y^2} = -2,\ \frac{\partial^2 f}{\partial x \partial y} = 0$

Consider $\frac{\partial f}{\partial x} = 0,\ 2x = 0 \Rightarrow x = 0$

$\frac{\partial f}{\partial y} = 0,\quad 2y = 0 \Rightarrow y = 0$

$\therefore$ The stationary point is (0, 0)

At (0,0)

$r = \frac{\partial^2 f}{\partial x^2} = 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = -2$

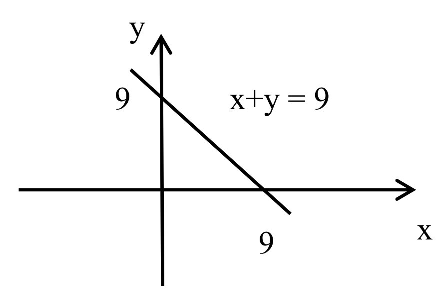
$rt - s^2 = 2(-2) - 0 = -4 < 0$

$\therefore$ f (x,y) has neither maximum nor minimum at (0,0)

Both 'b' and 'c' are correct.

38. Ans: (a and c)

Sol:



The given function can have extreme values either at critical points or at boundary points

$$\frac{\partial f}{\partial x} = 2 - 2x$$

$$\frac{\partial f}{\partial y} = 2 - 2y$$

Equating $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to zero for obtaining critical point.

$2 – 2x = 0 \Rightarrow x = 1$

$2 – 2y = 0 \Rightarrow y = 1$

Critical point (1, 1)

$f(1,1) = 2+2(1)+2(1) – 1 –1 = 4$

Let us check along the boundaries

Along x = 0, $f(0,y) = 2+2y – y^2,\ 0 \le y \le 9$

$f'(0, y) = 2 - 2y \Rightarrow 2 - 2y = 0 \Rightarrow y = 1$

$f(0,0) = 2$

$f(0,1) = 2$

$f(0,9) = 2 + 2(9) – 9^2 = – 61$

Along y = 0, $f(x,0) = 2+2x–x^2,\ 0 \le x \le 9$

$f(0,0) = 2,\ f(0,1) = 2,\ f(0,9) = – 61$

Along y = 9 – x, f(x,9–x)

$= – 61 + 18x – 2x^2,\ 0 \le x \le 9$

$f'(x,9–x) = 18 – 4x = 0 \quad x = \frac{9}{2}$

$$f\left(\frac{9}{2},\frac{9}{2}\right) = 2 + 2\left(\frac{9}{2}\right) + 2\left(\frac{9}{2}\right) - \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 = $$

–20.5

Absolute maximum = 4

Absolute minimum = –61

Correct answer a and c.

39. Ans: (c)

Sol: $\int_{-4}^{7} |x| dx + \int_{-4}^{0} -x\ dx + \int_{0}^{7} x\ dx$

$$= \left[-\frac{x^2}{2}\right]_{-4}^{0} + \left[\frac{x^2}{2}\right]_{0}^{7}$$

$$= 0 - \left[-\frac{16}{2}\right] + \left[\frac{49}{2} - 0\right]$$

$= 8 + 24.5 = 32.5$

40. Ans: (d)

Sol: $\int_{0}^{1.5} x[x^2]\, dx$

$$= \int_{0}^{1} x[x^2]dx + \int_{1}^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx$$

$$= 0 + \int_{0}^{\sqrt{2}} x\ dx + \int_{\sqrt{2}}^{1.5} 2x\ dx = \frac{3}{4}$$

41. Ans: (d)

Sol: $\int_{0}^{\pi} x \underbrace{\sin^8 x \cos^6 x}_{f(x)} dx$

$$\left[\because \int_{0}^{a} x f(x)dx = \frac{a}{2}\int_{0}^{a} f(x)dx \text{ if } f(a-x) = f(x)\right]$$

$$= \frac{\pi}{2}\int_{0}^{\pi} \sin^8 x \cos^6 x\ dx$$

$$= \frac{\pi}{2} \times 2 \times \int_{0}^{\pi/2} \sin^8 x \cos^6 x\ dx$$

$$= \pi\left[\frac{(7.5.3.1)(5.3.1)}{14.12.10.8.6.4.2}\right]\frac{\pi}{2} = \frac{5\pi^2}{4096}$$

42. Ans: (a)

Sol: Given that, $x\sin(\pi x) = \int_0^{x^2} f(t)dt$

Differentiating both sides, we get

$x\cos(\pi x).\pi + \sin(\pi x) = f(x).2x$

Putting x = 4

$4\pi\cos(4\pi) = f(4).8$

$\therefore f(4) = \frac{\pi}{2}$

43. Ans: (b)

Sol: $\lim_{x\to 0}\left[\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x^3}\right]\left(\frac{0}{0}\text{ form}\right)$

Using L′ Hospital Rule,

$= \lim_{x\to 0}\frac{(\sin x)2x-(\sin 0)(0)}{3x^2}\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{2\cos x}{3} = \frac{2}{3}$

44. Ans: 0.785 range 0.78 to 0.79

Sol: $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\cos^4 x+\sin^4 x}dx$

$= 2\int_0^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x(1+\tan^4 x)}dx$

$= \int_0^1 \frac{2t}{1+t^4}dt$ (by putting tan x = t)

$= \frac{\pi}{4} = 0.785$

45. Ans: (a)

Sol: Required area

$= \int_{\frac{1}{2}}^1 \frac{1}{x}dx - \int_{\frac{1}{2}}^1 x^2 dx$

$= \ell nx\Big|_{\frac{1}{2}}^1 - \frac{x^3}{3}\Big|_{\frac{1}{2}}^1$

$= \ell n2 - \left(\frac{1}{3}-\frac{1}{3}.\frac{1}{8}\right)$

$= \ell n2 - \frac{7}{24}$

46. Ans: (d)

Sol: $\int_{-\infty}^0 e^{x+e^x}dx = \int_{-\infty}^0 e^x.e^{e^x}dx$ Put $e^x = t$

$\Rightarrow e^x\,dx = 0$

$= \int_0^1 e^t dt = \left[e^t\right]_0^1 = e-1$

47. Ans: (a)

Sol: $\int_0^\infty \frac{1}{(x^2+4)(x^2+9)}dx = k\pi$

$\Rightarrow \int_0^\infty \frac{1}{5}\left[\frac{1}{x^2+4}-\frac{1}{x^2+9}\right]dx = k\pi$

$\Rightarrow \frac{1}{5}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^\infty = k\pi$

$\Rightarrow \frac{1}{5}\left[\frac{1}{2}.\frac{\pi}{2}-\frac{1}{3}.\frac{\pi}{2}\right] = k\pi$

$\Rightarrow \frac{1}{10}\left[\frac{1}{6}\right] = k \Rightarrow k = \frac{1}{60}$

48. Ans: (c)

Sol: $\int_0^1 x \log x \, dx = \log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx$

$$= \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1 = -\frac{1}{4}$$

49. Ans: (a)

Sol: $f(a) = \int_0^\infty e^{-x} \frac{\sin ax}{x} dx$

Differentiation on both w.r.t 'a'

$$f'(a) = \int_0^\infty e^{-x} \frac{(\cos ax)}{x}(x) dx$$

$$f'(a) = \int_0^\infty e^{-x} \cos ax \, dx$$

$$\therefore \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

$$f'(a) = \frac{e^{-x}}{(-1)^2 + a^2}(-\cos ax + a \sin ax) \Big|_0^\infty$$

$$= \left(0 - \frac{1}{1+a^2}(-1)\right)$$

Integrate on both sides

$f(a) = \tan^{-1}(a) + C$

option (c) is a correct.

50. Ans: (d)

Sol: Let $I = \int_{y=0}^{\infty} \int_{x=0}^{\infty} e^{-x^2} e^{-y^2} dx \, dy$ …… (1)

Put $x^2 = t$

$\Rightarrow x = \sqrt{t}$

$\Rightarrow dx = \frac{1}{2\sqrt{t}} dt$

If x = 0 then t = 0 and if x = ∞ then t = ∞

$$\int_{x=0}^{\infty} e^{-x^2} dx = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}} e^{-t} dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

[using gamma function]

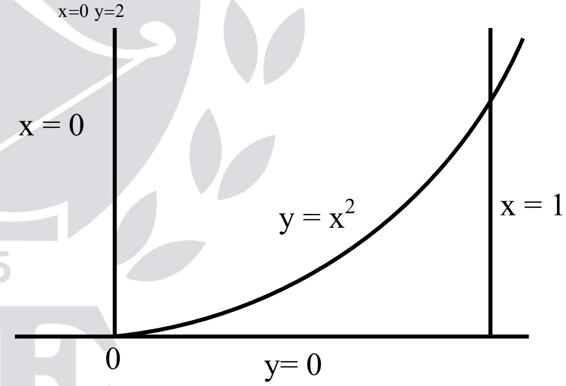
$$= \frac{\sqrt{\pi}}{2} \text{ .... (2)}$$

Using (2), (1) can be expressed as

$$\left(\int_{y=0}^{\alpha} e^{-y^2} dy\right)\left(\int_{x=0}^{\alpha} e^{-x^2} dx\right) = \frac{\sqrt{\pi}}{2} . \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

51. Ans: (c)

Sol: $I_1 = \int_{x=0}^{1} \int_{y=2}^{x^2} xy^2 dy \, dx$



Changing the order of integration

We get

$$\int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} xy^2 dy dx$$

$\Rightarrow I_1 = I_2$

$$I_1 = \int_{x=0}^{1} \frac{xy^3}{3}\Big|_0^{x^2} dx = \int_0^1 \frac{x^7}{3} dx = \frac{x^8}{24}\Big|_0^1 = \frac{1}{24}$$

Only option c is correct

52. Ans: (b)

Sol: $\int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} r\sin\theta\, dr\, d\theta$

$= \int_{0}^{\pi/2} \frac{r^2}{2}\Big|_0^{\cos\theta} \sin\theta\, d\theta$

$= \int_{0}^{\pi/2} \frac{\cos^2\theta}{2} \sin\theta\, d\theta$

$= -\int_{0}^{\pi/2} \frac{\cos^2\theta}{2}(-\sin\theta)\, d\theta$

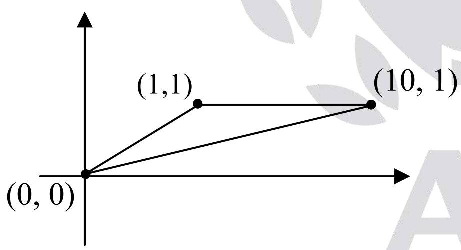
$= -\frac{\cos^3\theta}{2(3)}\Big|_0^{\pi/2}$

$= -\frac{1}{6}[0\ \ -1]$

$= \frac{1}{6}$

53. Ans: (6)

Sol:

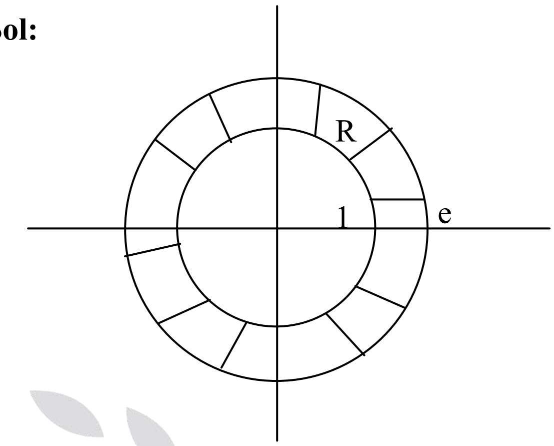


$\int_{y=0}^{1}\int_{x=y}^{10y}\sqrt{xy-y^2}\ dx\, dy$

$\int_{y=0}^{1} \frac{\left(xy-y^2\right)^{3/2}}{\frac{3}{2}.y}\Bigg|_{x=y}^{10y} dy = \int_0^1 \frac{27\,y^3}{\frac{3}{2}y} dy = 18\frac{y^3}{3}\Bigg|_0^1 = 6$

54. Ans: (2π)

Sol:



Let $x = r\cos\theta,\ y = r\sin\theta$

$\iint_R \frac{\ell n\left(x^2+y^2\right)}{x^2+y^2}dxdy = \int_{\theta=0}^{2\pi}\int_{r=1}^{e}\frac{\ell n\left(r^2\right)}{r^2}\, rdrd\theta$

$= \int_{\theta=0}^{2\pi}\int_{r=1}^{e} 2\frac{\ell n(r)}{r}\, drd\theta$

Let $\ell n(r) = t$

$\frac{1}{r}dr = dt$

Limits of t are from 0 to 1

$2\int_{\theta=0}^{2\pi}\int_{t=0}^{1} tdtd\theta = 2\int_{\theta=0}^{2\pi}\frac{t^2}{2}\Big|_0^1 d\theta$

$= \theta\Big|_0^{2\pi} = 2\pi$

55. Ans: 2

Sol: $I = \int_{y=0}^{\pi}\left[\int_{x=y}^{\pi}\frac{\sin x}{x}dx\right]dy$

Changing the order of integration, y varies from 0 to x and x varies from 0 to π, then

$I = \int_{x=0}^{\pi}\left[\int_{y=0}^{y=x}\frac{\sin x}{x}dy\right]dx$

$$= \int_{x=0}^{\pi} \frac{\sin x}{x} . (y)_0^x \, dx$$

$$= \int_{x=0}^{\pi} \frac{\sin x}{x} . x \, dx = -\cos x \Big|_0^{\pi}$$

$$= -(-1-1)$$

$$= 2$$

56. Ans: (1)

Sol: $$\int_{y=0}^{1} \int_{x=0}^{y} \int_{z=0}^{1+x+y} dz dx dy = \int_{y=0}^{1} \int_{x=0}^{y} (1+x+y) \, dx \, dy$$

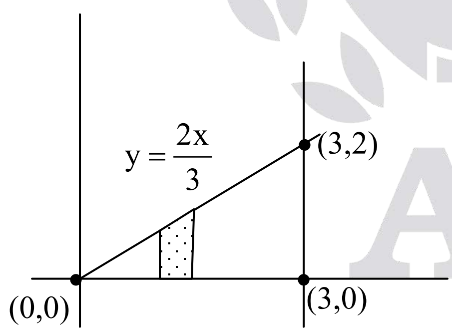
$$= \int_{y=0}^{1} \left(x + \frac{x^2}{2} + xy \right) \Big|_0^y dy$$

$$= \int_0^1 \left(y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \frac{y^2}{2} + \frac{3}{2} \frac{y^3}{3} \Big|_0^1 = 1$$

57. Ans: 10

Sol:



$$\text{Volume} = \int \int z dx dy$$

$$= \int_{x=0}^{3} \int_{y=0}^{\frac{2}{3}x} (6 - x - y) dx dy$$

$$= \int_0^3 \left[6y - xy - \frac{y^2}{2} \right]_0^{\frac{2x}{3}} dx$$

$$= \int_0^3 \left[4x - \frac{2}{3} x^2 - \frac{2}{9} x^2 \right] dx$$

$$= \left[2x^2 - \frac{2x^3}{9} - \frac{2x^3}{27} \right]_0^3$$

$$= [18 - 6 - 2]$$

$$= 10$$

58. Ans: (d)

Sol: $y = \log \sec x$, $\frac{dy}{dx} = \tan x$

$$\text{Length of curve } = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx$$

$$= \int_0^{\pi/4} \sec x \, dx = \log (\sec x + \tan x) \Big|_0^{\pi/4}$$

$$= \log \left(\sqrt{2} + 1 \right)$$

59. Ans: 25.12

Sol: $$\text{Volume} = \int_0^4 \pi y^2 \, dx$$

$$= \int_0^4 \pi x \, dx$$

$$= 8\pi \text{ cubic units}$$

60. Ans: 1.88

Sol: $$\text{Volume} = \int_0^1 \pi \, x^2 \, dy$$

$$= \pi \int_0^1 y^{\frac{2}{3}} \, dy \approx 1.88$$

61. Ans: (a)

Sol: The unit normal vector to the sphere $x^2+y^2+z^2=r^2$ at a point $(x_1,\ y_1,\ z_1)$ is given by $\dfrac{x_1\,i+y_1 j+z_1 k}{r}$

$$=\frac{4\,i+4j+4k}{\sqrt{48}}=\frac{i+j+k}{\sqrt{3}}$$

62. Ans: (1)

Sol: $\nabla f=\left(i\dfrac{\partial}{\partial x}+j\dfrac{\partial}{\partial y}\right)xe^y=e^y i+xe^y j$

$$\nabla f\Big|_{(2,0)}=i+2j$$

Vector along the direction of the straight line segment from (2,0) to $\left(\dfrac{1}{2},2\right)$ is given by $\bar{a}=\left(\dfrac{1}{2}-2\right)i+(2-0)\,j$

$$\bar{a}=-\frac{3}{2}i+2j$$

Unit vector along $\bar{a}$ is $\dfrac{-\frac{3}{2}i+2j}{\sqrt{\frac{9}{4}+4}}$

The value of the directional derivative is

$$\nabla f\bullet\frac{\bar{a}}{|\bar{a}|}=\frac{(i+2j).\left(-\frac{3}{2}i+2j\right)}{5/2}$$

$$=\frac{\frac{-3}{2}+4}{5/2}=\frac{5/2}{5/2}=1$$

Correct answer = 1

63. Ans: (c)

Sol: $\phi(xy)=e^{xy}\sin(x+y)$

$$\nabla\phi=i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{dz}$$

$\nabla\phi$ = i[e^{xy} (y) sin (1+y) + e^{xy}(b) (x + y) + j[e^{xy}(x) sin (x + y) + e^{xy}(x) (x+y)]+k[0]]

$$(\nabla\theta)_{\left(0,\ \frac{\pi}{2}\right)}$$

$$=i\left[e^0\frac{\pi}{2}\sin\frac{\pi}{2}+e^0 cs\frac{\pi}{2}+j\left[e^0 0\sin\frac{\pi}{2}+e^o\cos\pi\right]\right]$$

$$\nabla\phi=i\left[\frac{\pi}{2}\right]+j[0]+k[0]$$

Required direction = $\nabla\phi=\dfrac{\pi}{2}i$

64. Ans: (0)

Sol: Given $\nabla\times\bar{F}=\bar{0}$

$$\nabla\times\bar{F}=\begin{vmatrix} i & j & k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 4y-c_1 z & 4x+2z & 2y+z\end{vmatrix}$$

$$=i(2-2)-j(-c_1+0)+k(4-4)=0i+0j+0k$$

$$\Rightarrow c_1=0$$

65. Ans: (b)

Sol: $\nabla.\bar{f}=2x+2y+2z\neq 0$

Not divergence free

$$\nabla\times\bar{f}=\begin{vmatrix} i & j & k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2+yz & y^2+xz & z^2+xy\end{vmatrix}$$

$$=i(x-x)-j\,(y-y)+k(z-z)$$

$$=0i+0j+0k$$

$$\nabla\times\bar{f}=\bar{0}\Rightarrow\text{Curl free}$$

66. Ans: (a)

Sol: $\nabla[f(r)] = f'(r)\frac{\vec{r}}{r}$

$\nabla(\sin r) = (\cos r)\frac{\vec{r}}{r}$

67. Ans: (c)

Sol: div $[e^r.\vec{r}]$

$\nabla.(\phi\overline{A}) = (\nabla\phi).\overline{A} + \phi(\nabla.\overline{A})$ (Identity)

$\nabla.(e^r\ \bar{r}) = (\nabla e^r).\bar{r} + e^r(\nabla.\bar{r})$

$= e^r\frac{\vec{r}}{r}.\vec{r} + e^r(3)$

$= e^r(3) + e^r\frac{r^2}{r}$

$\nabla(e^2.\vec{r}) = e^r(3+r)$

68. Ans: (d)

Sol: $\text{curl}(r^4\vec{r}) = ?$

curl $[\phi\,\overline{F}]$

$= \phi\ \text{curl}\,\overline{F} + (\text{grad}\,\phi)\times\overline{F}$ (Identity)

$= \text{curl}\ (r^4\vec{r})$

$= r^4(\text{curl}\,\bar{r}) + \text{grad}(r^4)\times\bar{r}$

$= r^4\cdot 0 + 4r^3\frac{\vec{r}}{r}\times\vec{r}$

$= \overline{0} + \overline{0} = \overline{0}$

69. Ans: (1/2)

Sol: Along C

$x = r\cos\theta,\ y = r\sin\theta$

Where $r = 1, \theta = 0$ to $\frac{\pi}{4}$

$x = \cos\theta,\ y = r\sin\theta$

$dx = -\sin\theta\ dy = \cos\theta$

$\int_c F(r).dr = \int_c(-x\ i + yi).(dx\ i + dy\ j)$

$= \int_{\theta=0}^{\pi/4}\cos\theta\sin\theta d\theta + \sin\theta\cos\theta d\theta$

$= \int_0^{\pi/4}\sin 2\theta d\theta = -\frac{\cos 2\theta}{2}\Big|_0^{\pi/4} = \frac{1}{2}$

70. Ans: 139

Sol: $\int_C$ (grad f). dr $= \int_C df = (f)_{(-3,-3,2)}^{(2,6,-1)}$

$= [2x^3 + 3y^2 + 4z]_{(-3,-3,2)}^{(2,6,-1)}$

$= 139$

71. Ans: 202

Sol: Given $\overline{F} = (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$

$$\text{Curl }\overline{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$= \bar{i}[0-0] - \bar{j}[3z^2 - 3z^2] + \bar{k}[2x - 2x] = \overline{0}$

$\Rightarrow \overline{F}$ is irrotational

$\Rightarrow$ Work done by $\overline{F}$ is independent of path of curve

$\Rightarrow \overline{F} = \nabla\phi$

where $\phi(x, y, z)$ is scalar potential

$\Rightarrow$

$(2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\Rightarrow d\phi = (2xy + z^3)\, dx + x^2\, dy + 3xz^2\, dz$

$\Rightarrow \int d\phi = \int (2xy + z^3) dx + x^2\, dy + 3xz^2\, dz$

$\Rightarrow \int d\phi = \int d(x^2y + xz^3)$

$\Rightarrow \phi(x, y, z) = x^2y + xz^3$

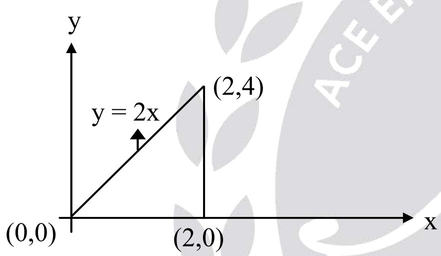
$\therefore$ Workdone $= \int_C \bar{F}.d\bar{r} = \phi(3,1,4) - \phi(1, -2,1)$

$= [9(1)+3(64)] - [1(-2) + 1(1)]$

$= 202$

72. Ans: (d)

Sol:



By Green's Theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

where $M = x + y$, $N = x^2$ and

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 1$

The given integral $= \int_{x=0}^{2} \int_{y=0}^{2x} (2x - 1)\, dy\, dx$

$= \int_0^2 [2xy - y]_0^{2x}\, dx$

$= \int_0^2 [4x^2 - 2x] dx$

$= \frac{20}{3}$

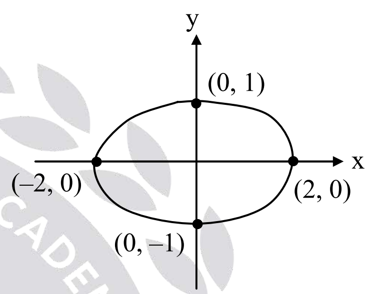
73. Ans: (c)

Sol: By Green's Theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here, $M = 2x - y$ and $N = x + 3y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$



The given integral $= \iint_R 2\, dx\, dy$

$= 2$ (Area of the given ellipse)

$= 2\,(\pi.\,2.\,1) = 4\pi$

74. Ans: 0

Sol: Given $\bar{A} = \nabla\phi$

$\Rightarrow$ Curl $\bar{A} = \bar{0}$

$\Rightarrow$ $\bar{A}$ is Irrotational

$\therefore$ Line integral of Irrotational vector function along a closed curve is zero

i.e. $\int_C \bar{A}.d\bar{r} = 0$, where $C : \frac{x^2}{4} + \frac{y^2}{9} = 1$ is a closed curve.

75. Ans: (d)

Sol: The given surface is a closed surface.

$$\oiint_s \overline{F}\,\overline{N}\,ds = \iiint_V \nabla \bullet \overline{F}\,dv$$

$$\nabla \bullet \overline{F} = 3$$

$$\iiint_V 3dv = 3\iiint dv$$

= 3 (Volume of Cylinder)

= 3 (π (4²) (2))

= 96 π

76. Ans: 264

Sol: Using Gauss–Divergence Theorem,

$$\iint_S xy\,dy\,dz + yz\,dzdx + zx\,dx\,dy = \iiint_V \text{div}\,\overline{F}\,dv$$

$$= \iiint_V (y+z+x)\,dv$$

$$= \int_{x=0}^{4}\int_{y=0}^{3}\int_{z=0}^{4}(x+y+z)\,dzdydx$$

$$= \int_{x=0}^{4}\int_{y=0}^{3}[4x+4y+8]\,dy\,dz$$

$$= \int_{x=0}^{4}[12x+18+24]\,dx = 264$$

77. Ans: 0

Sol: By Stokes' theorem, we have

$$\int_C \overline{f}.d\overline{r} = \iint_S (\nabla\times\overline{f}).\overline{n}\,ds$$

Here, $\nabla \times \overline{f}$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+yz) & (y^2+xz) & (z^2+xy) \end{vmatrix} = \overline{0}$$

$\Rightarrow \overline{f}$ is an irrotational vector

$$\therefore \int \overline{f}.d\overline{r} = 0$$

78. Ans: (d)

Sol: $$\text{Curl } \overline{f} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \overline{i}[-2yz+2yz]-\overline{j}[0]+\overline{k}[0+1]$$

$\Rightarrow \text{Curl } \overline{f} = \overline{k}$

Using Stokes' theorem,

$$\int_C \overline{f}.d\overline{r} = \int_S \text{curl } \overline{f}.\overline{N}\,ds = \int_S \overline{k}.\overline{N}\,ds$$

Let R be the protection of s on xy plane

$$\Rightarrow \int_S \overline{k}.\overline{N}\,ds = \iint_R \overline{k}.\overline{N}\frac{dxdy}{|\overline{N}.\overline{k}|} = \iint_R 1\,dx\,dy$$

= Area of Region

$= \pi r^2 = \pi(1)^2 = \pi$

79. Ans: (d)

Sol: The function f(x) = x^2 cos(x) is even function

∴ The fourier series of f(x) contain only cosine terms.

The coefficient of sin 2x = 0

80. Ans: –6.58

Sol: Let $f(x) = x - x^2$, $-\pi \le x \le \pi$

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{\pi}xdx - \int_{-\pi}^{\pi}x^2dx\right\}$$

$$= \frac{1}{\pi}\left[0 - 2\int_0^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\left[-2\left(\frac{x^3}{3}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi^3}{3}\right]$$

$$= -\frac{2}{3}\pi^2 = -6.58$$

81. Ans: (a)

Sol: $b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi}{\ell}x dx$

$$b_5 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{5\pi}{\pi}x\,dx$$

$$b_5 = \frac{2}{\pi}\int_0^{\pi} \pi \sin 5x\,dx$$

$\because f(x)\sin 5x$ is an even function

$$b_5 = \frac{-2}{5}\cos 5x\Big|_0^{\pi} = \frac{4}{5}$$

82. Ans: (b)

Sol: $f(x) = \sum_{n=1}^{\infty} \frac{k}{\pi}\left[\frac{2-2(-1)^n}{n}\right]\sin(nx)$

At $x = \frac{\pi}{2}$

$$k = \frac{k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right]$$

$$\therefore\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\infty = \frac{\pi}{4}$$

83. Ans: (c)

Sol: $f(0) = \frac{\pi}{4} + \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$ .... (1)

The convergence of f(x) at x = 0 is valid if

$$f(0) = \frac{f(0^-) + f(0^+)}{2} = \frac{\pi}{2}$$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

84. Ans: (c)

Sol: $f(x) = \pi x - x^2$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\left[(\pi x - x^2).\sin x\right]dx$$

$$\frac{2}{\pi}\left[\begin{matrix}(\pi x - x^2)(-\cos x) - (\pi - 2x)(-\sin x)\\ +(-2)\cos x\end{matrix}\right]_0^{\pi} = \frac{8}{\pi}$$

85. Ans: (b)

Sol: $f(x) = (x-1)^2$

The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x-1)^2 \cos(n\pi x)\,dx$$

$$\frac{2}{\pi}\left[(x-1)^2.\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1).\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}$$

Chapter 3 Probability & Statistics


C R Rao

01. Ans: (10)

Sol: Total cases = 6^4 = 1296

Favourable cases for sum is 22 = 10

i. (6, 6, 6, 4) = $\frac{4!}{3!} = 4$ cases

ii. (6, 6, 5, 5) = $\frac{4!}{2!\,2!} = 6$ cases

Required probability $= \frac{10}{1296}$

$\therefore$ x = 10

02. Ans: (c)

Sol: Given : P(j) α j

P(j) = kj, j = 1, 2, 3, 4, 5, 6

X = j	1	2	3	4	5	6
P(X = j)	k	2k	3k	4k	5k	6k

$\Sigma P(X = j) = 1$

$21\,k = 1$

$k = \frac{1}{21}$

$\therefore$ P(odd number of dots)

$= P(X = 1) + P(X = 3) + P(X = 5)$

$= k + 3k + 5k = 9\,k$

$= \frac{9}{21}$

$= \frac{3}{7}$

03. Ans: (c)

Sol: Total cases for selection of 3 integers out of 20 = $20C_3$ = 1140

Let A = Product is even

A^C = Product is odd

Probability of three integers is odd only when all are odd integers. Out of first 20 integers, we have 10 odd integers and 10 even integers.

Favourables of $A^C = 10C_3 = 120$

$P(A^C) = \frac{120}{1140} = \frac{2}{19}$

$P(A) = 1 - P(A^C)$

$= 1 - \frac{2}{19}$

$= \frac{17}{19}$

04. Ans: (0.027)

Sol: Let the four digit number be

↓	↓	↓	↓
Thousand place	Hundred place	Tens place	Units place

Total cases = 9 × 10 × 10 × 10 = 9000

Calyampudi Radhakrishna Rao, FRS known as C R Rao (born 10 September 1920) is an Indian-born, mathematician and statistician. The American Statistical Association has described him as "a living legend whose work has influenced not just statistics, but has had far reaching implications for fields as varied as economics, genetics, anthropology, geology, national planning, demography, biometry, and medicine.

Favourable cases = $3C_2 \times 9 \times 9 = 243$

$\therefore$ Required Probability $= \frac{243}{9000}$

$= 0.027$

05. Ans: (a)

Sol: Total cases for arranging six boys and six girls in a row = 12!

Assume six girls as one unit.

We have a total of six boys + 1 unit = 7

They can be arranged among themselves in 7! ways and six girls can be arranged among themselves in 6! ways.

Favourable cases = 6! × 7!

$\therefore$ Required probability $= \frac{6! \times 7!}{12!}$

06. Ans: (c)

Sol: P = P(IB $\cap$ IIB $\cap$ IIIR) + P (IR $\cap$ IIB $\cap$ IIIR) + P(IB $\cap$ IIR $\cap$ IIIR)

$$= \left(\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}\right)+\left(\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}\right)+\left(\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\right)$$

$$= \frac{12}{60}+\frac{6}{60}+\frac{6}{60}$$

$$= \frac{24}{60}$$

$$= \frac{2}{5}$$

07. Ans: 0.02

Sol: Candidates attend interview with 3 pens means they attend interview with 3 pens of same colour (or) 2 pens of same colour and one different colour (or) 3 pens of different colours.

Total number of cases

$= 4C_1 + 2(4C_2) + 4C_3 = 20$

Favourable number of cases for 3 pens having same colour = $4C_1 = 4$

Required Probability = $\frac{4}{20} = 0.2$

08. Ans: (d)

Sol: Number of ways, we can choose R = C(n, 3)

We have to count number of ways we can choose R, so that median (R) = median (S).

Each such set R contains median S, one of the $\left(\frac{n-1}{2}\right)$ elements of S less than median (S), and one of the $\left(\frac{n-1}{2}\right)$ elements of S greater than median (S).

So, there are $\left(\frac{n-1}{2}\right)^2$ choices for R.

Required probability = $\frac{\left(\frac{n-1}{2}\right)^2}{C(n,3)}$

$= \frac{3(n-1)}{2n(n-2)}$

09. Ans: 0.66

Sol: Let N = the number of families

Total No. of children $= \left(\frac{N}{2}\times 1\right)+\left(\frac{N}{2}\times 2\right)$

$= \frac{3N}{2}$

$\therefore$ Required Probability $= \frac{\left(\frac{N}{2}\times 2\right)}{\frac{3N}{2}}$

$= \frac{2}{3} = 0.66$

10. Ans: (c)

Sol: Given: $P(A)=\frac{1}{2}P(B),\ P(A^C\cap B^C)=\frac{1}{3}$

$A\cup B^C = A\cup(A^C\cap B^C)$

$P(A\cup B^C) = P(A)+P(A^C\cap B^C)$

$= \frac{1}{2}+\frac{1}{3}$

$= \frac{5}{6}$

$P(A^C\cup B^C) = P(A^C) + P(B^C) - P(A^C\cap B^C)$

$= \frac{1}{2}+\frac{1}{2}-\frac{1}{3}$

$= \frac{2}{3}$

11. Ans: (0.867)

Sol: Let A = Selected number divisible by 12

B = Selected number divisible by 15

$A\cap B$ = Selected number divisible by 60

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$= \frac{83}{1000}+\frac{66}{1000}-\frac{16}{1000}$

$P(A\cup B) = \frac{133}{1000} = 0.133$

$P(A\cup B)^C = 1 - P(A\cup B)$

$= 1 - 0.133$

$= 0.867$

12. Ans: (0.67)

Sol: Given : P(E) = 0.4, P(F) = 0.3

$P(F\mid E) = 3\,P(F\mid E^C)$

$\frac{P(E\cap F)}{P(E)} = 3\frac{P(E^C\cap F)}{P(E^C)}$

$\frac{P(E\cap F)}{0.4} = 3\frac{P(E^C\cap F)}{0.6}$

$P(E\cap F) = 2\{P(F) - P(E\cap F)\}$

$[\because E^C\cap F = F - E\cap F]$

$P(E\cap F) = 2P(F) - 2P(E\cap F)$

$3P(E\cap F) = 2P(F)$

$\frac{P(E\cap F)}{P(F)} = \frac{2}{3}$

$P(E\mid F) = 0.67$

13. Ans: (b)

Sol: Let A = First toss produces head

B = Second toss produces head

$$P(A)=\left(\frac{1}{3}\times\frac{1}{2}\right)+\left(\frac{1}{3}\times\frac{2}{2}\right)=\frac{3}{6}=\frac{1}{2}$$

$$P(B\,|\,A)=\frac{P(A\cap B)}{P(A)}$$

$$=\frac{\left(\frac{1}{3}\times\frac{1}{2}\times\frac{1}{2}\times\frac{2}{2}\right)+\left(\frac{1}{3}\times\frac{2}{2}\times\frac{1}{2}\times\frac{1}{2}\right)}{\frac{1}{2}}$$

$$=\frac{1}{3}$$

14. Ans: (a)

Sol: Let A = Number appearing are different

B = Sum is even

$A \cap B$ = Numbers appearing are different and sum is even

P(A) $=\frac{30}{36}$ [Other than (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) cases]

Sum of two integers is even only when

i. First integer is odd and second integer is odd.

ii. First integer is even and second integer is even.

$A \cap B$ = {(1, 3), (1, 5), (3, 1), (5, 1), (3, 5), (5, 3), (2,4),(2,6),(4, 2), (6, 2), (4, 6), (6, 4)}

$$P(A\cap B)=\frac{12}{36}$$

$$P(B\,|\,A)=\frac{P(A\cap B)}{P(A)}$$

$$=\frac{\frac{12}{36}}{\frac{30}{36}}$$

$$=\frac{2}{5}$$

15. Ans: (d)

Sol: The required probability

$$=\frac{1}{6}+\left(\frac{5}{6}\right)^2.\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^4.\left(\frac{1}{6}\right)+\ldots\ldots$$

$$=\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^2\right]^{-1}$$

$$=\frac{1}{6}.\left(\frac{36}{11}\right)=\frac{6}{11}$$

16. Ans: (0.248)

Sol: Total number of games played = 4

P(A gets atleast 6 points)

= P(6 points) + P(7 points) + P(8 points)

= P(3w, 1L) + P(2w, 2D) + P(3w, 1D) + P(4w)

$= (0.6)^3 (0.1) + (0.6)^2 (0.3)^2 + (0.6)^3 (0.3) + (0.6)^4$

= 0.2484

17. Ans: (a)

Sol: Given : $P(A) = P(B) = P(C) = \frac{1}{3}$

$$P(A \cap B \cap C) = \frac{1}{4}$$

A, B, C are pair wise independent events

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$= P(A) + P(B) + P(C) - P(A) P(B) - P(B) P(C) - P(A) P(C) + P(A \cap B \cap C)$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \left(\frac{1}{3} \times \frac{1}{3}\right) - \left(\frac{1}{3} \times \frac{1}{3}\right) - \left(\frac{1}{3} \times \frac{1}{3}\right) + \frac{1}{4}$$

$$= 1 - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{2}{3} + \frac{1}{4}$$

$$= \frac{11}{12}$$

18. Ans: (b)

Sol: Exactly two heads out of four tosses.

i. $C_1 = 2H$, $C_2 = 2T$

ii. $C_1 = 2T$, $C_1 = 2H$

iii. $C_1 = HT$, $C_2 = HT$

iv. $C_1 = TH$, $C_2 = TH$

v. $C_1 = TH$, $C_2 = HT$

vi. $C_1 = HT$, $C_2 = TH$

$$P = \left(\frac{2}{3} \times \frac{2}{3} \times \frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{3} \times \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4}\right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{3}{4} \times \frac{1}{4}\right)$$
$$+ \left(\frac{1}{3} \times \frac{2}{3} \times \frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{3} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{4}\right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{4} \times \frac{3}{4}\right)$$
$$= \frac{4}{144} + \frac{9}{144} + \frac{6}{144} + \frac{6}{144} + \frac{6}{144} + \frac{6}{144}$$
$$= \frac{37}{144}$$

19. Ans: (0.04)

Sol: Let H_T = Transmitted signal is H

H_R = Received signal is H

L_T = Transmitted signal is L

L_R = Received signal is L

$P(H_R) = P(H_T)\, P(H_R \mid H_T) + P(L_T)\, P(H_R \mid L_T)$

$= 0.1 \times 0.3 + 0.9 \times 0.8$

$= 0.75$

$$P(H_T \mid H_R) = \frac{P(H_T \cap H_R)}{P(H_R)}$$

$$= \frac{P(H_T) P(H_R \mid H_T)}{P(H_R)}$$

$$= \frac{0.1 \times 0.3}{0.75}$$

$= 0.04$

20. Ans: (d)

Sol: Let B_1 = Transferred ball from bag A to bag B is red

B_2 = Transferred ball from bag A to bag B is white

R = Selecting a red ball from bag B

W = Selecting a white ball from bag B

$$P(B_1) = \frac{3}{10}, \quad P(B_2) = \frac{7}{10}$$

$$P(R \mid B_1) = \frac{6}{10}, \quad P(W \mid B_2) = \frac{5}{10}$$

P(R) = P(B_1) P(R | B_1) + P(B_2) P(R | B_2)

$$P(R) = \left(\frac{3}{10} \times \frac{6}{10}\right) + \left(\frac{7}{10} \times \frac{5}{10}\right) = \frac{53}{100}$$

$$P(W \mid B_1) = \frac{4}{10}, \quad P(W \mid B_2) = \frac{5}{10}$$

P(W) = P(B_1) P(W | B_1) + P(B_2) P(W | B_2)

$$= \frac{3}{10} \times \frac{4}{10} + \frac{7}{10} \times \frac{5}{10}$$

$$= \frac{47}{100}$$

$$P(B_1 \mid W) = \frac{P(B_1 \cap W)}{P(W)}$$

$$= \frac{P(B_1)P(W \mid B_1)}{P(W)}$$

$$= \frac{\frac{3}{10} \times \frac{4}{10}}{\frac{47}{100}} = \frac{12}{47}$$

$$P(B_2 \mid R) = \frac{P(B_2 \cap R)}{P(R)}$$

$$= \frac{P(B_2)P(R \mid B_2)}{P(R)}$$

$$= \frac{\frac{7}{10} \times \frac{5}{10}}{\frac{53}{100}} = \frac{35}{53}$$

21. Ans: (a)

Sol: Let B_1 = selection of urn I

B_2 = selection of urn II

B_3 = selection of urn III

A = selection of whit ball

P(B_1) = P(2 heads) =

$$= 0.2 \times 0.3 = \frac{6}{100}$$

P(B_2) = P(HT) + P(TH)

$$= 0.2 \times 0.7 + 0.8 \times 0.3 = \frac{38}{100}$$

$$P(B_3) = P(\text{2 tails}) = 0.8 \times 0.7 = \frac{56}{100}$$

$$P(A \mid B_1) = \frac{2}{4}, \quad P(A \mid B_2) = \frac{1}{4},$$

$$P(A \mid B_3) = \frac{3}{4}$$

P(A) = P(B_1) P(A | B_1) + P(B_2) P(A | B_2) + P(B_3) P(A | B_3)

$$= \frac{6}{100} \times \frac{2}{4} + \frac{38}{100} \times \frac{1}{4} + \frac{56}{100} \times \frac{3}{4}$$

$$= \frac{12}{400} + \frac{38}{400} + \frac{168}{400}$$

$$= \frac{218}{400}$$

$$P(B_1 \mid A) = \frac{P(A \cap B_1)}{P(A)}$$

$$= \frac{P(B_1)P(A \mid B_1)}{P(A)}$$

$$= \frac{\frac{6}{100} \times \frac{2}{4}}{\frac{218}{400}}$$

$$= \frac{12}{218}$$

$$= \frac{6}{109}$$

22. Ans: (c)

Sol: Let x = number of times a coin is tossed until first head appears

	1	2	3	……	…….
P(X)	P	qP	q^2P	……	……

$E(X) = P + 2qP + 3q^2P + \ldots\ldots\ldots\ldots$

$E(X) = P\{1+2q + 3q^2 + \ldots\ldots\ldots\}$

$= P\{1 - q\}^{-2}$

$\{\because (1 - x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots\ldots\}$

$= P(P)^{-2}$

$= P^{-1}$

$= \frac{1}{P}$

23. Ans: (a, b, c)

Sol:

X	1	2	3
P(X)	$\frac{11}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

$P(X = 1) =$

$$\frac{1}{3}\times\frac{3}{4}\times\frac{1}{2}+\frac{2}{3}\times\frac{1}{4}\times\frac{1}{2}+\frac{2}{3}\times\frac{3}{4}\times\frac{1}{2}=\frac{11}{24}$$

$P(X = 2) =$

$$\frac{1}{3}\times\frac{1}{4}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}+\frac{2}{3}\times\frac{1}{4}\times\frac{1}{2}=\frac{6}{24}$$

$$P(X = 3) = \frac{1}{3}\times\frac{1}{4}\times\frac{1}{2}=\frac{1}{24}$$

$$E(X) = \frac{11}{24}+\frac{12}{24}+\frac{3}{24}=\frac{26}{24}=\frac{13}{12}$$

$$E(X^2) = \frac{11}{24}+\frac{24}{24}+\frac{9}{24}=\frac{44}{24}=\frac{11}{6}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{11}{6}-\left(\frac{13}{12}\right)^2$$

$$= \frac{11}{6}-\frac{169}{144}$$

$$= \frac{264-169}{144}$$

$$= \frac{95}{144}$$

$$P(X = odd) = P(X = 1) + P(X = 3)$$

$$= \frac{11}{24}+\frac{1}{24}$$

$$= \frac{1}{2}$$

24. Ans: (b)

Sol: Let x = Amount (in rupees) to be won by player

X	–1	–4	–9	–16	–25	36
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = -\frac{1}{6}-\frac{4}{6}-\frac{9}{6}-\frac{16}{6}-\frac{25}{6}+\frac{36}{6}$$

$$E(X) = -\frac{19}{6}$$

25. Ans: (a, c, d)

Sol: Given : $f(x)=\begin{cases} ax+bx^2 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 (ax+bx^2)dx = 1$$

$$\left(\frac{ax^2}{2}+\frac{bx^3}{3}\right)_0^1 = 1$$

$$\frac{a}{2}+\frac{b}{3} = 1$$

3a + 2b = 6 …………………..(1)

E(x) = 0.6

$$\int_0^1 x(ax+bx^2)dx = 0.6$$

$$\int_0^1 (ax^2+bx^3)dx = 0.6$$

$$\left(\frac{ax^3}{3}+\frac{bx^4}{4}\right)_0^1 = 0.6$$

$$\frac{a}{3}+\frac{b}{4} = 0.6$$

4a + 3b = 7.2 …………………(2)

On solving equations (1) and (2), we get

a = 3.6 b = –2.4

$$E(X^2) = \int_0^1 x^2(ax+bx^2)dx$$

$$= \int_0^1 (ax^3+bx^4)dx$$

$$= \left(\frac{ax^4}{4}+\frac{bx^5}{5}\right)_0^1$$

$$= \frac{a}{4}+\frac{b}{5}$$

$$= \frac{3.6}{4}-\frac{2.4}{5}$$

= 0.9 – 0.48

= 0.42

var(X) = $E(X^2) - (E(X))^2$

= $0.42 - (0.6)^2$

= 0.42 – 0.36

= 0.06

$$P\left(X<\frac{1}{2}\right) = \int_0^{\frac{1}{2}} f(x)dx$$

$$= \int_0^{\frac{1}{2}} (ax + bx^2) dx$$

$$= \left(\frac{ax^2}{2} + \frac{bx^3}{3}\right)_0^{1/2}$$

$$= \frac{a}{8} + \frac{b}{24}$$

$$= \frac{3.6}{8} - \frac{2.4}{24}$$

$$= 0.45 - 0.1$$

$$= 0.35$$

26. Ans: (10)

Sol: $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$E(X) = E\left(e^{\frac{3x}{4}} + 6\right)$$

$$= E\left(e^{\frac{3x}{4}}\right) + 6$$

$$= \int_0^{\infty} e^{\frac{3x}{4}} f(x) dx + 6$$

$$= \int_0^{\infty} e^{-x}.e^{\frac{3x}{4}} dx + 6$$

$$= \int_0^{\infty} e^{-\frac{1}{4}x} dx + 6$$

$$= \left(\frac{e^{-\frac{1}{4}x}}{\frac{-1}{4}}\right)_0^{\infty} + 6$$

$$= 4 + 6$$

$$= 10$$

27. Ans: (a)

Sol: $n = 5,\quad P = \frac{2}{3},\qquad q = \frac{1}{3}$

P(even number of heads)

= P(0 heads) + P(2 heads) + P(4 heads)

$= q^n + nC_2\, P^2\, q^{n-2} + nC_4\, P_4\, q^{n-4}$

$$= \left(\frac{1}{3}\right)^5 + 5C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 + 5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)$$

$$= \frac{1}{243} + \frac{40}{243} + \frac{80}{243}$$

$$= \frac{121}{243}$$

28. Ans: 0.042

Sol: $n = 10,\quad p = \frac{1}{2}\ \&\ q = \frac{1}{2}$

Let x = number of heads

$P(x = 4) = nC_x\, p^x\, q^{n-x}$

$$= 10C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^6$$

$$= \frac{10C_4}{(2)^{10}} = \frac{210}{1024} = 0.205$$

$\therefore$ Required probability $= (0.205)^2 = 0.042$

29. Ans: (d)

Sol: $P = \frac{5}{6},\qquad q = \frac{1}{6}$

Let A = Total of 7 successful attempts

B = Last attempts

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{5}{6} \times \frac{5}{6} \times 8C_5 \left(\frac{5}{6}\right)^5 \left(\frac{1}{6}\right)^3}{10C_7 \left(\frac{5}{6}\right)^7 \left(\frac{1}{6}\right)^3}$$

$$= \frac{8C_5}{10C_7} \times \frac{\left(\frac{(5)^7}{6^{10}}\right)}{\left(\frac{5^7}{6^{10}}\right)} = \frac{7}{15}$$

30. Ans: 0.865 range 0.86 to 0.87

Sol: Let X = number of cashew nuts per biscuit.

We can use Poisson distribution with mean

$$= \lambda = \frac{2000}{1000} = 2$$

$$P(X = k) = \frac{e^{-\lambda}.\lambda^k}{\angle k} \quad (k = 0,1,2...)$$

Probability that the biscuit contains no cashew nut = P (X = 0)

$$= e^{-\lambda} = e^{-2} = 0.135$$

Required probability = 1 – 0.135 = 0.865

31. Ans: 0.27

Sol: Given that λ = 240 veh/h

$$= \frac{240}{60} \text{veh/min} = 4 \text{ veh/min}$$

$$= 2 \text{ veh/30 sec}$$

$\therefore$ The required probability = P(X = 1)

$$= \lambda\, e^{-\lambda} = 2\, e^{-2}$$

$$= 0.27$$

32. Ans: (12)

Sol: $P(X) = \frac{\lambda^x e^{-\lambda}}{x\,!}$, x = 0, 1, 2, ………..

$P(y) = \frac{\lambda^y e^{-\lambda}}{y\,!}$, y = 0, 1, 2, ………..

$$P(X = 1) = P(X = 2)$$

$$\lambda e^{-\lambda} = \frac{\lambda^2 e^{-\lambda}}{2}$$

$$\Rightarrow \lambda_x = 2$$

$$P(Y = 3) = P(Y = 4)$$

$$\frac{\lambda^3 e^{-\lambda}}{3\,!} = \frac{\lambda^4 e^{-\lambda}}{4\,!}$$

$$\frac{1}{6} = \frac{\lambda}{24}$$

$$\lambda_y = 4$$

$$\text{Var}(2x - Y) = (2)^2 \text{ var}(X) + (-1)^2 \text{ var}(Y)$$

$$= 4 \times 2 + (1) \times 4$$

$$= 12$$

33. Ans: (d)

Sol: $P(X) = \frac{\lambda^x e^{-\lambda}}{x\,!}$

$P(X = \text{odd}) = P(X = 1) + P(X = 3) + P(X= 5) + \ldots\ldots\ldots$

$$= \lambda e^{-\lambda} + \frac{\lambda^3 e^{-\lambda}}{3\,!} + \frac{\lambda^5 e^{-\lambda}}{5\,!} + \ldots\ldots\ldots$$

$$= e^{-1} + \frac{1}{3!}e^{-1} + \frac{1}{5!}e^{-1} + \ldots\ldots\ldots$$

$$= e^{-1}\left\{1 + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots\ldots\right\}$$

$$= e^{-1}\sinh(1)\ldots\ldots\ldots..(1)$$

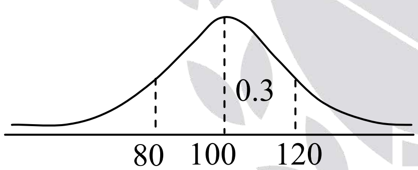
$$= e^{-1}\left\{\frac{e - e^{-1}}{2}\right\}$$

$$= \frac{1 - e^{-2}}{2}$$

$$= \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$$

34. Ans: 0.2

Sol: The area under normal curve is 1 and the curve is symmetric about mean.



$\therefore$ P(100 < X < 120) = P(80 < X < 120)

= 0.3

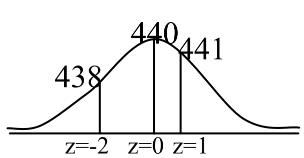
Now, P(X < 80) = 0.5 – P(80 < X < 120)

= 0.5 – 0.3 = 0.2

35. Ans: (a)

Sol: The standard normal variable Z is given by

$$Z = \frac{x - \mu}{\sigma}$$



When x = 438

$$Z = \frac{438 - 440}{1} = -2$$

When x = 441

$$Z = \frac{441 - 440}{1} = 1$$

The percentage of rods whose lengths lie between 438 mm and 441 mm

= P(438 < x < 441)

= P(–2 < Z < 1)

= P(–2 < Z< 0) + P(0 < Z< 1)

$$= \frac{0.9545}{2} + \frac{0.6826}{2} = 0.81855$$

$\approx$ 81.85 %

36. Ans: (d)

Sol: The parameters of normal distribution are μ = 68 and σ = 3

Let X = weight of student in kgs

Standard normal variable = $Z = \dfrac{X - \mu}{\sigma}$

(a) When X = 72, we have Z = 1.33

Required probability = P(X > 72)

= Area under the normal curve to the right of Z = 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh greater than 72 kgs = 300 × 0.0918

= 28

(b) When X = 64, we have Z = –1.33

Required probability = $P(X \le 64)$

= Area under the normal curve to the left of Z = – 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

(By symmetry of normal curve)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh less than 68 kgs = 300 × 0.0918

= 28

(c) When X = 65, we have Z = –1

When X = 71, we have Z = +1

Required probability = $P(65 < X < 71)$

= Area under the normal curve to the left of Z = – 1 and Z = +1

= 0.6826

(By Property of normal curve)

Expected number of students who weighs between 65 and 71 kgs

= 300 × 0.6826

≈ 205

37. Ans: 0.8051

Sol: The probability of population has Alzheimer's disease is

p = 0.04, q = 0.96, n = 3500

$\mu = np = (3500)(0.04) = 140$

$\sigma^2 = npq = (3500)(0.04)(0.96)$

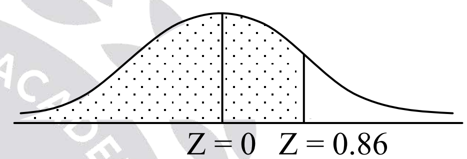
$\sigma^2 = 134.4$, $\sigma \cong 11.59$

Let X = number of people having Alzheimer's disease

$$P(X < 156) = P\left(\frac{X-\mu}{\sigma} < \frac{150-\mu}{\sigma}\right)$$

$$= P\left(Z < \frac{150-140}{11.59}\right)$$

$$= P(Z < 0.86)$$



= 0.5 + Area between z = 0 & z = 0.86

= 0.5 + 0.3051

= 0.8051

38. Ans: 0.0228

Sol: Given x_1, x_2, x_3 are independent normal random variables with means 47, 55, 60 and variances 10, 15 and 14 respectively.

Let $U = x_1 + x_2 - 2x_3$

$E(U) = \mu_U = E(x_1) + E(x_2) - 2E(x_3)$

$= 47 + 55 - 2 \times 60$

$= -18$

$Var(U) = \sigma_U^2 = var(x_1 + x_2 - 2x_3)$

$= \sigma_1^2 + \sigma_2^2 + 4\sigma_3^2$

$= 10 + 15 + 4 \times 14$

$$\sigma_U^2 = 81$$

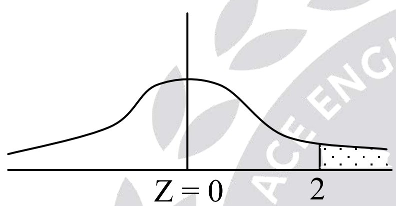
$$\sigma_U = 9$$

$$P(x_1 + x_2 \geq 2x_3) = P(x_1 + x_2 - 2x_3 \geq 0)$$

$$= P(U \geq 0)$$

$$= P\left[\frac{U-\mu_U}{\sigma_U} \geq \frac{0-\mu_U}{\sigma_U}\right]$$

$$= P\left(Z \geq \frac{18}{9}\right)$$



$$= P(Z \geq 2)$$

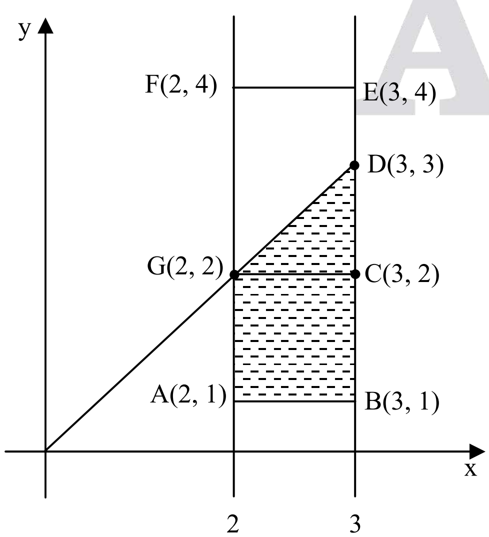
$$= 0.5 - P(0 < Z < 2)$$

$$= 0.5 - 0.4772$$

$$= 0.0228$$

39. Ans: 0.5

Sol:



$$P(Y \geq X) = \frac{\text{Area of the shaded portion}}{\text{Area of the rectangle ABCDEFG}}$$

$$= \frac{\text{Area of ABCG} + \text{Area of CDG}}{\text{Area of ABCDEFG}}$$

$$= \frac{1\times1+\frac{1}{2}\times1\times1}{1\times3} = \frac{\frac{3}{2}}{3}$$

$$= \frac{1}{2} = 0.5$$

40. Ans: 0.4

Sol: $x \sim$ UNIF $(-5, 5)$

$$f(x) = \frac{1}{10},\ -5 < x < 5$$

$$= 0, \quad \text{elsewhere}$$

$P[100t^2 + 20tx + 2x+3 = 0$ has complex solutions]

$$= P\{[(20x)^2 - 4(100)(2x+3)] \leq 0\}$$

$$= P\{[400x^2 - 800x - 1200] \leq 0\}$$

$$= P\{(x^2 - 2x - 3) \leq 0\}$$

$$= P\{[(x-3)(x+1) \leq 0\}$$

$$= P(-1 < x < 3)$$

$$= \int_{-1}^{3} \frac{1}{10}\,dx$$

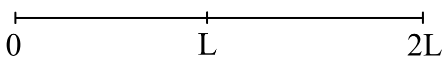
$$= \frac{1}{10}(x)_{-1}^{3}$$

$$= \frac{4}{10}$$

$$= 0.4$$

41. Ans: (c)

Sol:



To get a shorter piece, it can be broken anywhere between 0 and L.

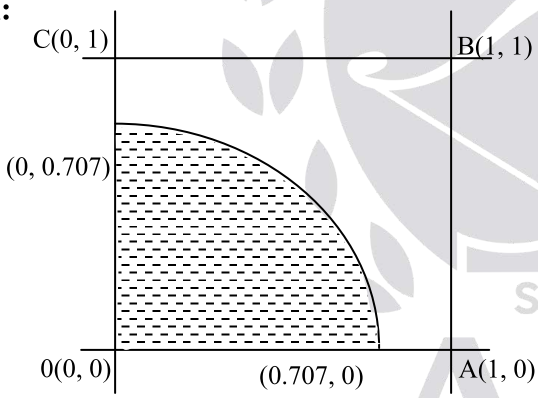
Let X be a random variable uniformly distributed in [0, L]

Mean $E(X) = \frac{0+L}{2} = \frac{L}{2}$

$$var(X) = \frac{(L-0)^2}{12} = \frac{L^2}{12}$$

42. Ans: 0.393

Sol:



$$P\left(x^2 + y^2 < \frac{1}{2}\right) = P\left(x^2 + y^2 < \left(\frac{1}{\sqrt{2}}\right)^2\right)$$

$$= \frac{\text{Area of shaded portion}}{\text{Area of OABC}}$$

$$= \frac{\frac{1}{4} \times \pi \times \left(\frac{1}{\sqrt{2}}\right)^2}{1 \times 1}$$

$$= \frac{\pi}{8} = 0.393$$

43. Ans: 2

Sol: x ~ UNIF (0, 1)

$f(x) = 1, \quad 0 < x < 1$

$\quad = 0, \quad$ otherwise

$E(y) = E(-2 \log x)$

$$= \int_0^1 -2 \log x \; f(x) dx$$

$$= \int_0^1 -2 \log x \; dx$$

$$= -2\{x \log x - x\}_0^1$$

$$= -2\{[\log(1) - 1] - \underset{x \to 0}{Lt}[x \log x - x]\}$$

$$= -2\{(0-1) - \underset{x \to 0}{Lt}[x \log x - x]\}$$

$$= -2\left\{-1 - \underset{x \to 0}{Lt} \frac{\log x}{\frac{1}{x}}\right\}$$

$$= -2\left\{-1 - \underset{x \to 0}{Lt} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}\right\}$$

($\because$ By L-Hospital rule)

$$= -2\{-1 + \underset{x \to 0}{Lt}\, x\}$$

$$= 2$$

44. Ans: 0.1

Sol: $\frac{1}{\lambda_1} = 1 \Rightarrow \lambda_1 = 1$

$\frac{1}{\lambda_2} = \frac{1}{2} \Rightarrow \lambda_2 = 2$

$\frac{1}{\lambda_3} = \frac{1}{3} \Rightarrow \lambda_3 = 3$

$\frac{1}{\lambda_4} = \frac{1}{4} \Rightarrow \lambda_4 = 4$

The distribution y = min(x_1, x_2, x_3, x_4) is also an exponential distribution with mean

$$\frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} = 0.1$$

45. Ans: (b)

Sol: x ~UNIF(0, 2)

y ~ EXP(λ)

$f(x) = \frac{1}{2}, \quad 0 < x < 2$

$= 0, \quad$ otherwise

$f(y) = \lambda e^{-\lambda y}, \quad 0 < y < 2$

$= 0, \quad$ otherwise

$P(x < 1) = P(y < 1)$

$$\int_0^1 f(x)dx = 1 - P(y \geq 1)$$

$$\int_0^1 \frac{1}{2}dx = 1 - \int_1^\infty \lambda e^{-\lambda y}\, dy$$

$$\frac{1}{2}(x)_0^1 = 1 - \lambda\left(\frac{-e^{-\lambda y}}{\lambda}\right)_1^\infty$$

$$\frac{1}{2} = 1 - \left[\left(-e^{-\infty}\right) - \left(-e^{-\lambda}\right)\right]$$

$$\frac{1}{2} = 1 - e^{-\lambda}$$

$$e^{-\lambda} = \frac{1}{2}$$

$$\ell n e^{-\lambda} = \ell n\left(\frac{1}{2}\right)$$

$-\lambda = -\,ln2$

$\lambda = ln(2)$

46. Ans: 0.0025

Sol: $\lambda = 2$

$$P(x > 3) = \int_3^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda\left(\frac{-e^{-\lambda x}}{\lambda}\right)_3^\infty$$

$= (-e^{-\infty}) - (-e^{-3\lambda})$

$= e^{-3(2)}$

$= e^{-6}$

$= 0.0025$

47. Ans: 0.367

Sol: $P(x > E(x)) = P\left(x > \frac{1}{\lambda}\right)$

$$= \int_{\frac{1}{\lambda}}^{\infty} f(x)\,dx$$

$$= \int_{\frac{1}{\lambda}}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left(\frac{-e^{-\lambda x}}{\lambda}\right)_{\frac{1}{\lambda}}^{\infty}$$

$$= (-e^{-\infty}) - (-e^{-1})$$

$$= e^{-1}$$

$$= 0.367.$$

48. Ans: –1

Sol: $\frac{1}{\lambda_1} = \frac{1}{2} \Rightarrow \lambda_1 = 2$

$\frac{1}{\lambda_2} = \frac{1}{2} \Rightarrow \lambda_2 = 2$

var(z) = var(x + y)

0 = var(x) + var(y) + 2cov(x, y)

$= \text{var}(x) + \text{var}(y) + 2r\sqrt{\text{var}(x)}\,\sqrt{\text{var}(y)}$

$= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + 2r\sqrt{\frac{1}{\lambda_1^2}}\,\sqrt{\frac{1}{\lambda_2^2}}$

$= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{2r}{\lambda_1\lambda_2}$

$0 = \frac{1}{4} + \frac{1}{4} + 2x\frac{1}{2} \times \frac{1}{2} \times r$

$0 = \frac{1}{2} + \frac{r}{2}$

$\therefore r = -1$

49. Ans: 43.33

Sol:

$$\text{Mean} = \frac{5\times4+15\times5+25\times7+35\times10+45\times12+55\times8+65\times4}{4+5+7+10+12+8+4} = 37.2$$

For median

Class	f	Cf	
0 – 10	4	4	
10 – 20	5	9	
20 – 30	7	16	
30 – 40	10	26	→Median class
40 – 50	12	38	
50 – 60	8	46	
60 – 70	4	50	

Here : $N = \sum f = 50$, $\frac{N+1}{2} = 25.5$

l = lower limit of the class internal of the median class = 30

m = Cumulative frequency preceding the median class = 16

f = frequency of the median class = 10

C = size of the class = 10

$$\text{median} = \ell + \left\{\frac{\frac{N}{2} - m}{f}\right\}C$$

$$= 30 + \left\{\frac{\frac{50}{2} - 16}{10}\right\}10 = 39$$

For Mode :

Class	Freq
0 – 10	4
10 – 20	5
20 – 30	7
30 – 40	10
40 – 50	12
50 – 60	8
60 – 70	4

→Modal class (40 – 50)

l = lower limit of the modal class = 40

f = frequency of modal class = 12

f_{-1} = frequency preceding the modal class

f_1 = frequency succeeding the modal class

C = size of the class = 10

$f_{-1} = 10$, $f_1 = 8$

$\Delta_1 = f - f_{-1} = 12 - 10 = 2$

$\Delta_2 = f - f_1 = 12 - 8 = 4$

$$\text{mode} = \ell + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$$

$$= 40 + \left(\frac{2}{2+4}\right)10$$

$$= 43.33$$

50. Ans: (i) a (ii) c (iii) d

Sol: The regression line of x and y is

$2x - y - 20 = 0$

$2x = y + 20$

$x = \frac{1}{2}y + 10$

The regression coefficient of x and y is

$b_{xy} = \frac{1}{2}$

The regression line of y on x is

$2y - x + 4 = 0$

$2y = x - 4$

$y = \frac{1}{2}x - 2$

The regression coefficient of y on x is

$b_{yx} = \frac{1}{2}$

(i) The correlation coefficient is

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{\frac{1}{4}}$$

$$r = \frac{1}{2}$$

(ii) Given $\sigma_y = \frac{1}{4}$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$\frac{1}{2}=\frac{1}{2}\frac{\frac{1}{4}}{\sigma_x}$$

$$\therefore\ \sigma_x=\frac{1}{4}$$

(iii) Both regression lines passing through $(\bar{x},\bar{y})$, we have

$$2\bar{x}-\bar{y}-20=0$$

$$2\bar{y}-\bar{x}+4=0$$

By solving these two equations, we get

$\bar{x}=12$ and $\bar{y}=4$

51. Ans: 0.18

Sol: Given: b_{yx} = 1.6 and b_{xy} = 0.4

$$r=\sqrt{b_{yx}\,b_{xy}}$$

$$r=\sqrt{1.6\times0.4}$$

$$r=0.8$$

Now, $b_{yx}=r\dfrac{\sigma_y}{\sigma_x}$

$$1.6=0.8\frac{\sigma_y}{\sigma_x}$$

$$\frac{\sigma_y}{\sigma_x}=\frac{1.6}{1.8}=\frac{2}{1}$$

$\Rightarrow \sigma_x=1$ and $\sigma_y=2$

The angle between two regression lines is

$$\tan\theta=\left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x\ \sigma_y}{\sigma_x^{\ 2}\ \sigma_y^{\ 2}}\right)$$

$$=\left\{\frac{1-(0.8)^2}{0.8}\right\}\left\{\frac{(1)\,(2)}{(1)^2+(2)^2}\right\}=0.18$$

52. Ans: 0.33

Sol: $P(X+Y\le1)=\int_R f(x,y)\,dxdy$

$$=\int_{x=0}^{1}\int_{y=0}^{1-x}(x+y)\,dxdy$$

$$=\int_0^1\left(xy+\frac{y^2}{2}\right)_0^{1-x}dx$$

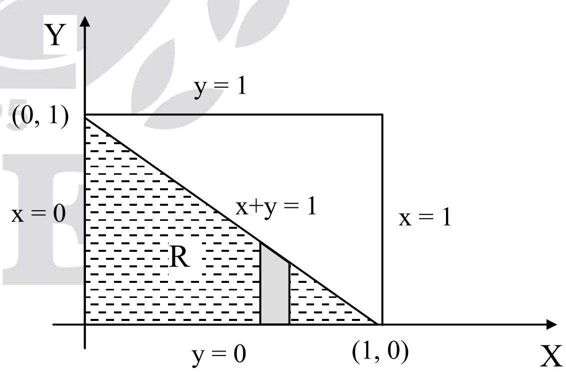
$$=\int_0^1\left[x(1-x)+\frac{(1-x)^2}{2}\right]dx$$

$$\int_0^1\left[x-x^2+\frac{(1+x^2-2x)}{2}\right]dx=\frac{1}{2}\int_0^1\left[1-x^2\right]dx$$

$$=\frac{1}{2}\left[x-\frac{x^3}{3}\right]_0^1$$

$$=\frac{1}{2}\left[\frac{2}{3}\right]$$

$$=0.33$$



53. Ans: (c)

Sol: Given :

$$f(x,y)=\begin{cases}\frac{1}{8}(6-x-y) & 0\le x\le 2, 2\le y\le 4\\ 0, & \text{otherwise}\end{cases}$$

$$P(x<1 \mid y<3)=\frac{P\{(x<1)\cap(y<3)\}}{P(y<3)}$$

$$(x<1\cap y<3)=\int_{x=0}^{1}\int_{y=2}^{3}\frac{1}{8}(6-x-y)\,dy\,dx$$

$$=\frac{1}{8}\int_0^1\left\{6y-xy-\frac{y^2}{2}\right\}_{y=2}^{y=3}dx$$

$$=\frac{1}{8}\int_0^1\left\{\left(18-3x-\frac{9}{2}\right)-(12-2x-2)\right\}dx$$

$$=\frac{1}{8}\left\{\left(18x-3\frac{x^2}{2}-\frac{9x}{2}\right)-(12x-x^2-2x)\right\}_0^1$$

$$=\frac{1}{8}\left\{\left(18-\frac{3}{2}-\frac{9}{2}\right)-(12-1-2)\right\}$$

$$=\frac{1}{8}\{(18-6)-9\}$$

$$=\frac{1}{8}(3)$$

$$P(y<3)=\int_{x=0}^{2}\int_{y=2}^{3}f(x,y)\,dy\,dx$$

$$=\int_{x=0}^{2}\int_{y=2}^{3}\frac{1}{8}(6-x-y)\,dy\,dx$$

$$=\frac{1}{8}\int_0^2\left\{6y-xy-\frac{y^2}{2}\right\}_{y=2}^{3}dx$$

$$=\frac{1}{8}\left\{\left(18-3x-\frac{9}{2}\right)-(12-2x-2)\right\}dx$$

$$=\frac{1}{8}\left\{\left(18x-\frac{3x^2}{2}-\frac{9x}{2}\right)-(12x-x^2-2x)\right\}_0^2$$

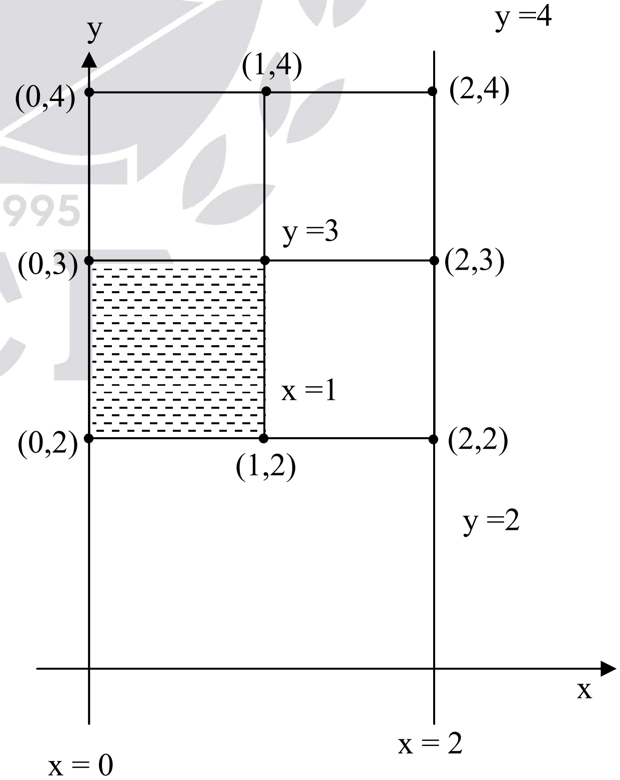
$$=\frac{1}{8}\{(36-6-9)-(24-4-4)\}$$

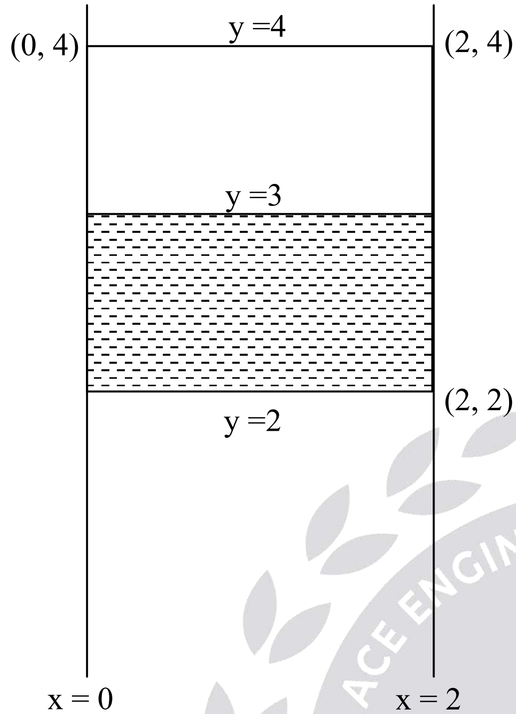
$$=\frac{1}{8}\{21-16\}$$

$$=\frac{5}{8}$$

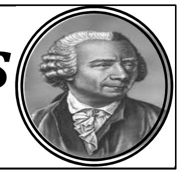
$$\therefore P(x<1 \mid y<3)=\frac{\frac{3}{8}}{\frac{5}{8}}$$

$$=\frac{3}{5}$$




y =4
(0, 4)
(2, 4)
y =3
(2, 2)
y =2
x = 0
x = 2

Chapter 4 Differential Equations
(With Laplace Transforms)



Leonhard Euler
(1707 – 1783)

01. Ans: (a)

Sol: Given $\frac{d^3y}{dx^3}+4\sqrt{\left(\frac{dy}{dx}\right)^3+y^2}=0$

$$\Rightarrow \frac{d^3y}{dx^3}=-4\left(\left(\frac{dy}{dx}\right)^3+y^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^2=16\left(\frac{dy}{dx}\right)^3+16y^2$$

$\therefore$ Order = 3 and Degree = 2

02. Ans: (a)

Sol: $\frac{dy}{dx}+\frac{2x}{3y}=0$

$$\frac{dy}{dx}=-\frac{2x}{3y}$$

$$3ydy=-2x\,dx$$

$$\Rightarrow \int 3ydy=\int -2xdx$$

$$\frac{3y^2}{2}=\frac{-2x^2}{2}+c$$

$$\frac{3}{2}y^2+x^2=c$$

$$3y^2+2x^2=2c$$

$\therefore$ ellipse

03. Ans: (c)

Sol: Given that $\frac{dy}{dx}=\frac{1+\cos 2y}{1-\cos 2x}$

$$\Rightarrow \frac{dy}{dx}=\frac{1+2\cos^2 y-1}{1-\left(1-2\sin^2 x\right)}=\frac{\cos^2 y}{\sin^2 x}$$

$$\Rightarrow \int \sec^2 y\,dy=\int \operatorname{cosec}^2 x\,dx+c$$

$$\Rightarrow \tan y=-\cot x+c$$

$\therefore$ The general solution is $\tan y+\cot x=c$

04. Ans: (c)

Sol: Given $(1+y)\frac{dy}{dx}=y$

$$\left(\frac{1+y}{y}\right)dy=dx$$

$$\int \frac{1+y}{y}dy=\int dx$$

$$\int\left(\frac{1}{y}+1\right)dy=\int dx$$

$\ln y+y=x+c$ ………….. (1)

$y(1)=1$, (1) $\Rightarrow \ln(1)+1=1+c$

$\Rightarrow c=0$

(1) $\Rightarrow \ln y+y=x$

$y\,e^y=e^x$

05. Ans: 0.886 (Range: 0.8774 to 0.8952)

Sol: $\frac{dy}{dx}=2x-y$

$$\frac{dy}{dx}+y=2x$$

Leonhard Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He made important discoveries in every branch of mathematic

I.F = $e^{\int 1.dx} = e^x$

$\therefore ye^x = 2\int xe^x dx = 2e^x(x-1) + C$

$y(0) = 1 \Rightarrow 1 = -2 + C$

$\therefore C = 3$

$\therefore ye^x = 2e^x(x-1) + 3$

at $x = \ln 2 \to 2y = 4(\ln 2 - 1) + 3$

$= 4(0.693 - 1) + 3$

$\therefore y = 0.886$

06. Ans: (d)

Sol: Given $t\dfrac{dx}{dt} + x = t$

$\Rightarrow \dfrac{dx}{dt} + \dfrac{1}{t}x = 1$ .... (1)

$\therefore$ I.F = $e^{\int \frac{1}{t}dt}$

$= e^{\log t} = t$

Now, the general solution of (1) is

$xt = \int t\,dt$

$= \dfrac{t^2}{2} + c$

$\because x(1) = 0.5$

$\Rightarrow 0.5 = \dfrac{1}{2} + c$ (or) $c = 0$

$\therefore$ The solution of equation (1) is

$$x = \frac{t}{2}$$

07. Ans: (c)

Sol: The given D.E is in the form of $\dfrac{dy}{dx} + py = Q$

Integrating factor = $e^{\int p dx} = e^{-\int \frac{1}{x}dx}$

$= e^{-\log x} = \dfrac{1}{x}$

Solution to the D.E is given by

$y.\dfrac{1}{x} = \int \dfrac{1}{x}dx + e$

$y.\dfrac{1}{x} = \log x + \log k \Rightarrow y = x\log kx$

08. Ans: (c)

Sol: Given differential equation is

$x(y\,dx + x\,dy)\cos\left(\dfrac{y}{x}\right) = y(x\,dy - y\,dx)\sin\left(\dfrac{y}{x}\right)$

$\Rightarrow x\,d(xy)\cos\left(\dfrac{y}{x}\right) = y(x\,dy - y\,dx)\sin\left(\dfrac{y}{x}\right)$

$\Rightarrow d(xy) = \left(\dfrac{y}{x}\right)(x\,dy - y\,dx)\tan\left(\dfrac{y}{x}\right)$

Dividing both sides by 'xy', we get

$\dfrac{d(xy)}{xy} = \left(\dfrac{x\,dy - y\,dx}{x^2}\right)\tan\left(\dfrac{y}{x}\right)$

$\Rightarrow \int \dfrac{d(xy)}{xy} = \int \tan\left(\dfrac{y}{x}\right) d\left(\dfrac{y}{x}\right)$

$\Rightarrow \ln(xy) = \ln\left[\sec\left(\dfrac{y}{x}\right)\right] + \ln c$

$\Rightarrow \ln(xy) = \ln\left[c\sec\left(\dfrac{y}{x}\right)\right]$

$\Rightarrow xy = c\sec\left(\dfrac{y}{x}\right)$

$\therefore xy\cos\left(\dfrac{y}{x}\right) = c$ is a required solution of (1)

09. Ans: (a)

Sol: $(x^2y^2 + y)dx + (2x^3y - x)dy = 0$

$(x^2y^2\,dx + 2x^3y\,dy) + (y\,dx - x\,dy) = 0$

$(y^2\,dx + 2xy\,dy) + \left(\dfrac{y\,dx - x\,dy}{x^2}\right) = 0$

$$\int d(xy^2) - \int d\left(\frac{y}{x}\right) = c$$

$$xy^2 - \left(\frac{y}{x}\right) = c$$

10. Ans: (b)

Sol: $(y - xy^2)dx + (x + x^2y)\,dy = 0$

$(y\,dx + x\,dy) + xy\,(x\,dy - y\,dx) = 0$

$$\frac{(y\,dx + x\,dy)}{xy} + (x\,dy - y\,dx) = 0$$

$$\frac{d(xy)}{(xy)^2} + \left(\frac{x\,dy - y\,dx}{xy}\right) = 0$$

$$\int \frac{d(xy)}{(xy)^2} + \int d\log\left(\frac{y}{x}\right) = c$$

$$-\frac{1}{xy} + \log\left(\frac{y}{x}\right) = c$$

11. Ans: (a)

Sol: $2xy^3\,dx + (3x^2y^2 + x^2y^3 + 1)dy = 0$

$(3x^2y^2 + x^2y^3 + 1)\,dy = -\,2xy^3\,dx$

$$\frac{dx}{dy} = -\frac{3x}{2y} - \frac{x}{2} - \frac{1}{2xy^3}$$

$$\frac{dx}{dy} + \left(\frac{3}{2y} + \frac{1}{2}\right)x = \frac{-1}{2xy^3}$$

$$2x\frac{dx}{dy} + \left(\frac{3}{y} + 1\right)x^2 = \frac{-1}{y^3} \quad ----(1)$$

Let $x^2 = z \Rightarrow 2x\dfrac{dx}{dy} = \dfrac{dz}{dy}$

(1) becomes

$$\frac{dz}{dy} + \left(\frac{3}{y} + 1\right)z = \left(-\frac{1}{y^3}\right)$$

$$\text{I.F} = e^{\int\left(\frac{3}{y}+1\right)dy} = e^{3\log y + y} = y^3e^y$$

$$\therefore\ z(y^3e^y) = \int\left(\frac{-1}{y^3}\right)y^3e^y\,dy$$

$$= -\,e^y + c$$

$$x^2y^3e^y + e^y = c$$

12. Ans: (a)

Sol: $x^2\dfrac{dy}{dx} + 2xy - x + 1 = 0$

$$x^2\frac{dy}{dx} + 2xy = (x - 1)$$

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$$\text{I.F} = e^{\int\frac{2}{x}dx} = e^{2\log x} = x^2$$

$$\therefore\ \text{solution } y.x^2 = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)x^2\,dx$$

$$= \int(x - 1)\,dx$$

$$= \frac{x^2}{2} - x + C$$

Given that $y(1) = 0$

i.e., $0 = \dfrac{1}{2} - 1 + C$

$\Rightarrow C = \dfrac{1}{2}$

$\therefore\ y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{1}{2x^2}$

$\therefore$ Option (a) is correct.

13. Ans: (a)

Sol: Given equation

$(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$

$$\frac{dy}{dx} = \frac{-(x^2y - 2xy^2)}{(3x^2y - x^3)} = \frac{2y^2 - xy}{3xy - x^2}$$

The above equation is homogenous

Put y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{2v^2 - v}{3v - 1} - v = \frac{-v^2}{3v-1}$$

$$\frac{3v-1}{v^2}dv + \frac{dx}{x} = 0$$

Integrating $3\log v + \frac{1}{v} + \log x = c$

$$\left(\frac{x}{y}\right) + \log\left(\frac{y^3}{x^2}\right) = c$$

14. Ans: (b)

Sol: Given $\frac{dy}{dx} + \left(\frac{x}{1-x^2}\right)y = x\sqrt{y}$

Dividing both sides by $\sqrt{y}$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x}{(1-x^2)}\frac{y}{\sqrt{y}} = x$$

$$y^{\frac{-1}{2}}\frac{dy}{dx} + \frac{x}{(1-x^2)}y^{\frac{1}{2}} = x \text{ …….. (1)}$$

$$\frac{1}{2}y^{\frac{-1}{2}}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{\frac{-1}{2}}\frac{dy}{dx} = \frac{2dt}{dx}$$

$$(1) \Rightarrow 2\frac{dt}{dx} + \frac{x}{(1-x^2)}t = x$$

$$\Rightarrow \frac{dt}{dx} + \frac{x}{2(1-x^2)}t = \frac{x}{2}$$

$$IF = e^{\int \frac{x}{2(1-x^2)}dx}$$

$$= e^{-\frac{1}{4}\int \frac{-2x}{(1-x^2)}dx}$$

$$= e^{-\frac{1}{4}\ell n(1-x^2)}$$

$$= (1-x^2)^{\frac{-1}{4}}$$

15. Ans: (a)

Sol: $\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$

$I.F = e^{\int \cot x dx} = \sin x$

$\therefore$ solutions

sinx y = ∫cosecx[sinx] dx +c

sinx y = ∫dx + c

sinx y = x + c

$$y = \frac{x+c}{\sin x}$$

$$y\,\pi/2 = \frac{\pi/2 + c}{1} = \pi/2$$

$\therefore$ c = 0

$$y(x) = \frac{x}{\sin x}$$

$$y\left(\frac{\pi}{6}\right) = \frac{\pi/6}{\sin(\pi/6)} = (2)\frac{\pi}{6} = \frac{\pi}{3}$$

16. Ans: (a)

Sol: Given $\frac{dy}{dx}+\frac{1}{x}y = x^3$ .... (1)

$$y(1)=\frac{6}{5} \quad \text{........ (2)}$$

$$I.F = e^{\int \frac{1}{x}dx}$$

$$= e^{\log x} = x$$

The solution is

$$y(x) = \int x^3\,(x)\,dx + c$$

$$xy = \frac{x^5}{5} + c \quad \text{......... (3)}$$

Using (2), (3) becomes

$$\frac{6}{5} = \frac{1}{5} + c \Rightarrow c = 1$$

$\therefore$ The solution of equation (1) is

$$y = \frac{x^4}{5} + \left(\frac{1}{x}\right)$$

$\therefore$ Option (a) is correct.

17. Ans: (a)

Sol: Given $\frac{dy}{dx} = \frac{x^2+y^2}{2y} + \frac{y}{x}$

$$= \frac{x^2}{2y} + y\left(\frac{1}{2}+\frac{1}{x}\right) \quad \text{....... (1)}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{1}{2}+\frac{1}{x}\right)y = \frac{x^2}{2y}$$

$$\left(\because \frac{dy}{dx} + p(x).y = Q(x).y^n\right)$$

$$\Rightarrow 2y\frac{dy}{dx} - \left(1+\frac{2}{x}\right)y^2 = x^2 \quad \text{................ (2)}$$

Let $y^2 = z$ .... (3)

Then $2y\frac{dy}{dx} = \frac{dz}{dx}$ .... (4)

Using (3) and (4), (2) becomes

$$\frac{dz}{dx} - \left(1+\frac{2}{x}\right)z = x^2 \quad \text{................ (5)}$$

Now, $I.F = e^{-\int\left(1+\frac{2}{x}\right)dx} = e^{-(x+2\log x)}$

$$\therefore \quad I.F = e^{-x-\log x^2} = \frac{e^{-x}}{e^{\log x^2}} = \frac{e^{-x}}{x^2}$$

The general solution of (5) is given by

$$z\left(\frac{e^{-x}}{x^2}\right) = \int x^2\left(\frac{e^{-x}}{x^2}\right)dx + C$$

$$\Rightarrow z\left(\frac{e^{-x}}{x^2}\right) = \int e^{-x}\,dx + C$$

$$\Rightarrow z\left(\frac{e^{-x}}{x^2}\right) = -e^{-x} + C$$

$\therefore \frac{y^2e^{-x}}{x^2} = C - e^{-x}$ .... (6), is a general solution of (1)

For x = 1 and y = 0, (6) becomes

$$0 = C - e^{-1}$$

$$\Rightarrow C = e^{-1}$$

Now, the solution of (1) passing through (1, 0) is

$$\frac{y^2e^{-x}}{x^2} = e^{-1} - e^{-x}$$

$$\Rightarrow \left(1+\frac{y^2}{x^2}\right)e^{-x} = e^{-1}$$

$$\Rightarrow \left(1+\frac{y^2}{x^2}\right) = e^{x-1}$$

$$\therefore \ln\left(1+\frac{y^2}{x^2}\right) = (x-1)$$

18. Ans: (d)

Sol: $\frac{dy}{dx}+\frac{2y}{x}=\frac{2\ell nx}{x^3}, y(1)=0$

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\ell nx} = x^2$$

$$y[x^2]=\int \frac{2\ell nx}{x^3}(x^2)\,dx + c$$

$$y(x)[x^2]=2\int \frac{\ell nx}{x}dx + c$$

$$x^2 y(x) = 2\left[\frac{1}{2}\ell n^2(x)\right]+c$$

$$y(x)=\frac{\ell n^2(x)+c}{x^2}$$

$$y(1)=\frac{0+c}{1}=0$$

c = 0

$$y(e)=\frac{\ell n^2(e)}{e^2}=\frac{1}{e^2}$$

19. Ans: (b)

Sol: $\frac{d^2u}{dx^2}-2x^2u+\sin x=0$

It is a linear non-homogeneous equation.

20. Ans: (d)

Sol: Given:

$$\frac{d^2x}{dt^2}-5\frac{dx}{dt}+6x=0,$$

The auxiliary equation is

$D^2 - 5D + 6 = 0$

Roots are 2, 3

The general solution is

$x = C_1 e^{2t} + C_2 e^{3t}$ --------- (1)

$\frac{dx}{dt}=2C_1e^{2t}+3C_2e^{3t}$ --------- (2)

$x(0) = 0$, (1) $\Rightarrow 0 = C_1 + C_2$

$\Rightarrow C_2 = - C_1$

$\frac{dx(0)}{dt}=10$, (2) $\Rightarrow 10 = 2C_1 + 3C_2$

$\Rightarrow 10 = 2C_1 - 3C_1 = -C_1$

$\Rightarrow C_1 = -10$ & $C_2 = 10$

$\therefore$ The particular solution is

$$x = -10e^{2t} + 10e^{3t}$$

Note: Among four options, only option (d) satisfies x(0) = 0. So option (d) is correct

21. Ans: (b)

Sol: Then auxiliary equation is $D^2 + 2D + 1 = 0$

$\Rightarrow (D + 1)^2 = 0$

$\Rightarrow D = -1, -1$

General solution is $y = (C_1 + C_2x)e^{-x}$ ……………..(1)

$y(0) = 1$, (1) $\Rightarrow 1 = C_1$

$y(1) = 3$, (1) $\Rightarrow 3 = (C_1 + C_2)e^{-1}$

$3e = C_1 + C_2$

$3e = 1 + C_2$

$C_2 = 3e-1$

$\therefore$ The solution is $y = [1 + 3e-1)e^{-x}$

$$y = e^{-x} + (3e-1)xe^{-x}$$

22. Ans: (b)

Sol: Given that $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=0$ .... (1)

with $y(0) = 1$ ....(2)

and $y(1) = 3e^{-1}$ .... (3)

The auxiliary equation corresponding to given differential equation is $m^2+2m+1 = 0$

$\Rightarrow m = -1, -1$

$\therefore$ The general solution of (1) is

$y(t) = (C_1+C_2t)e^{-t}$ .... (4)

Using (2), (4) becomes

$1 = C_1$

Using (3), (4) becomes

$3e^{-1} = (1+C_2)e^{-1}$

$\Rightarrow 3 = 1 + C_2$

$\Rightarrow C_2 = 2$

Substituting the values of C_1 & C_2 in equation (1), we get

$y(t) = (1+2t)e^{-t}$

$\therefore y(2) = 5e^{-2}$

23. Ans: – 0.21

Sol: Given $\frac{d^2y}{dt^2}+\frac{dy}{dt}-\frac{5}{4}y=0$ .... (1)

$\Rightarrow \left(D^2+D-\frac{5}{4}\right)y=0$, where $D=\frac{d}{dx}$

$\Rightarrow f(D)\, y = 0$, when $f(D) = D^2 + D - \frac{5}{4}$

Consider Auxiliary equation $f(m) = 0$

$\Rightarrow m^2+m-\frac{5}{4}=0$

$\Rightarrow m = \frac{-1}{2}\pm i$

$\therefore$ The general solution of (1) is

$y = e^{-\frac{t}{2}}\left(C_1\cos t + C_2\sin t\right)$ .... (2)

$\Rightarrow y^1 = \frac{dy}{dt} = \frac{-1}{2}e^{\frac{-t}{2}}\left[C_1\ \cos t + C_2\sin t\right]$

$+\ e^{\frac{-t}{2}}\left[-C_1\ \sin t + C_2\cos t\right]$ .... (3)

$\because y(0) = 1$

$\Rightarrow 1 = C_1$

$\because \left(\frac{dy}{dt}\right)_{t=0} = 0$ & $C_1 = 1$

$\Rightarrow \frac{-1}{2}[\ 1+0\ ] + [\ 0 + C_2] = 0$

$\Rightarrow C_2 = \frac{1}{2}$

$\therefore$ The solution of (1) with initial conditions is $y = e^{\frac{-t}{2}}\left[\cos t + \frac{1}{2}\sin t\right]$

Hence, $y = y(\pi) = e^{\frac{-\pi}{2}}\left[-1+0\right] = -0.21$

24. Ans: (–1)

Sol: Given that $y'' + 9y = 0$ .... (1)

with $y(0) = 0$ .... (2)

and $y\left(\frac{\pi}{2}\right) = \sqrt{2}$ .... (3)

Consider Auxiliary equation $f(D) = 0$

$\Rightarrow D^2 + 9 = 0$ where $D = \frac{d}{dx}$

$\Rightarrow$ $D = \pm 3i$ are different imaginary roots

$\therefore$ The general solution of (1) is

$y = C_1\cos 3x + C_2\sin 3x$ …….. (4)

Using (2), (4) becomes

$0 = C_1$

Using (3), (4) becomes

$\sqrt{2} = c_2 \sin\frac{3\pi}{2}$

$\Rightarrow$ $C_2 = -\sqrt{2}$

Substituting the values of C_1, C_2 in (1), we get

$y = -\sqrt{2}\sin 3x$

$\therefore$ $y\left(\frac{\pi}{4}\right) = -\sqrt{2}\sin\left(\frac{3\pi}{4}\right) = -1$

25. Ans: (b)

Sol: $c_1 e^x + e^{\frac{-x}{2}}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\right)x + c_3 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$

$\Rightarrow$ AE has roots $1, \left(\frac{-1}{2} \pm \frac{\sqrt{3}}{2}i\right)$

$\Rightarrow (D - 1)(D^2 + D + 1)\, y = 0$

$(D^3 - 1)\, y = 0$

26. Ans: (b)

Sol: $y = (c_1e^x + c_2 e^x \cos x + c_3 e^x \sin x)$ is the general solution from the given independent solutions

$\therefore$ $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$

$\therefore$ A.E. has roots 1, $(1 \pm i)$

$\therefore$ $(D - 1)(D^2 - 2D + 2)\, y = 0$

$(D^3 - 3D^2 + 4D - 2)\, y = 0$

27. Ans: 5

Sol: $y^{iv} + 4y''' + 8y'' + 8y' + 4y = 20$

$(D^4 + 4D^3 + 8D^2 + 8D + 4)\, y = 20\, e^{0.x}$

$$y_p = \frac{20.e^{0.x}}{(D^4 + 4D^3 + 8D^2 + 8D + 4)}$$

$$= \frac{20.e^{0.x}}{4} = 5$$

28. Ans: (a)

Sol: $y^{v} - y' = 12e^x$

$(D^5 - D)\, y = 12\, e^x$

$$\therefore \quad y_p = \frac{12e^x}{D(D^4 - 1)}$$

$$= \frac{12e^x}{D(D-1)(D+1)(D^2+1)}$$

$$= \frac{12xe^x}{2.2}$$

$y_p = 3x\, e^x$

29. Ans: (c)

Sol: $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=-2\cosh(x)$

$(D^2 + 4D + 5)\, y = -(e^x + e^{-x})$

$y_p=\frac{-\left(e^x+e^{-x}\right)}{\left(D^2+4D+5\right)}$

$=\frac{-e^x}{(1+4+5)}-\frac{e^{-x}}{(1-4+5)}$

$=\frac{-e^x}{10}-\frac{e^{-x}}{2}$

30. Ans: 18

Sol: Given that $\frac{d^2y}{dx^2}=-12x^2+24x-20$ …… (1)

with $y(0) = 5$ ………. (2)

and $y(2) = 21$ ………. (3)

Integrating both sides of above differential equation (1), we get

$\frac{dy}{dx}=-4x^3+12x^2-20x+c_1$ …….. (4)

Again integrating both sides of (4) w.r.t 'x', we get

$y=-x^4+4x^3-10x^2+c_1x+c_2$ ……..(5)

Using (2), (5) becomes

$5 = c_2$

Using (3), (5) becomes

$c_1 = 20$

Substituting c_1, c_2 values in (5), we get

$y = -x^4 + 4x^3 - 10x^2 + 20x + 5$

$\therefore$ $y(1) = 18$

31. Ans: (a)

Sol: $\frac{d^2y}{dx^2}+y=\cos(x)$

$(D^2 + 1)\, y = \cos x$

A.E has roots $\pm i$

$\therefore$ $y_c = (c_1 \cos x + c_2 \sin x)$

$\therefore$ $y = (y_c + y_p)$

$= (c_1 \cos x + c_2 \sin x + \frac{x}{2}\sin x)$

$y(0) = 1 \Rightarrow 1 = c_1$

$y(\pi/2) = 0 \Rightarrow 0 = c_2+\frac{\pi}{4}\Rightarrow c_2=-\frac{\pi}{4}$

$\therefore$ $y=\left(\cos x-\frac{\pi}{4}\sin x+\frac{x}{2}\sin x\right)$

32. Ans: (*)

Sol: $\frac{d^3y}{dx^3}+4\frac{dy}{dx}=\sin(2x)$

$(D^3 + 4D)\, y = \sin 2x$

$\therefore y_p=\frac{\sin 2x}{\left(D^3+4D\right)}$

$=\frac{1}{D}\frac{\sin 2x}{\left(D^2+4\right)}$

$y_p=\frac{1}{D}\left(\frac{-x}{4}\cos 2x\right)$

$=-\frac{1}{4}\left(\frac{x}{2}\sin 2x+\frac{\cos 2x}{4}\right)$

$=-\frac{1}{8}\left(2x\sin 2x+\cos 2x\right)$

33. Ans: (a)

Sol: $P.I = \frac{1}{D^2+3D+2}[5\cos x]$

$\therefore \quad D^2 = -a^2 \quad \therefore \quad a = 1$

$= \frac{5}{-1+3D+2}\cos x$

$= \frac{5}{3D+1}\cos x$

$= \left[\frac{5}{3D+1}\times\frac{3D-1}{3D-1}\right]\cos x$

$= \frac{5}{9D^2-1}(3D-1)\cos x$

$= \frac{5}{-10}[-3\sin x - \cos x]$

$P.I = \frac{5}{10}[3\sin x + \cos x]$

34. Ans: (a)

Sol: Given $y^{11} - 4y^1 + 3y = 2t - 3t^2$ .... (1)

$\Rightarrow \left(D^2 - 4D + 3\right)y = 2t - 3t^2$

Now, the particular integral of (1) is

$PI = \frac{1}{(D^2-4D+3)}(2t-3t^2)$

$= \frac{1}{3\left[1+\left(\frac{D^2-4D}{3}\right)\right]}(2t-3t^2)$

$= \frac{1}{3}\left\{1+\left(\frac{D^2-4D}{3}\right)\right\}^{-1}(2t-3t^2)$

$= \frac{1}{3}\left\{1-\left(\frac{D^2-4D}{3}\right)+\left(\frac{D^2-4D}{3}\right)^2\right\}(2t-3t^2)$

(Expanding by binomial theorem up to D^2 terms)

$= \frac{1}{3}\left\{(2t-3t^2)-\left[\frac{-6-4(2-6t)}{3}\right]+\frac{16}{9}(-6)\right\}$

$\therefore$ P.I $= -2 - 2t - t^2$

35. Ans: (c)

Sol: Given $(D^2 + 6D + 9)\,y = 9x + 6$

Consider Auxiliary equation f(D) = 0

$\Rightarrow D^2 + 6D + 9 = 0$

$\Rightarrow D = -3, -3$ are real and equal roots

$\therefore$ The complementary function of (1) is

$CF = (C_1x + C_2)\,e^{-3x}$

Now, the particular integral of (1) is

$PI = \frac{1}{(D+3)^2}(9x+6)$

$\Rightarrow P.I = \frac{1}{9}\left[1+\frac{D}{3}\right]^{-2}(9x+6)$

$\Rightarrow P.I = \frac{1}{9}\left[1-2\frac{D}{3}+3\left(\frac{D^2}{9}\right)-\right](9x+6)$

$\Rightarrow P.I = \frac{1}{9}(9x+6)-\frac{2}{3}.\frac{1}{9}(9)$

$\Rightarrow P.I = x+\frac{2}{3}-\frac{2}{3} = x$

$\therefore$ The solution of equation (1) is

$y = (C_1x + C_2)\,e^{-3x} + x$

36. Ans: (d)

Sol: $P.I = \frac{x^2e^{-x}}{D^2+2D+1}$

When $X = e^{ax}V$, where V is a function of X.

$$y_P = \left(\frac{1}{f(D)}e^{ax}V\right)$$

$$= e^{ax}\left[\frac{1}{f(D+a)}V\right]$$

$\therefore$ a = –1 & V = x^2

$$P.I = e^{-x}\left[\frac{1}{(D-1)^2+2(D-1)+1}\right]x^2$$

$$P.I = e^{-x}\left[\frac{1}{D^2}\right]x^2$$

$$= e^{-x}\left[\frac{1}{D}\frac{x^3}{3}\right] = e^{-x}\frac{1}{12}x^4$$

$$= \frac{1}{12}e^{-x}x^4$$

37. Ans: (c)

Sol: Given $(D^2 + 4)\,y = 2e^x \sin^2 x$

Auxiliary equation is $D^2 + 4 = 0$

Complementary function is

$C_1\cos(2x) + C_2\sin(2x)$

$$P.I = \frac{1}{D^2+4}2e^x\sin^2 x$$

$$= 2e^x\left\{\frac{1}{(D+1)^2+4}\left[\frac{1-\cos 2x}{2}\right]\right\}$$

$$= e^x\left\{\frac{1}{D^2+2D+5}(1-\cos 2x)\right\}$$

$$= e^x\left\{\frac{1}{D^2+2D+5}e^{0x} - \frac{1}{D^2+2D+5}\cos 2x)\right\}$$

$$= \frac{e^x}{2}\left\{\frac{1}{5}e^{0x} - \frac{1}{(-4+2D+5)}\cos 2(x)\right\}$$

$$= e^x\left\{\frac{1}{5} - \frac{1}{2D+1}\cos(2x)\right\}$$

$$= e^x\left\{\frac{1}{5} - \frac{2D-1}{4D^2-1}\cos(2x)\right\}$$

$$= e^x\left\{\frac{1}{5} - \frac{[-4\sin(2x)-\cos(2x)]}{4\times -4-1}\right\}$$

$$= e^x\left\{\frac{1}{5} - \frac{1}{17}[4\sin(2x)+\cos(2x)]\right\}$$

The solution is y = CF + PI

$Y = C_1\cos(2x)+C_2\sin(2x)$

$$+\ e^x\left\{\frac{1}{5} - \frac{1}{17}[4\sin(2x)+\cos(2x)]\right\}$$

38. Ans: (b)

Sol: $y'' + 4y = x\sin(x)$

$$(D^2+4)y = x\sin x$$

$$y_p = \frac{x\sin x}{(D^2+4)}$$

$$= x\left(\frac{\sin x}{D^2+4}\right) - \left(\frac{2D}{(D^2+4)^2}\right)\sin x$$

$$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

39. Ans: 5.25

Sol: Given $(x^2D^2 - 3xD + 3)y = 0$ .... (1)

where $D = \dfrac{d}{dx}$

with y(1) =1 and y(2) = 14 .... (2)

Let $x = e^x$ (or) $\log x = z$

and $xD = \theta$, $x^2D^2 = \theta(\theta-1)$ ..... (3)

where $\theta = \frac{d}{dz}$

Using (3), (1) becomes

$[\theta(\theta - 1) - 3\theta + 3]y = 0$

$\Rightarrow (\theta^2 - 4\theta + 3)y = 0$

$\Rightarrow f(\theta)y = 0$, where $f(\theta) = \theta^2 - 4\theta + 3$

The auxiliary equation is f(m) = 0

$\Rightarrow m^2 - 4m + 3 = 0$

$\Rightarrow m = 1, 3$

$\Rightarrow y_c = c_1e^z + c_2e^{3z}$

$\therefore$ The general solution (1) is

$y = c_1x + c_2 x^3$ .... (4)

$\because y(1) = 1$

$\Rightarrow c_1 + c_2 = 0$ .... (5)

$\because y(2) = 14$

$\Rightarrow 2c_1 + 8c_2 = 0$ .... (6)

Solving (5) and (6), we get $c_1 = -1$, $c_2 = 2$

$\therefore$ The solution of (1) with (2) is

$y = y(x) = -x + 2x^3$

Hence, y = y(1.5) = 5.25

40. Ans: (c)

Sol: Given $(x^2 D^2 - 7x D + 16) y = 0$ ....(1),

where $D = \frac{d}{dx}$

Let $x = e^x$ (or) $\log x = z$
and $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$ ..... (2)

where $\theta = \frac{d}{dz}$

Using (2), (1) becomes

$[\theta(\theta - 1) - 7\theta + 16] y = 0$

$\Rightarrow [\theta^2 - 8\theta + 16) y = 0$

$\Rightarrow f(\theta) y = 0$, where $f(\theta) = \theta^2 - 8\theta + 16$

The auxiliary equation is f(m) = 0

$\Rightarrow m^2 - 3m + 16 = 0$

$\Rightarrow m = 4, 4$

$\Rightarrow y_c = (c_1 + c_2 z)e^{4z}$

$\Rightarrow y_c = (c_1 + c_2 \log x) x^4$

$\therefore$ The general solution of (1) is

$y = (c_1 + c_2 \log x)x^4$

Hence, option (c) is correct.

41. Ans: 6

Sol: Given $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ .... (1)

with y (1) = 0, y(2) = 2 .... (2)

Let $x = e^x$ (or) $\log x = z$
and $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$ ..... (3)

Using (3), (1) becomes

$\theta(\theta - 1)y - 2\theta y + 2y = 0$

$\Rightarrow (\theta^2 - 3\theta + 2) y = 0$

$\Rightarrow f(\theta)y = 0$, where $f\theta) = \theta^2 - 3\theta + 2$

The auxiliary equations is

$m^2 - 3m + 2 = 0$

$\Rightarrow m = 1, 2$

The general solutions of (1) is

$y = c_1 e^z + c_2 e^{2z} = c_1x + c_2x^2$

$\because y(1) = 0$

$\Rightarrow 0 = c_1 + c_2$ .... (4)

$\because y(2) = 2$

$\Rightarrow 2 = 2c_1 + 4c_2$ .... (5)

Solving (4) and (5), we get $c_1 = -1$, $c_2 = 1$

$\therefore$ The solution of (1) with (2) is $y = -x + x^2$

Hence, $y(3) = -3 + 9 = 6$

42. Ans: (a)

Sol: Given $(D^2 + a^2)\,y = \tan(ax) = X$ (say)

Auxiliary equation is $D^2 + a^2 = 0$

$CF = c_1\cos(ax) + c_2\sin(ax) = c_1 y_1 + c_2 y_2$

where $y_1 = \cos(ax)$ and $y_2 = \sin(ax)$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a\cos^2(ax) + a\sin^2(ax) = a$$

$$P.I = -y_1\int \frac{Xy_2}{w}dx + y_2\int \frac{Xy_1}{w}dx$$

$$= -\cos ax\int \frac{\tan(ax)\times\sin(ax)}{a}dx$$

$$+\sin(ax)\int \frac{\tan(ax)(\cos(ax))}{a}dx$$

$$= \frac{-\cos ax}{a}\int \frac{\sin^2(ax)}{\cos(ax)}dx + \frac{\sin(ax)}{a}\int \sin(ax)\,dx$$

$$= \frac{-\cos ax}{a}\int \frac{1-\cos^2(ax)}{\cos(ax)}dx + \frac{\sin(ax)}{a}\left[\frac{-\cos(ax)}{a}\right]$$

$$= \frac{-\cos ax}{a}\int [\sec(ax) - \cos(ax)]dx - \left[\frac{\sin(ax)\cos(ax)}{a^2}\right]$$

$$= \frac{-\cos ax}{a}\left[\frac{1}{a}\log(\sec(ax)+\tan(ax)) - \frac{\sin(ax)}{a}\right]$$

$$-\left[\frac{\sin(ax)\cos(ax)}{a^2}\right]$$

$$= \frac{-\cos ax}{a^2}\log[\sec(ax)+\tan(ax)] + \frac{\sin(ax)\cos(ax)}{a^2}$$

$$-\frac{\sin(ax)\cos(ax)}{a^2}$$

The general solution is

$$y = c_1\cos(ax) + c_2\sin(ax)$$

$$-\frac{\cos ax}{a^2}\log[\sec(ax)+\tan(ax)]$$

43. Ans: (b)

Sol: The given equation is

$$\left(\frac{d^2y}{dx^2}\right) - 4\left(\frac{dy}{dx}\right) + 4y = \frac{e^{2x}}{x}$$

The auxiliary equation is

$(D-2)^2 = 0 \Rightarrow D = 2, 2$

$C.F = (C_1 + C_2x)e^{2x}$

$= C_1e^{2x} + C_2\,xe^{2x}$

$= C_1\,y_1 + C_2y_2$

where, $C_1 = e^{2x}$ & $C_2 = xe^{2x}$

$P.I = A.y_1 + B.y_2$ .... (i)

where, $A = -\int \frac{Py_2}{W}dx$

where, $W = y_1.y_2' - y_2.y_1' = e^{4x}$

$$A = -\int \frac{e^{2x}}{x}.\frac{x\,e^{2x}}{e^{4x}}\,dx = -x$$

$$B = \int \frac{Py_1}{W}dx = \int \frac{e^{2x}}{x}.\frac{e^{2x}}{e^{4x}}\,dx = \log x$$

Substituting the values of A & B in (i)

$P.I = -xe^{2x} + xe^{2x}\log x$

The solution is

$$y = (C_1 + C_2x + x\log x - x)\,e^{2x}$$

44. Ans: (a)

Sol: Given

$$(x-a)^2+(y-b)^2=z^2\cot^2\alpha \quad \ldots\ldots (1)$$

Differentiating equation (1) wrt 'x' partially, we get

$$2(x-a)=2z\frac{\partial z}{\partial x}\cot^2\alpha$$

$$\Rightarrow (x-a)=pz\cot^2\alpha \quad \ldots\ldots. (2)$$

where $p=\frac{\partial z}{\partial x}$

Differentiating equation (1) wrt 'y' partially, we get

$$2(y-b)=2z\frac{\partial z}{\partial y}\cot^2\alpha$$

$$\Rightarrow (y-b)=qz\cot^2\alpha \quad \ldots.. (3)$$

where $q=\frac{\partial z}{\partial y}$

Equation (1) can be re-written as

$$p^2z^2\cot^4\alpha+q^2z^2\cot^4\alpha=z^2\cot^2\alpha$$

$$\Rightarrow p^2\cot^2\alpha+q^2\cot^2=1$$

$$\Rightarrow \cot^2\alpha\left(p^2+q^2\right)=1$$

$\therefore\ p^2+q^2=\tan^2\alpha$ is the solution

45. Ans: (d)

Sol: Given $z=(x+y)\phi(x^2-y^2) \quad \ldots\ldots (1)$

Differentiating (1) wrt 'x' partially, we get

$$p=\frac{\partial z}{\partial x}=2x(x+y)\phi'(x^2-y^2)+\phi(x^2-y^2)$$

Differentiating (1) wrt 'y' partially, we get

$$q=\frac{\partial z}{\partial y}=-2y(x+y)\phi'(x^2-y^2)+\phi(x^2-y^2)$$

$$\Rightarrow py+qx=\left[2xy(x+y)\varphi'(x^2-y^2)+y\varphi(x^2-y^2)\right]$$

$$+\left[-2xy(x+y)\varphi'(x^2-y^2)+x\varphi\left(x^2-y^2\right)\right]$$

$$\Rightarrow py+qx=(x+y)\varphi(x^2-y^2)$$

$\therefore\ py+qx=z$ is the required solution

46. Ans: (c)

Sol: The partial differential equation

$$A\frac{\partial^2 p}{\partial x^2}+B\frac{\partial^2 p}{\partial x\partial y}+C\frac{\partial^2 p}{\partial y^2}+D\frac{\partial p}{\partial x}+E\frac{\partial p}{\partial y}+Fu=G$$

is Hyperbolic if $B^2-4AC>0$.

From the given equation, we have

A = 1, B = 3, C = 1

$B^2-4AC=9-4=5>0$

$\therefore$ The given equation is Hyperbolic

47. Ans: 36

Sol: Given $3\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+3\frac{\partial^2\phi}{\partial y^2}+4\phi=0 \ldots(1)$

Comparing given partial different equation with general second order linear partial different equation, we get

A = 3, B = B and C = 3

The P.D.E (1) is said to be parabolic if $B^2-4AC=0$

$\Rightarrow\quad B^2-36=0$

$\therefore\quad B^2=36$

48. Ans: (c)

Sol: $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ --------- (1)

$u(x, 0) = 6e^{-3x}$ -------- (2)

$u = XT$ ------- (3) where X is a function of 'x' only and T is a function of 't' only

Sub (3) in (1)

$X'T = 8\,XT' + XT$

$X'T = X\,(2T' + T)$

$\frac{X'}{X} = \frac{2T'+T}{T} = K$

$\frac{X'}{X} = k$ & $\frac{8T'+T}{T} = K$

$\frac{X'}{X} = k \Rightarrow \frac{dX}{dx} = kX$

$\frac{dX}{X} = k\,dx$

On integrating

$\log X = kx + \log C_1$

$X = c_1\,e^{kx} \rightarrow (4)$

$\frac{2T'+T}{T} = k \Rightarrow 2T' + T = kT$

$\frac{dT}{dt} = \frac{(k-1)T}{2}$

$\frac{1}{T}dT = \left(\frac{k-1}{2}\right)dt$

On integrating

$\text{Log } T = \left(\frac{k-1}{2}\right)t + \log C_2$

$T = c_2\,e^{\left(\frac{k-1}{2}\right)t} \rightarrow (5)$

Sub (4) & (5) in (3)

$u = c_1\,e^{kx}\;c_2\,e^{\left(\frac{k-1}{2}\right)t}$

Given $u(x, 0) = 6e^{-3x}$

$6e^{-3x} = u(x, 0) = C_1C_2\,e^{kx}$

$c_1c_2 = 6$ & $k = -3$

$u = 6e^{-3x}\,.\,e^{\left(\frac{-3-1}{2}\right)t}$

$u = 6e^{-3x}\,e^{-2t}$

49. Ans: (a)

Sol: Given $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ .... (i)

Let $u = X(x).Y(y)$ be the solution of (i)

Then $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Substituting in equation (i)

$X^1Y = 4XY'$

$\frac{X'}{X} = \frac{4Y'}{Y} = k$

$\frac{X'}{X} = k$ and $\frac{4Y'}{Y} = k$

$\Rightarrow X = c_1\,e^{kx}$ and $Y = c_2\,e^{\frac{ky}{4}}$

Now, the solution is,

$u = c_1c_2\,e^{kx}\;e^{\frac{ky}{4}}$

$u = c_3\,e^{kx}\;e^{\frac{ky}{4}}$ .... (ii)

Given $u(0, y) = 8e^{-3y}$

$\Rightarrow 8e^{-3y} = u(0,y) = c_3\;e^{\frac{k}{4}y}$

$\Rightarrow c_3 = 8,\; k = -12$

$\therefore\; u = 8\,e^{-12x-3y}$

50. Ans: (c)

Sol: Given yzp –xzq = xy

The linear partial differential equation is Pp + Qq = R, where P,Q,R are functions of x,y,z

P = yz, Q = – xz, R = xy

The lagrange's auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

$$\Rightarrow \frac{dx}{yz} = \frac{dy}{-xz}$$

$$\Rightarrow ydy + xdx = 0$$

Integrating both sides

$$\frac{y^2}{2} + \frac{x^2}{2} = c_1$$

$$\Rightarrow x^2 + y^2 = k_1$$

Now $\frac{dy}{-xz} = \frac{dz}{xy}$

$$ydy + zdz = 0$$

Integrating both sides

$$\frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$y^2 + z^2 = k_2$$

∴ The solution is

$$\varphi\left[x^2 + y^2, y^2 + z^2\right] = 0$$

(or) $f\left[x^2 + y^2, y^2 + z^2\right] = 0$

51. Ans: (a)

Sol: Given p(mz – ny) + q(nx – lz) = ly – mx,

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

The auxiliary equation is

$$\frac{dx}{(mz - ny)} = \frac{dy}{(nx - lz)} = \frac{dz}{(ly - mx)} \quad \ldots\ldots (1)$$

choose x,y,z are as lagrange's multipliers

By algebra, Each ratio

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - \ln z) + z(ly - mx)}$$

Each ratio of (1) $= \frac{xdx + ydy + zdz}{0}$

$$\Rightarrow xdx + ydy + zdz = 0$$

Integrating both sides, we get

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$\Rightarrow x^2 + y^2 + z^2 = k_1$$

choose l,m,n as another set of lagrange's multipliers,

Each ratio of

$$(1) = \frac{l dx + mdy + ndz}{l(mz - ny) + m(nx - lz) + z(ly - mx)}$$

$$= \frac{l dx + mdy + ndz}{0}$$

$$\Rightarrow l dx + mdy + ndz = 0$$

Integrating both sides, we get

$$l x + my + nz = k_2$$

∴ The solution is

$$f\left[x^2 + y^2 + z^2, l x + my + nz\right] = 0$$

or $x^2 + y^2 + z^2 = f(l x + my + nz)$

52. Ans: (c)

Sol: Given $p^2 + q^2 = npq$

The complete solution is

$Z = ax + by + c$ ……. (1)

where a & b are arbitrary constants

$$p = \frac{\partial z}{\partial x} = a \quad \& \quad q = \frac{\partial z}{\partial y} = b$$

substitute these values in the equation

$$p^2 + q^2 = npq$$

$$a^2 + b^2 = nab$$

$$a^2 + b^2 - nab = 0$$

$$b = \frac{na \pm a\sqrt{n^2-4}}{2} = \frac{na}{2} \pm \frac{a}{2}\sqrt{n^2-4}$$

∴ The solution is

$$Z = ax + \frac{nay}{2} \pm \frac{a}{2}\sqrt{n^2-4y} + c$$

$$Z = ax + \frac{ay}{2}\left[n \pm \sqrt{n^2-4}\right] + c$$

53. Ans: (d)

Sol: Given $p(1 + q) = qz$

…….. (1)

consider a trial solution

$z = g(t)$

…….. (2)

where $t = x + ay$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}.\frac{\partial t}{\partial x} = g'(t) = \frac{dz}{dt}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial t}.\frac{\partial t}{\partial y} = g'(t).a = a\frac{dz}{dt}$$

substituting these values in (1), we get

$$\frac{dz}{dt}\left[1 + a\frac{dz}{dt}\right] = z.a\frac{dz}{dt}$$

$$1 + a\frac{dz}{dt} = az$$

$$\Rightarrow a\frac{dz}{dt} = az - 1$$

Integrating both sides

$$\Rightarrow \int\frac{dz}{(az-1)} = \int\frac{dt}{a}$$

$$\Rightarrow \frac{\log(az-1)}{a} = \frac{t}{a} + c$$

$$\Rightarrow \frac{\log(az-1)}{a} = \left(\frac{x+ay}{a}\right) + c$$

$$\Rightarrow \frac{\log(az-1)}{a} = \frac{x+ay}{a} + c$$

$$\Rightarrow \log(az-1) = (x+ay) + ac$$

∴ The solution is

$\log(az-1) = x + ay + b$ [∵ $b = ac$]

54. Ans: (a)

Sol: Given $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$

Dividing both sides by x^2q^2, we get

$$\frac{p^2}{x^2} + \frac{y^2}{q^2} = x^2 + y^2$$

$$\Rightarrow \frac{p^2}{x^2} - x^2 = y^2 - \frac{y^2}{q^2} = a^2 \qquad [\text{ say }]$$

$$\Rightarrow \frac{p^2}{x^2} - x^2 = a^2$$

$$\Rightarrow \frac{p^2}{x^2} = (x^2 + a^2)$$

$$\Rightarrow p^2 = x^2(x^2 + a^2)$$

$$\Rightarrow p = x(x^2 + a^2)^{\frac{1}{2}}$$

$$\Rightarrow y^2 - \frac{y^2}{q^2} = a^2$$

$$\Rightarrow y^2 - a^2 = \frac{y^2}{q^2}$$

$$\Rightarrow q^2 = \frac{y^2}{y^2 - a^2}$$

$$\Rightarrow q = \frac{y}{(y^2 - a^2)^{\frac{1}{2}}}$$

we know that dz = pdx + qdy

$$\Rightarrow dz = x(x^2 + a^2)^{\frac{1}{2}} dx + \frac{ydy}{(y^2 - a^2)^{\frac{1}{2}}}$$

Integrating both sides, we get

$$\int dz = \int x(x^2 + a^2)^{\frac{1}{2}} dx + \int \frac{y}{(y^2 - a^2)^{\frac{1}{2}}}$$

$\therefore \quad z = \frac{1}{3}(x^2 + a^2)^{\frac{3}{2}} + (y^2 - a^2)^{\frac{1}{2}} + b$ is the required solution.

55. Ans: (b)

Sol: Given $pqz = p^2(xq + p^2) + q^2(yp + q^2)$

dividing both sides by pq, we get

$$z = \frac{p}{q}(xq + p^2) + \frac{q}{p}(yp + q^2)$$

$$\Rightarrow z = px + \frac{p^3}{q} + qy + \frac{q^3}{p}$$

$$\Rightarrow z = px + qy + \left(\frac{p^3}{q} + \frac{q^3}{p}\right)$$

$\therefore$ The solution is

$$z = ax + by + \left(\frac{a^3}{b} + \frac{b^3}{a}\right)$$

56. Ans: (d)

Sol: Given $\frac{\partial^2 u}{\partial x^2} = 25\frac{\partial^2 u}{\partial t^2}$

(or) $\frac{\partial^2 u}{\partial t^2} = \frac{1}{25}\frac{\partial^2 u}{\partial x^2}$ .... (1)

with $u(0) = 3x$ .... (2)

and $\frac{\partial u(0)}{\partial t} = 3$ .... (3)

If the given one dimensional wave equation is of the form $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial u^2}{\partial x^2}$, $-\infty < x < \infty$, t > 0 and c > 0, satisfying the conditions u(x, 0) = f(x) and $\frac{\partial u(x,0)}{\partial t} = g(x)$, where f(x) & g(x) are given functions representing the initial displacement and initial velocity, respectively then its general solution is given by

$$u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

Comparing the given problem with above general problem, we have

$c = \frac{1}{5}$, f(x) = 3x, g(x) = 3

Now,

$$u(1, 1) = \frac{1}{2}\left[f\left(1-\frac{1}{5}\right)+f\left(1+\frac{1}{5}\right)\right]+\frac{1}{2\left(\frac{1}{5}\right)}\int_{1-\frac{1}{5}}^{1+\frac{1}{5}} 3\ ds$$

$$\Rightarrow u(1, 1) = \frac{1}{2}\left[3\left(\frac{4}{5}\right)+3\left(\frac{6}{5}\right)\right]+\frac{5}{2}(3)\ (s)_{\frac{4}{5}}^{\frac{6}{5}}$$

$$\Rightarrow u(1, 1) = \frac{1}{2}\left[\frac{3}{5}\times(4+6)\right]+\frac{15}{2}\left[\frac{6}{5}-\frac{4}{5}\right]$$

$$\Rightarrow u(1, 1) = 3+\frac{15}{2}\left(\frac{2}{5}\right)$$

$\therefore$ u(1, 1) = 6

57. Ans: (a)

Sol: Given that $\frac{\partial u}{\partial t} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}$ .... (1)

$\left(\because \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}\right)$

with B.C's : u(0, t) = 0 ($\because$ u(0, t) = 0)

u(1, t) = 0 ($\because u(l, t) = 0$)

and I.C's : u(x, 0) = sin(πx) .... (2)

($\because$ u(x, 0) = f(x))

Now, the solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n.\sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n^2\pi^2c^2}{\ell^2}\right)t}$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x).e^{-n^2 t} \text{ .... (3)}$$

where $a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

Put t = 0 in (3), we get

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$\Rightarrow \sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$\Rightarrow$ sin(πx) = a_1 sin(πx) + a_2 sin(2πx) + ...

Comparing coefficients of sin on both sides of above, we get

$a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 0$, ...(4)

$\therefore$ The solution of (1) with (2) from (3) and (4) is

$$u(x, t) = \sin(\pi x).\ e^{-\left[\frac{\pi^2}{1}.\left(\frac{1}{\pi^2}\right)\right].t} = e^{-t}\sin(\pi x)$$

58. Ans: (c)

Sol: Given $u_t = \left(\sqrt{2}\right)^2 u_{xx}$ ....(1)

($\because u_t = c^2 u_{xx}$)

with B.C's: u(0, t) = 0

($\because$ u(0, t) = 0)

u(π, t) = 0 ($\because u(\ell, t) = 0$)

and I.C: u(x, 0) = sin(x) ....(2)

($\because$ u(x, 0) = f(x))

The solution of (1) is given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n.\sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n\pi c}{\ell}\right)^2 t}$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} a_n.\sin(nx)\ e^{-2n^2 t} \text{ .... (3)}$$

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} a_n.\sin(nx) \qquad \text{(for t = 0)}$$

$\Rightarrow$ sin(x) = $a_1$sinx + $a_2$sin(2x) + ...

$\Rightarrow a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, ...$ (4)

$\therefore$ The solution of (1) with (2) from (3) and (4) is

$u(x, t) = a_1 \sin(x).\, e^{-2t}$

Hence, $u(\pi/2, \log 5) = 1.\sin(\frac{\pi}{2}).\, e^{-2\log 5}$

$= 5^{-2} = 0.04$

59. Ans: (b)

Sol: Given $u_{tt} = 2^2 u_{xx}$ .... (1)

$(\because u_{tt} = c^2 u_{xx})$

with B.C's : $u(0,t) = 0$ $(\because u(0,t) = 0)$

$u(\pi,t) = 0$ $(\because u(\ell,t) = 0)$

and I.C's : $u(x, 0) = 0$

$(\because u(x,0) = 0)$

$\frac{\partial}{\partial t}u(x,0) = 2\sin(x)$ $\left(\because \frac{\partial u}{\partial t}(x,0) = g(x)\right)$

The solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{\ell}\right).\sin\left(\frac{n\pi ct}{\ell}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx).\sin(n2t) .... (2)$$

$$\Rightarrow \frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx).\cos(2nt).2n$$

$$\Rightarrow \frac{\partial}{\partial t}u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)(2n) \quad \text{for } t = 0$$

$\Rightarrow \sin(x) = b_1\sin(x).\, 2 + b_2.\, 2(2).\, \sin(2x) ...$

$\Rightarrow b_1 = \frac{1}{2}, b_2 = 0, b_3 = 0,$ .... (3)

$\therefore$ The solution of (1) with given conditions from (2) and (3) is given by

$u(x, t) = b_1.\sin(x).\sin(2t) + 0 + 0.......$

$= \frac{1}{2}\sin(x).\sin(2t)$

Hence, $u\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \frac{1}{2}\sin\left(\frac{\pi}{3}\right).\sin\left(\frac{2\pi}{6}\right)$

$= \frac{1}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} = \frac{3}{8}$

60. Ans: (a)

Sol: Given $u_{tt} = u_{xx}$ ....(1) $(\because u_{tt} = c^2 u_{xx})$

with B.C's: $u(0, t) = 0$

$(\because u(0,t) = 0)$

$u(\pi,t) = 0$ $(\because u(\ell,t) = 0)$

and I.C's: $u(x,0) = 2\sin(x)$ ....(2)

$(\because u(x,0) = f(x))$

$\frac{\partial}{\partial t}u(x, 0) = 0$

$\left(\because \frac{\partial u}{\partial t}(x,0) = 0\right)$

Now, the solution of the wave equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right).\cos\left(\frac{n\pi ct}{\ell}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx).\cos(nt) ......(3)$$

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{for } t = 0$$

$\Rightarrow 2\sin(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$

$\Rightarrow 2\sin(x) = a_1 .\sin x + a_2 \sin(2x) + \ldots\ldots\ldots$

$\Rightarrow a_1 = 2, a_2 = 0. a_3 = 0$ ……….(4)

$\therefore$ The solution of (1) with (2) from (3) and (4) is given by

$u(x, t) = a_1.\sin(x)\cos(t) = 2.\sin(x)\cos(t)$

61. Ans: (a)

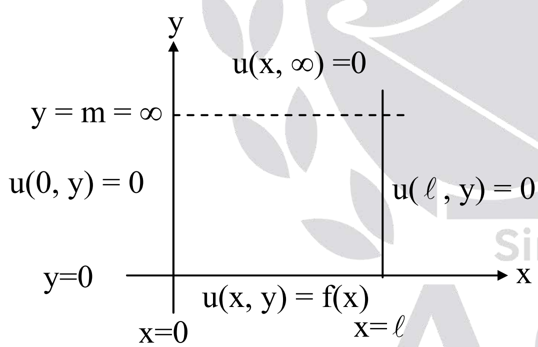
Sol: Given $u_{xx} + u_{yy} = 0 \rightarrow (1)$

$u(0, y) = 0 \quad \rightarrow (2) \quad \forall\, y > 0$

$u(l, y) = 0 \quad \rightarrow (3) \quad \forall\, y > 0$

$u(x, 0) = f(x) = u_0 \rightarrow (4) \quad 0 < x\, l$

$u(x, \infty) = 0 \quad \rightarrow (5) \quad 0 < x < l$



The G.S of (1) satisfying above all boundary conditions is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n\pi y}{\ell}\right)} \rightarrow (6)$$

where $b_n = \frac{2}{\ell}\int_0^{\ell} f(x).\sin\left(\frac{n\pi x}{\ell}\right)dx$

Now, $b_n = \frac{2}{\ell}\int_0^{\ell} u_0.\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$\Rightarrow b_n = \frac{2u_0}{\ell}\left[\frac{-\cos\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}\right]_0^{\ell}$$

$$\Rightarrow b_n = \frac{2u_0}{n\pi}\left[1 - \cos(n\pi)\right]$$

$$\Rightarrow b_n = \frac{2u_0}{n\pi}\left[1 - (-1)^n\right] \rightarrow (7)$$

Using (7) (i.e. the value of b_n in (6), the required solution is), the equation (6) becomes

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}\left[1-(-1)^n\right].\sin\left(\frac{n\pi x}{\ell}.\right)e^{\left(\frac{-2\pi y}{\ell}\right)}$$

(or)

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2u_0}{(2n-1)\pi}(2).\sin\left[\frac{(2n-1)\pi x}{\ell}\right].e^{-\left[\frac{(2n-1)\pi y}{\ell}\right]}$$

62. Ans: (a)

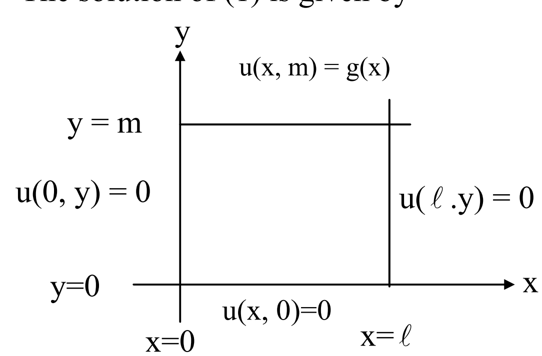
Sol: Given $u_{xx} + u_{yy} = 0$ …..(1)

with B.C's

$\left.\begin{array}{l} u(0,y) = 0 \\ u(\ell,y) = 0 \end{array}\right\} \quad 0 \le y \le m$

$\left.\begin{array}{l} u(x,0) = 0 \\ u(x,a) = \sin\frac{n\pi x}{\ell} \end{array}\right\} \quad 0 \le x \le \ell$

The solution of (1) is given by



$$u(x, y) = \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{\ell}\right).\sinh\left(\frac{n\pi y}{\ell}\right)$$

where

$$b_n = \frac{2}{\ell \sinh\left(\frac{n\pi m}{\ell}\right)} \int_0^{\ell} g(x).\sin\left(\frac{n\pi x}{\ell}\right) dx$$

Now,

$$b_n = \frac{2}{\ell.\sinh\left(\frac{n\pi m}{\ell}\right)} \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right).\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\ell.\sinh\frac{n\pi m}{\ell}} \int_0^{\ell} \frac{1+\cos\frac{2n\pi x}{\ell}}{2} dx$$

$$= \frac{2}{\ell.\sinh\left(\frac{n\pi m}{\ell}\right)} \left[\frac{x}{2} + \frac{1}{2}\frac{\sin\left(\frac{2n\pi x}{\ell}\right)}{\frac{2n\pi}{\ell}}\right]_0^{\ell}$$

$$= \frac{1}{\ell.\sin h\left(\frac{n\pi m}{\ell}\right)} \left[(\ell+0)-(0+0)\right]$$

$$b_n = \frac{1}{\sinh\left(\frac{n\pi m}{\ell}\right)} = \frac{1}{\sinh\frac{n\pi a}{\ell}} \quad \text{for } m = a$$

$\therefore$ The solution of (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{1}{\sinh\left(\frac{n\pi a}{\ell}\right)}.\sin\left(\frac{n\pi x}{\ell}\right).\sinh\frac{n\pi y}{\ell}$$

63. Ans: (d)

Sol: Given that $L\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$

$$\Rightarrow \quad L\{\cos 4t\} = \frac{s}{(s^2+16)}$$

$$\therefore \quad L\{e^{-2t}\cos 4t\} = \frac{s+2}{(s+2)^2+16}$$

(from first shifting theorem)

64. Ans: 0.5

Sol: Given that $f(t) = 2t^2e^{-t}$

$$\because \ L\{t^n\} = \frac{n!}{s^{n+1}}, n \in N$$

$$\text{Now, } L\{t^2\} = \frac{2!}{s^{2+1}}$$

$$= \frac{2}{s^3}$$

$$L(2t^2) = \frac{4}{s^3}$$

By 1st shifting theorem, we have

$$L(2t^2e^{-t}) = F(s) = \frac{4}{(s+1)^3}$$

$$\text{Then } F(1) = \frac{4}{(1+1)^3}$$

$$= \frac{4}{8} = \frac{1}{2} = 0.5$$

65. Ans: (c)

Sol: $L\left\{\frac{\sin at}{t}\right\}$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} \frac{a}{s^2+a^2} ds$$

$$= \left(a.\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right)_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{a} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

66. Ans: (d)

Sol: $L\left\{\int_0^t e^{-t}\sin t\,dt\right\}$

$L\{\sin t\} = \dfrac{1}{s^2+1}$

$L\{e^{-t}\sin t\} = \dfrac{1}{(s+1)^2+1}$

$L\left\{\int_0^t e^{-t}\sin t dt\right\} = \dfrac{1}{s}\left(\dfrac{1}{(s+1)^2+1}\right)$

67. Ans: (b)

Sol: L {t e^{-t} sint}

$L\{\sin t\} = \dfrac{1}{s^2+1}$

$L\{t\sin t\} = (-1)\dfrac{d}{ds}\left[\dfrac{1}{s^2+1}\right] = (-1)\dfrac{(-1)2s}{(s^2+1)^2}$

$= \dfrac{2s}{(s^2+1)^2}$

$L\{e^{-t}\,t\sin t\} = \dfrac{2(s+1)}{[(s+1)^2+1]^2}$

68. Ans: (c)

Sol: $L(\sin t) = \dfrac{1}{s^2+1}$

$L\left(\dfrac{\sin t}{t}\right) = \int_s^{\infty}\dfrac{1}{s^2+1}ds$

$= [\tan^{-1} s]_s^{\infty}$

$= \dfrac{\pi}{2} - \tan^{-1} s$

$= \cot^{-1} s$

$L\{f^1(t)\} = s.\,L\{f(t)\} - f(0)$

$\Rightarrow L\{f^1(t)\} = s.L\left\{\dfrac{\sin t}{t}\right\} - f(0)$

$L\{f^1(t)\} = s\cot^{-1} s - f(0) = s\cot^{-1} s - 1$

69. Ans: (a)

Sol: $f(t) = \begin{cases} t, & 0 < t \le 1 \\ 0, & 1 < t < 2 \end{cases}$

$\because$ f(t) is periodic function with period 2

$L\{f(t)\} = \dfrac{1}{1-e^{-2s}}\int_0^2 e^{-st}\, f(t)\, dt$

$= \dfrac{1}{1-e^{-2s}}\int_0^1 t.e^{-st}\, dt$

$= \dfrac{1}{1-e^{-2s}}\left[t.\left(\dfrac{e^{-st}}{-s}\right) - 1\left(\dfrac{e^{-st}}{s^2}\right)\right]_0^1$

$= \dfrac{1}{1-e^{-2s}}\left[\left(\dfrac{e^{-s}}{-s}\right) - \left(\dfrac{e^{-s}}{s^2}\right) + \dfrac{1}{s^2}\right]$

70. Ans: (d)

Sol: $L\{U(t-1)\,(t^2-2t)\}$

Use Laplace Transform: $L\{U(t-c)\,f(t)\} = e^{-cs}\,L\{f(t+c)\}$

For $(t^2-2t)\,U(t-1)$: $f(t) = (t^2-2t)$, c =1

$= e^{-1-s}\,L\{((t+1)^2 - 2(t+1))\}$

$L\{((t+1)^2 - 2(t+1))\}\ \dfrac{2}{s^3} - \dfrac{1}{s}$

$= e^{-1-s}\left(\dfrac{2}{s^3} - \dfrac{1}{s}\right)$

$= e^{-s}\left(\dfrac{2}{s^3} - \dfrac{1}{s}\right)$

71. Ans: (b)

Sol: $L\{e^t\} = \dfrac{1}{s-1}$

$e^t\, u(t-4) = [e^{t-4}.\, u(t-4)]e^4$

By second shifting property

$L[e^t.u(t-4)] = e^4.\, L[e^{t-4}.\, u(t-4)]$

$$= e^4.\left(\frac{e^{-4s}}{s-1}\right) = \frac{e^{4-4s}}{s-1} = \frac{e^{-4(s-1)}}{s-1}$$

72. Ans: (b)

Sol: $f(t) = 2\sqrt{\dfrac{t}{\pi}}$

$\Rightarrow f'(t) = \dfrac{1}{\sqrt{\pi t}} = g(t)$

$\therefore L\{g(t)\} = L\{f'(t)\}$

$= s\,\bar{f}(s) - f(0)$

$= s.s^{-3/2} - 0 = s^{-1/2}$

73. Ans: (d)

Sol: Expand

$$\frac{s+9}{s^2+6s+13} = \frac{s+3}{(s+3)^2+4} + 6\frac{1}{(s+3)^2+4}$$

$$= L^{-1}\left\{\frac{s+3}{(s+3)^2+4} + 6\frac{1}{(s+3)^2+4}\right\}$$

Use the linearity property of inverse laplace transform

$$= L^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} + 6L^{-1}\left\{\frac{1}{(s+3)^2+4}\right\}$$

$$L^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} = e^{-3t}\cos(2t)$$

$$L^{-1}\left\{\frac{1}{(s+3)^2+4}\right\} = \frac{1}{2}e^{-3t}\sin(2t)$$

$$= e^{-3t}\cos(2t) + 6e^{-3t}\frac{1}{2}\sin(2t)$$

$$= e^{-3t}\cos(2t) + 3e^{-3t}\sin(2t)$$

74. Ans: (a)

Sol: Given $H(s) = \dfrac{s+3}{s^2+2s+1}$

$$= \frac{(s+1)+2}{(s+1)^2}$$

Now, $L^{-1}\{H(s)\} = L^{-1}\left\{\dfrac{1}{(s+1)} + \dfrac{2}{(s+1)^2}\right\}$

$$= e^{-t}\, L^{-1}\left\{\frac{1}{s} + \frac{2}{s^2}\right\}$$

(By First shifting property)

$= e^{-t}\,(1+2t)$

75. Ans: (a)

Sol: $L^{-1}\left\{\dfrac{s+5}{(s+1)(s+3)}\right\}$

Take the partial fraction of

$$\frac{s+5}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{1}{s+3}$$

$$= L^{-1}\left\{\frac{2}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+3}\right\}$$

$\therefore\ L^{-1}\left\{\dfrac{2}{s+1}\right\} = 2e^{-t}$

$\therefore\ L^{-1}\left\{\dfrac{1}{s+3}\right\} = e^{-3t}$

$= 2e^{-t} - e^{-3t}$

76. Ans: (c)

Sol: $L^{-1}\left\{\dfrac{1}{s^2(s+1)}\right\} = L^{-1}\left\{\dfrac{-1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s+1}\right\}$

(By Partial fractions)

$= -1 + t + e^{-t}$

77. Ans: (a)

Sol: $L^{-1}\left(\dfrac{1}{s+3}\right) = e^{-3t}$

$L^{-1}\left(\dfrac{e^{-4s}}{s+3}\right) = e^{-3(t-4)}u(t-4)$

By 2nd shifting property

$$\begin{cases} e^{-3(t-4)} & \text{when } t \ge 4 \\ 0 & \text{other wise} \end{cases}$$

78. Ans: (b)

Sol: Let $L^{-1}\left(\log\left(\dfrac{s-a}{s-b}\right)\right) = f(t)$

$\Rightarrow L[f(t)] = \log\left(\dfrac{s-a}{s-b}\right)$

$= \log(s-a) - \log(s-b)$

$\Rightarrow L[t.f(t)] = (-1)\dfrac{d}{ds}\left(\log(s-a) - \log(s-b)\right)$

$= \dfrac{1}{s-b} - \dfrac{1}{s-a}$

$t.\ f(t) = L^{-1}\left(\dfrac{1}{s-b} - \dfrac{1}{s-a}\right)$

$= e^{bt} - e^{at}$

$\therefore f(t) = \dfrac{e^{bt} - e^{at}}{t}$

79. Ans: (b)

Sol: Let $L\{f(t)\} = \bar{f}(s) = \dfrac{s+3}{(s+1)(s+2)}$

Then $f(t) = L^{-1}\left\{\dfrac{s+3}{(s+1)(s+2)}\right\}$

$\Rightarrow f(t) = L^{-1}\left\{\dfrac{2}{s+1} - \dfrac{1}{s+2}\right\}$

$\Rightarrow f(t) = 2L^{-1}\left\{\dfrac{1}{s+1}\right\} - L^{-1}\left\{\dfrac{1}{s+2}\right\}$

$\Rightarrow f(t) = 2e^{-t} - e^{-2t}$

$\therefore f(0) = 2 - 1 = 1$

80. Ans: (a)

Sol: Given $\dfrac{d^2y}{dt^2} - y = 1,$

$y''(t) - y(t) = 1$

Taking laplace transforms both sides

$L[y''(t) - y(t)] = L(1)$

$s^2\bar{y}(s) - sy(0) - y'(0) - \bar{y}(s) = \dfrac{1}{s}$

$s^2\bar{y}(s) - \bar{y}(s) = \dfrac{1}{s}$

$(s^2 - 1)\bar{y}(s) = \dfrac{1}{s}$

$\therefore \bar{y}(s) = \dfrac{1}{s(s^2-1)}$

$= \dfrac{1}{s(s+1)(s-1)}$

81. Ans: (a)

Sol: Given that $y''(t) + 2y'(t) + y(t) = 0$ …..(1)

and $y(0) = 0, \quad y'(0) = 1$

Applying Laplace transform on both sides of (1), we get

$L\{y''(t)\} + 2\,L\{y'(t)\} + L\{y(t)\} = L\,\{0\}$

$\Rightarrow [s^2 L\{y(t)\} - s\,y(0) - y'(0)]$

$+ 2\,[s\,L\{y(t)\} - y(0)\} + L\{y(t)\} = 0$

$\Rightarrow (s^2 + 2s + 1)\,L\{y(t)\} - (s)\,(0) - 1 - (2)\,(0) = 0$

$$\Rightarrow L\{y(t)\} = \frac{1}{s^2 + 2s + 1}$$

$$= \frac{1}{(s+1)^2}$$

$$\Rightarrow \quad y(t) = L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= e^{-t}\ L\left\{\frac{1}{s^2}\right\}$$

$\therefore \quad y(t) = e^{-t}\,t\,u(t)$

82. Ans: (c)

Sol: Given that $\dfrac{d^2f}{dt^2} + f = 0$ …. (1)

and $f(0) = 0, \quad f'(0) = 4$

Applying Laplace transform on both sides of (1), we get

$$L\{f^{11}(t)\} + L\{f(t)\} = L(0)$$

$$\Rightarrow s^2\bar{f}(s) - sf(0) - f^1(0) + \bar{f}(s) = 0$$

$$\Rightarrow (s^2 + 1)\,\bar{f}(s) - (s)(0) - 4 = 0$$

$$\Rightarrow \bar{f}(s) = \frac{4}{s^2 + 1}$$

$$\therefore \ f(t) = L^{-1}\{\bar{f}(s)\}$$

$$= L^{-1}\left\{\frac{4}{s^2+1}\right\} = 4\sin t$$

Hence, $L\{f(t)\} = 4L\{\sin t\} = \dfrac{4}{s^2 + 1}$

Chapter 5 Complex Variables

Augustin-louis Cauchy
(1789 –1857)

01. Ans: (d)

Sol: Let $u+iv = w = f(z) = e^{-y}\cos x + ie^{-y}\sin x$

Then $u = e^{-y}\cos x$ and $v = e^{-y}\sin x$

$\Rightarrow u_x = e^{-y}(-\sin x)$, $u_y = -e^{-y}\cos x$, $v_x = e^{-y}\cos x$ and $v_y = -e^{-y}\sin x$

Here, C-R equation , $u_x = v_y$ and $v_x = -u_y$ are satisfied at every point and also u, v, u_x, u_y, v_x, v_y are continuous at every point.

$\Rightarrow$ f(z) = u+iv is differentiable at every point.

$\Rightarrow$ f(z) = u+iv is analytic at every point

$\therefore$ f(z) = u+iv is everywhere analytic

Hence, f(z) is also an entire function.

02. Ans: (a)

Sol: Let $u + iv = f(z) = z\,\text{Im}(z) = (x + iy)\,y$

Then $u + iv = f(z) = xy + iy^2$

$\Rightarrow u = xy$ and $v = y^2$

$\Rightarrow u_x = y$, $u_y = x$, $v_x = 0$ and $v_y = 2y$

Here, $u_x = v_y$ and $v_x = -u_y$ only at one point origin. i.e., C.R equations $u_x = v_y$ and $v_x = -u_y$ are satisfied only at origin.

Further u, v, v_x, v_y, u_x, u_y are also continuous at origin.

$\therefore$ f(z) = z Im(z) is differentiable only at origin (0,0).

03. Ans: (d)

Sol: Let $u+iv = f(z) = \bar{z} = x - iy$

Then $u = x$ and $v = -y$

$\Rightarrow u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$

Here, one of the C-R equation $u_x = v_y$ is not satisfied at any point.

$\Rightarrow$ f(z) is not differentiable at any point

$\Rightarrow$ f(z) is not analytic at any point.

$\therefore$ f(z) is nowhere analytic

04. Ans: (d)

Sol: $\frac{1}{1-z}$ is not analytic at z = 1,

$e^{\frac{1}{z}}$ is not analytic at z = 0,

*l*n(z) is not analytic at z = 0,

cos(z) is analytic at every point over the entire complex plane.

$\therefore$ Option (d) is correct.

05. Ans: (b)

Sol: Let $u + iv = f(z) = (x^2 + ay^2) + i\,bxy$

Then $u = x^2 + ay^2$ and $v = bxy$

$\Rightarrow u_x = 2x$, $v_x = by$, $u_y = 2ay$, $v_y = bx$

Consider C-R equations,

$u_x = v_y$ & $u_y = -v_x$ ($\because$ f(z) = u + iv is analytic)

$\Rightarrow 2x = bx$ & $2ay = -by$

$\therefore b = 2$, $a = -1$

Augustin-Louis Cauchy was a French mathematician. "More concepts and theorems have been named for Cauchy than for any other mathematician". Cauchy was a prolific writer; he wrote approximately eight hundred research articles and almost single handedly founded complex analysis.

06. Ans: (a)

Sol: Given u = sinhx cosy

$\Rightarrow u_x = \cosh x.\cos y$ and $u_y = -\sinh x.\ \sin y$

Consider $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$\Rightarrow dv = (-u_y)dx + (u_x)dy$

($\because u_x = v_y$ and $v_x = -u_y$)

$\Rightarrow dv = (\sinh x.\sin y)\ dx + (\cosh x.\cos y)dy$

$\Rightarrow dv = d\ (\cosh x.\sin y)$

$\Rightarrow \int dv = \int d(\cosh x.\sin y) + k$, where k is a real integral constant.

$\therefore$ v(x, y) = cosh(x) . sin(y) + k is a required harmonic conjugate function.

07. Ans: (b)

Sol: Let $v(x,y) = 4xy - 2x^2 + 2y^2$

Then $v_x = 4y - 4x$ and $v_y = 4x + 4y$

Consider $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$\Rightarrow du = (v_y)dx + (-v_x)dy$

($\because u_x = v_y$ and $v_x = -u_y$)

$\Rightarrow du = (4x+4y)dx + (4x-4y)dy$

$\Rightarrow du = 4xdx - 4ydy + 4(ydx+xdy)$

$\Rightarrow \int du = \int (4x)\ dx + \int(-4y)\ dy + \int 4\ d(xy) + k$

$\Rightarrow u = 2x^2 - 2y^2 + 4xy + k$, where k is a real integral constant.

$\therefore$ $u\ (x,y) = 2x^2 - 2y^2 + 4xy + k$ is a required function

08. Ans: (c)

Sol: Given u = log r

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r} \text{ and } \frac{\partial u}{\partial \theta} = 0$$

Consider $dv = \frac{\partial v}{\partial \theta}d\theta + \frac{\partial v}{\partial r}dr$

$$\Rightarrow dv = \frac{\partial v}{\partial \theta}d\theta + \frac{\partial v}{\partial r}dr$$

$$\Rightarrow dv = \left(r\frac{\partial u}{\partial r}\right)d\theta + \left(\frac{-1}{r}\frac{\partial u}{\partial \theta}\right)dr$$

$$\left(\because u_r = \frac{1}{r}v_\theta \ \&\ v_r = \frac{-1}{r}u_\theta\right)$$

$$\Rightarrow \int dv = \int\left(r\times\frac{1}{r}\right)d\theta + \int\left(-\frac{1}{r}\times 0\right)dr = \theta + c$$

$\therefore$ $v(r, \theta) = \theta + c$

09. Ans: (a)

Sol: Given $u = x^3 - 4xy - 3xy^2$

$\Rightarrow u_x = 3x^2 - 4y - 3y^2$ and $u_y = -4x - 6xy$

Consider $f'(z) = u_x - iu_y$ for analytic function f(z) .

$\Rightarrow f'(z) = (3x^2 - 4y - 3y^2) - i(-4x - 6xy)$

$\Rightarrow f'(z) = 3z^2 - 0 - 0 + i4z$ ($\because$ 'x' by 'z' and 'y' by '0')

$\Rightarrow \int f'(z)\ dz = \int 3z^2 dz + i\ 4\int z\ dz + c$, where $c = c_1 + ic_2$

$$\Rightarrow f(z) = \frac{3z^3}{3} + i4\frac{z^2}{2} + c$$

$\therefore$ $f(z) = z^3 + 2iz^2 + c$ is a required analytic function.

10. Ans: (a)

Sol: Given that $v = e^x[y \cos y + x \sin y]$

$\Rightarrow v_x = e^x[0 + \sin y] + e^x[y \cos y + x \sin y]$

and $v_y = e^x[-y \sin y + \cos y + x \cos y]$

Consider $f^1(z) = u_x - iu_y$

$\Rightarrow f^1(z) = v_y + i v_x$ ($\because u_x = v_y$ & $v_x = -u_y$)

$\Rightarrow f^1(z) = e^x[-y \sin y + \cos y + x \cos y] + i e^x[\sin y + y\cos y + x\sin y]$

$\Rightarrow \int f^1(z) = ze^z - e^z + e^z + c$

$\therefore$ $f(z) = z e^z + c$, $c = c_1 + ic_2$ is a required analytic function.

11. Ans: (a)

Sol: Given $3u+2v = y^2-x^2+16xy$..........(1)

Differentiating (1) partially w.r.t. x, we get

$3u_x+2v_x = -2x+16y$ ....(2)

Differentiating (1) partially w.r.t. 'y' , we get

$3u_y+2v_y = 2y + 16x$

$\Rightarrow 3(-v_x) + 2(u_x) = 2y + 16x$ ....(3)
($\because u_x = v_y$ & $v_x = -u_y$)

Solving (2) and (3), we get

$u_x = 2x+4y$ and $v_x = 2y-4x$

Consider $f'(z) = u_x-iu_y$

$\Rightarrow f'(z) = u_x + iv_x$ ($\because v_x = -u_y$)

$\Rightarrow f'(z) = (2x+4y) + i(2y-4x)$

$\Rightarrow f'(z) = 2z + i(-4z)$

($\because$ x by z and y by 0)

$\Rightarrow \int f'(z)\,dz = \int (2z-i\,4z)\,dz + c$,

where $c = c_i + ic_2$

$\therefore$ $f(z) = z^2-i\,2z^2 + c$ is a required analytic function

12. Ans: (a)

Sol: Let $I = \int_C (x^2 + iy^2)dz$, where C : y = x from (0,0) to (1,1)

Then $I = \int_{z=(0,0)}^{(1,1)} (x^2 + iy^2)(dx + idy)$

$\Rightarrow I = \int_{x=0}^{1} (x^2 + ix^2)(dx + idx)$ ($\because$ y =x)

$\Rightarrow I = \int_{x=0}^{1} (1+i)^2 x^2 dx$

$\Rightarrow I = (2i)\left(\frac{x^3}{3}\right)_0^1$

$\therefore$ $I = \frac{2i}{3}$

13. Ans: (c)

Sol: Let $I = \int_C z\,dz$, where c: $y = x^3$ from (0,0) to $1 + i$

Then $I = \int_{z=0}^{1+i} z\,dz$

$\Rightarrow I = \left(\frac{z^2}{2}\right)_0^{1+i}$ ($\because$ (z) is analytic function)

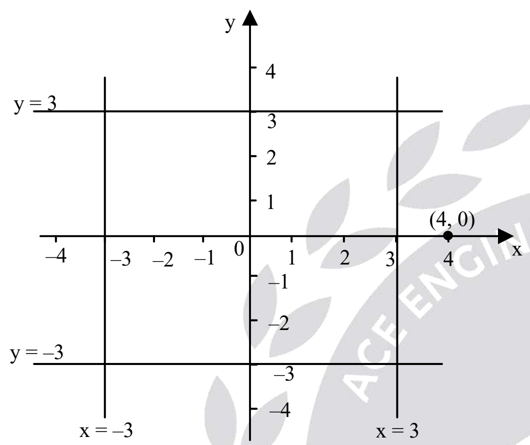
$\Rightarrow I = \frac{1}{2}(1+i)^2 - \frac{1}{2}(0)$

$\therefore$ $I = i$

14. Ans: (c)

Sol: Let $f(z)=\dfrac{e^{z}+\cos(z)}{(z-4)^{2}}$

Then the singular point of f(z) is given by equating the denominator to zero



i.e $(z-4)^{2}=0$

$\Rightarrow$ z = 4 is a singular point of f(z)

$\Rightarrow$ z = 4 lies outside the given region.

$\therefore$ By Cauchy's integral theorem, we have

$$\oint_{C} f(z)\,dz=0$$

15. Ans: (d)

Sol: Let $f(z)=\dfrac{\sin\left(\dfrac{\pi z}{2}\right)}{z-1}$

Then the singular point of the function f(z) is given by equating the denominator to zero.

i.e $z-1=0$

$\Rightarrow$ z = 1 is a singular point of f(z)

$\Rightarrow$ z = 1 lies inside the circle $|z-2|=4$

So, we can evaluate it by using Cauchy's integral formula.

Now, $\oint_{c} f(z)\,dz=\oint_{c}\dfrac{\sin\left(\dfrac{\pi z}{2}\right)}{(z-1)}\,dz$

$$=2\pi i\left[\sin\left(\frac{\pi z}{2}\right)\right]_{z=1}$$

$$\Rightarrow \oint_{c} f(z)\,dz=2\pi i\left[\sin\left(\frac{\pi}{2}\right)\right]$$

$$\therefore \oint_{c} f(z)\,dz=2\pi i$$

16. Ans: (d)

Sol: Let $I=\oint_{c}\dfrac{\cosh(3z)}{2z}dz$, where C: $|z|=1$

Then $I=\oint_{c}\dfrac{\left(\dfrac{\cosh(3z)}{2}\right)}{[z-0]}dz$

$$\Rightarrow I=2\pi i\left(\frac{\cosh(3z)}{2}\right)_{z=0}$$

$\Rightarrow$ I = πi[cosh(0)]

$$\left(\because \cosh(0)=\frac{e^{0}+e^{0}}{2}=1\right)$$

$\therefore$ I = πi

17. Ans: (d)

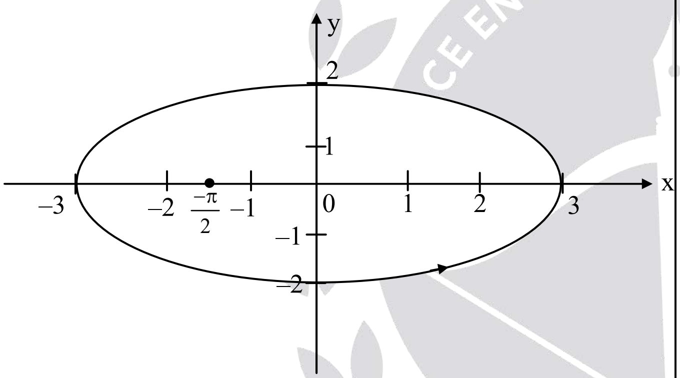
Sol: Let $f(z)=\dfrac{\cos(z)+ie^{iz}}{\left(z+\frac{\pi}{2}\right)^3}$

Then the singular point of f(z) is given by

$\left(z+\frac{\pi}{2}\right)^3=0$

$\Rightarrow z=\frac{-\pi}{2}$ is a singular point of f(z)

$\Rightarrow z=\frac{-\pi}{2}$ lies inside the given region $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$



So, we evaluate the given integral by using Cauchy's integral formula of derivatives.

Now,

$$\oint_C f(z)dz=\oint_C \frac{\cos(z)+ie^{iz}}{\left[z-\left(\frac{-\pi}{2}\right)\right]^{2+1}}dz=\frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\left(\cos z+ie^{iz}\right)\right]_{z=\frac{-\pi}{2}}$$

$$\Rightarrow \oint_C f(z)dz=\pi i\left[\frac{d}{dz}\left\{-\sin z-e^{iz}\right\}\right]_{z=\frac{-\pi}{2}}$$

$$\Rightarrow \oint_C f(z)dz=\pi i\left[-\cos(z)-ie^{iz}\right]_{z=\frac{-\pi}{2}}$$

$$\Rightarrow \oint_C f(z)dz=\pi i\left[-\cos\left(\frac{-\pi}{2}\right)-ie^{-i\pi/2}\right]$$

$$\therefore \oint_C f(z)dz=\pi i\left[0-i(-i)\right]=-\pi i$$

18. Ans: (a)

Sol: Let $f(z)=\dfrac{\cos(\pi z^2)}{(z-1)(z-3)}$

Then the singular points of f(z) are given by (z–1)(z–3) = 0,

$\Rightarrow$ z = 1 and z = 3 are singular points.

$\Rightarrow$ only one singular point z = 1 lies inside the given region $|z|=2$

So, we evaluate the given integral by using Cauchy's integral formula.

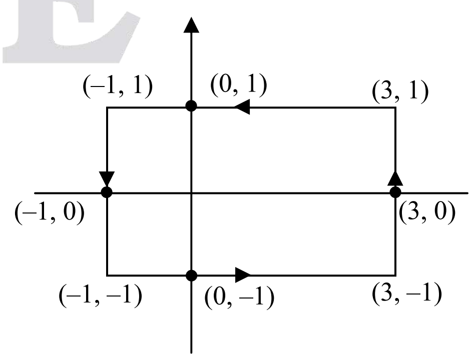
$$\text{Now, } \oint_C f(z)dz=\oint_C \frac{\left(\frac{\cos \pi z^2}{z-3}\right)}{(z-1)}dz$$

$$\Rightarrow \oint_C f(z)dz=2\pi i\left[\frac{\cos(\pi z^2)}{z-3}\right]_{z=1}$$

$$\therefore \oint_C f(z)dz=2\pi i\left[\frac{\cos(\pi)}{1-3}\right]=2\pi i\left(\frac{-1}{-2}\right)=\pi i$$

19. Ans: (b)

Sol:



Let $f(z)=\dfrac{1}{z^2(z-4)}$

z = 0, 4 are singular points

z = 0 lies inside the rectangle

z = 4 lies outside the rectangle

$$\int_c \frac{dz}{z^2(z-4)} = \int_c \frac{\left(\frac{1}{z-4}\right)}{(z-0)^{1+1}} dz$$

$$= 2\pi j \frac{F'(0)}{1!}$$

$$\left\{F(z) = \frac{1}{(z-4)},\ F'(z) = \frac{-1}{(z-4)^2}\right.$$

$$\left.\therefore\ F'(0) = \frac{-1}{(0-4)^2} = \frac{1}{16}\right\}$$

$$= 2\pi j\ F'(0)$$

$$= 2\pi j\left(\frac{-1}{16}\right) = \frac{-\pi j}{8}$$

20. Ans: (b)

Sol: Let $f(z) = \frac{e^z}{(z-2)(z-1)}$

Then the singular points of f(z) are given by (z–2) (z–1) = 0

⇒ z = 1 and z = 2 are singular point of f(z)

⇒ both singular points z = 1 and z = 2 lie inside the given region $|z| = 3$

Consider $f(z) = e^z\left[\frac{1}{(z-1)(z-2)}\right]$

$$\Rightarrow f(z) = e^z\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$\Rightarrow \oint_C f(z)dz = \oint_C \frac{e^z}{z-2}dz - \oint_C \frac{e^z}{z-1}dz$$

$$\Rightarrow \oint_C f(z)dz = 2\pi i\left(e^z\right)_{z=2} - 2\pi i\left(e^z\right)_{z=1}$$

$$\Rightarrow \oint_C f(z)dz = 2\pi i\left(e^2\right) - 2\pi i(e)$$

$$\therefore \oint_C f(z)dz = 2\pi i\left(e^2 - e\right)$$

21. Ans: (d)

Sol: Given $f(z) = \frac{1}{z}$ and $z_0 = 4$

The Taylor series expansion of f(z) about a point $z = z_0$ is given by

$$f(z) = f(z_0) + (z-z_0).f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \frac{(z-z_0)^3}{3!}f'''(z_0) + \ldots\ldots..(1)$$

Now, $f(z) = \frac{1}{z}$ and $f(z_0) = f(4) = \frac{1}{4}$

$$\Rightarrow f'(z) = \frac{-1}{z^2} \text{ and } f'(z_0) = f'(4) = \frac{-1}{16}$$

$$\Rightarrow f''(z) = \frac{2}{z^3} \text{ and } f''(z_0) = f''(4) = \frac{2}{4^3} = \frac{1}{32}$$

$$\Rightarrow f'''(z) = \frac{-6}{z^4} \text{ and}$$

$$f'''(z_0) = f'''(4) = \frac{-6}{4^4} = \frac{-3}{128}$$

Substituting above all in (1), we get

i.e $$f(z) = \frac{1}{4} + (z-4)\left(\frac{-1}{16}\right) + \frac{(z-4)^2}{2!}\left(\frac{1}{32}\right) + \frac{(z-4)^3}{3!}\left(\frac{-3}{128}\right) + ----$$

$$\therefore \frac{1}{z} = \frac{1}{4} - \frac{1}{16}(z-4) + \frac{1}{64}(z-4)^2 - \frac{1}{256}(z-4)^3 + -----$$

The above series is a Taylor series expansion of $f(z) = \frac{1}{z}$ about a point z = 4 (or) in powers of (z – 4)

22. Ans: (c)

Sol: Given $f(z) = \frac{1}{z(z+2)^3}$ and $z_0 = -2$

Let z–(–2) = t
Then z = t–2

Now, $f(z) = \frac{1}{(t-2)\,t^3}$

$$\Rightarrow f(z) = \frac{1}{t^3}\frac{1}{(-2)\left(1-\frac{t}{2}\right)} = \frac{1}{-2t^3}\left[1-\left(\frac{t}{2}\right)\right]^{-1}$$

$$\Rightarrow f(z) = \frac{1}{-2t^3}\left[1+\left(\frac{t}{2}\right)+\left(\frac{t}{2}\right)^2+\left(\frac{t}{2}\right)^3+---\right], \left|\frac{t}{2}\right|<1$$

$$\left(\because (1-x)^{-1} = 1+x+x^2+x^3+----, |x|<1\right)$$

$$\Rightarrow f(z) = \frac{(-1)}{2t^3} + \frac{(-1)}{4t^2} + \frac{(-1)}{8t} + \frac{(-1)}{16} + \frac{(-1)}{32}t + \frac{(-1)}{64}t^2 + ---$$

$$\therefore \frac{1}{z(z+2)^3} = \left(\frac{-1}{2}\right)\frac{1}{(z+2)^3} + \left(\frac{-1}{4}\right)\frac{1}{(z+2)^2} + \left(\frac{-1}{8}\right)\frac{1}{(z+2)} + \left(\frac{-1}{16}\right) + \left(\frac{-1}{32}\right)(z+2) + \left(\frac{-1}{64}\right)(z+2)^2 + --$$

Hence the above expansion of f(z) is a Laurent series expansion of f(z) about z = –2 (or) in $0 < |z+2| < 2$

23. Ans: (d)

Sol: Given $f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$ and $|z+1| > 3$

Let z+1 = t

Then z = t–1 and $|t| > 3$

Now, $f(z) = \frac{1}{t-1} - \frac{3}{t} + \frac{2}{t-3}$

But $|t| > 3$

$\Rightarrow |t| > 3 > 1$

$\Rightarrow |t| > 3$ and $|t| > 1$

$\therefore \left|\frac{3}{t}\right| < 1$ and $\left|\frac{1}{t}\right| < 1$

Consider , $f(z) = \frac{1}{t-1} - \frac{3}{t} + \frac{2}{t-3}$

$$\Rightarrow f(z) = \frac{1}{t\left(1-\frac{1}{t}\right)} - \frac{3}{t} + \frac{2}{t\left(1-\frac{3}{t}\right)}$$

$$\Rightarrow f(z) = \frac{1}{t}\left[1-\frac{1}{t}\right]^{-1} - \frac{3}{t} + \frac{2}{t}\left[1-\frac{3}{t}\right]^{-1}$$

$$\Rightarrow f(z) = \frac{(1)}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+---\right] - \frac{3}{t} + \left(\frac{2}{t}\right)\left[1+\left(\frac{3}{t}\right)+\left(\frac{3}{t}\right)^2+----\right]$$

$$\Rightarrow f(z)=\frac{1}{(z+1)}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+---\right]$$
$$-\frac{3}{(z+1)}+\frac{2}{(z+1)}\left[1+\frac{3}{z+1}+\frac{3^2}{(z+1)^2}+--\right]$$

$$\Rightarrow f(z)=\frac{7}{(z+1)^2}+\frac{19}{(z+1)^3}+\frac{55}{(z+1)^4}+------$$

∴ The above series is a Laurent series expansion of f(z) about z = –1 (or) in $|z+1|>3$

24. Ans: (d)

Sol: Given $f(z)=\frac{1}{(z+1)(z+3)}$ in $1<|z|<3$

$$\Rightarrow f(z)=\frac{1}{2}\frac{1}{(z+1)}-\frac{1}{2(z+3)} \text{ in } 1<|z|<3$$

or $1<|z|$ & $|z|<3$

$$\Rightarrow f(z)=\frac{1}{2z\left(1+\frac{1}{z}\right)}-\frac{1}{6\left(1+\frac{z}{3}\right)} \text{ in } \left|\frac{1}{z}\right|<1$$

and $\left|\frac{z}{3}\right|<1$

$$\Rightarrow f(z)=\frac{1}{2z}\left[1+\left(\frac{1}{z}\right)\right]^{-1}-\frac{1}{6}\left[1+\left(\frac{z}{3}\right)\right]^{-1}$$

$$\Rightarrow f(z)=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}---\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{3^2}----\right]$$

$$\therefore f(z)=\frac{1}{2}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}----\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}+---\right]$$

which is a Laurent series expansion of f(z) in the given region $1<|z|<3$

25. Ans: (c)

Sol: Given that $X(z)=\frac{1-2z}{z(z-1)(z-2)}$

⇒ The poles of X(z) are 0 , 1 , 2 which are simple poles.

$R_1=\text{Res}[X(z):z=0]$

$$=\underset{Z\to 0}{\text{Lt}}\left[(z-0)\frac{(1-2z)}{z(z-1)\,(z-2)}\right]=\frac{1}{2}$$

$R_2=\text{Res}[X(z):z=1]$

$$=\underset{Z\to 1}{\text{Lt}}\left[(z-1)\frac{(1-2z)}{z(z-1)\,(z-2)}\right]=\frac{-1}{-1}=1$$

$R_3=\text{Res}[X(z):z=2]$

$$=\underset{Z\to 2}{\text{Lt}}\left[(z-2)\frac{(1-2z)}{z(z-1)\,(z-2)}\right]=\frac{-3}{2}$$

26. Ans: (b)

Sol: Given $f(z)=\frac{3\sin(z)}{z-\frac{3\pi}{2}}$

$$\left(\because f(z)=\frac{\phi(z)}{[z-z_0]}\right)$$

$\Rightarrow z=\frac{3\pi}{2}$ is a singular point of f(z)

$\Rightarrow z=\frac{3\pi}{2}$ is a pole of order one

Now, $R=\text{Res}\left(f(z):z=\frac{3\pi}{2}\right)=\phi\left(\frac{3\pi}{2}\right)$

$$\therefore R=3\sin\left(\frac{3\pi}{2}\right)=-3$$

27. Ans: (c)

Sol: Given $f(z)=\dfrac{e^z+z}{(z-4)^3}$

$\Rightarrow$ z = 4 is a singular point of f(z)

$\Rightarrow$ z = 4 is a pole of order 3

Now, R = Res (f(z) : z = 3) = $\dfrac{1}{(3-1)!}\lim_{z\to4}\left[\dfrac{d^2}{dz^2}\{(z-4)^3.f(z)\}\right]$

$$\Rightarrow R=\frac{1}{2!}\lim_{z\to4}\left[\frac{d^2}{dz^2}(z-4)^3.\frac{e^z+z}{(z-4)^3}\right]$$

$$\Rightarrow R=\frac{1}{2}\lim_{z\to4}\left[\frac{d}{dz}(e^z+1)\right]$$

$$\Rightarrow R=\frac{1}{2}\lim_{z\to4}e^z$$

$$\therefore R=\frac{1}{2}e^4$$

28. Ana: (a)

Sol: Given f(z) = cot (z) and z = 0

$$\Rightarrow f(z)=\frac{\cos(z)}{\sin(z)} \qquad \left(\because f(z)=\frac{\phi(z)}{\Psi(z)}\right)$$

Here, z = 0 is a pole of order one

$$\text{Now, } R=\text{Res}(f(z):z=z_0)=\frac{\phi(z_0)}{\Psi'(z_0)}$$

$$\Rightarrow R = \text{Res } (f(z) : z = 0) = \frac{\cos(0)}{\cos(0)}$$

$\therefore$ R = 1

29. Ans: (a)

Sol: Given $f(z)=\dfrac{z-\sin z}{z^2}$ and z = 0

$$\Rightarrow f(z)=\frac{1}{z^2}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+-----\right)\right]$$

$$\Rightarrow f(z)=\frac{1}{z^2}\left[\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}----\right]$$

$$\Rightarrow f(z)=\frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}--------$$

$$\Rightarrow f(z)=\frac{1}{3!}(z-0)-\frac{1}{5!}(z-0)^3+\frac{1}{7!}(z-0)^5----$$

$\therefore$ z = 0 is a removable singular point of f(z) and residue of f(z) at z = 0 is zero.

30. Ans: (d)

Sol: Given $f(z)=e^{\frac{1}{z-4}}$ and z = 4

$$\Rightarrow f(z)=1+\frac{(1/z-4)}{1!}+\frac{(1/z-4)^2}{2!}+\frac{(1/z-4)^3}{3!}+--$$

$$\Rightarrow f(z)=1+\frac{1}{(z-4)}+\frac{1}{2!}\frac{1}{(z-4)^2}+\frac{1}{3!}\frac{1}{(z-4)^3}+---$$

Here, the expansion is containing infinite number of terms in the negative power of (z–4).

$\therefore$ The singular point z = 4 of f(z) is an essential singular point and the residue of f(z) at z = 4 is one.

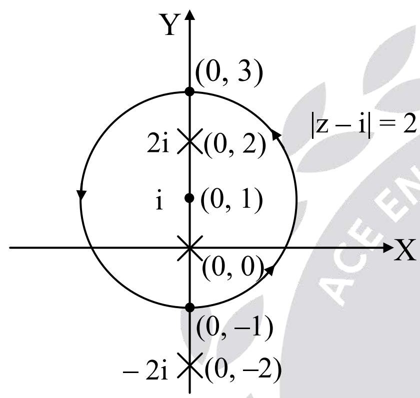
31. Ans: no option

Sol: Let $f(x)=\dfrac{\sin(x)}{x^2(x^2+4)}$

$$f(x)=\frac{\sin(x)}{x^2(x+2i)(x-2i)}$$

x = 0 is a pole of order 1 (or) simple pole

x = 2i, –2i are simple poles



x = – 2i lies outside |z – i| = 2

$$\underset{x=0}{\text{Res}}\, f(x)=\underset{x\to 0}{\text{Lt}}\; x\left\{\frac{\sin x}{x^2(x^2+4)}\right\}$$

$$=\underset{x\to 0}{\text{Lt}}\left\{\frac{\sin x}{(x^3+4x)}\right\}\left(\frac{0}{0}\right)$$

Applying L Hospital rule

$$\underset{x=0}{\text{Res}}\, f(x)=\underset{x\to 0}{\text{Lt}}\left\{\frac{\cos x}{3x^2+4}\right\}=\frac{1}{4}=0.25$$

$$\underset{x=2i}{\text{Res}}\, f(x)=\underset{x\to 2i}{\text{Lt}}\left\{\frac{(x-2i)\sin x}{x^2(x+2i)(x-2i)}\right\}$$

$$=\underset{x\to 2i}{\text{Lt}}\left\{\frac{\sin x}{x^2(x+2i)}\right\}=\frac{\sin(2i)}{(2i)^2(2i+2i)}$$

$$=\frac{\sin(2i)}{-4(4i)}=\frac{-1}{16i}\sin(2i)$$

Sum of residues $=\dfrac{1}{4}-\dfrac{1}{16i}\sin(2i)$

$$\oint_C f(z)dz=2\pi i(\text{sum of residues})$$

$$\oint_C f(z)dz=2\pi i\left\{\frac{1}{4}-\frac{1}{16i}\sin(2i)\right\}$$

$$=\frac{\pi i}{2}-\frac{\pi}{8}\sin(2i)$$

32. Ans: (d)

Sol: Let f(z) = $e^z\tan z$ is not analytic at $z=\pm\dfrac{\pi}{2},\ \pm\dfrac{3\pi}{2},\ \pm\dfrac{5\pi}{2}$, ...

Then f(z) = $\dfrac{e^z\sin z}{\cos z}$ $\left(\because f(z)=\dfrac{\phi(z)}{\Psi(z)}\right)$

$\Rightarrow$ $z=\pm\dfrac{\pi}{2},\ \pm\dfrac{3\pi}{2},\ \pm\dfrac{5\pi}{2}$, ... are singular points of f(z).

$\Rightarrow$ Of these points only $z=\pm\dfrac{\pi}{2}$ lie inside C

$\Rightarrow$ $z=\dfrac{\pi}{2},\dfrac{-\pi}{2}$ are first order poles

Now, $R_1=\text{Res}\left(f(z):z=\dfrac{\pi}{2}\right)=\dfrac{\phi\left(\frac{\pi}{2}\right)}{\Psi'\left(\frac{\pi}{2}\right)}$

$$\Rightarrow R_1=\frac{e^{\frac{\pi}{2}}\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)}$$

$$\therefore R_1=-e^{\frac{\pi}{2}}$$

Again,

$$R_2 = \text{Res}\left(f(z) : z = \frac{-\pi}{2}\right) = \frac{\phi\left(\frac{-\pi}{2}\right)}{\Psi'\left(\frac{-\pi}{2}\right)}$$

$$\Rightarrow R_2 = \frac{e^{\frac{-\pi}{2}}\sin\left(\frac{-\pi}{2}\right)}{-\sin\left(\frac{-\pi}{2}\right)}$$

$$\therefore R_1 = -e^{\frac{-\pi}{2}}$$

Now, by Cauchy's residue theorem, we have,

$$\oint_C f(z)dz = 2\pi i(R_1 + R_2)$$

$$\Rightarrow \oint_C f(z)dz = 2\pi i\left(-e^{\frac{\pi}{2}} - e^{\frac{-\pi}{2}}\right)$$

$$\Rightarrow \oint_C f(z)\,dz = -2\pi i\left(e^{\frac{\pi}{2}} + e^{\frac{-\pi}{2}}\right)$$

$$\therefore \oint_C f(z)\,dz = -4\pi i\ \cosh\left(\frac{\pi}{2}\right)$$

33. Ans: (a)

Sol: Given I = $\oint_c \frac{e^{\frac{1}{z}}}{z}dz$, where c is |z| = 2

Consider

$$f(z) = \frac{e^{1/z}}{z} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + - - - - -\right)$$

$$\Rightarrow f(z) = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + - - - - -\right)$$

$$\Rightarrow f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{2!z^3} + - - - - - -$$

$\Rightarrow$ z = 0 is essential singularity of f(z)

Now, R_1 = Res(f(z)) : z = 0) = coefficient of $\frac{1}{z} = 1$

Now, by Cauchy's residue theorem, we have,

$$\oint_C f(z)dz = 2\pi i(R_1)$$

$$\Rightarrow \oint_C f(z)dz = 2\pi i(1)$$

$$\therefore \oint_C f(z)\,dz = 2\pi i$$

34. Ans: (a, c, d)

Sol: Given series is $\sum \frac{1}{n^n}z^n$

Comparing the given series with general series $\sum a_n(z - z_0)^n$, we get

$$a_n = \frac{1}{n^n}, z_0 = 0$$

$$\text{Now, } r = \frac{1}{\lim_{n\to\infty}|a_n|^{1/n}} = \frac{1}{\lim_{n\to\infty}\left|\frac{1}{n^n}\right|^{1/n}}$$

$$\Rightarrow r = \frac{1}{\ell im_{n\to\infty}\left|\frac{1}{n}\right|} = \frac{1}{0} = \infty$$

$\therefore$ The radius of convergence is r = ∞, the circle of convergence is $|z - 0| = \infty$ and the region of convergence is $|z - 0| < \infty$

35. Ans: (a, b, c)

Sol: Given series is $\sum \frac{n!\, z^n}{n^n}$

Comparing the given series with general series $\sum a_n (z - z_0)^n$, we get

$a_n = \frac{n!}{n^n}$ and $z_0 = 0$

$\Rightarrow$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)(n!)}{(n+1)^n.(n+1)} = \frac{n!}{(n+1)^n}$$

$$\text{Consider } r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{n!}{n^n}}{\frac{n!}{(n+1)^n}}\right|$$

$$\Rightarrow r = \lim_{n\to\infty}\left|\frac{(n!).(n+1)^n}{(n^n)(n!)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^n}{n^n}\right|$$

$$\Rightarrow r = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)^n}{1}\right| = e$$

$$\left(\because \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a\right)$$

$\therefore$ The radius of convergence is r =e,

the circle of convergence is $|z-0| = e$ and

the region of convergence is $|z-0| < e$

36. Ans: (a, b, c)

Sol: Given series is $\sum \frac{(z-2)^n}{n}$

Comparing the given series with general series $\sum a_n (z - z_0)^n$, we get

$a_n = \frac{1}{n}$ and $z_0 = 2$

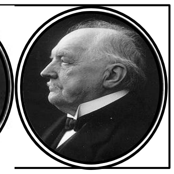
$$\text{Consider } r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$\Rightarrow r = \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = 1$$

$\therefore$ The radius of convergence is r = 1, the circle of convergence is $|z-2| = 1$ and the region of convergence is $|z-2| < 1$

Chapter 6 Numerical Methods

Carl David Tolme Runge (1856 – 1927)



Martin Wilhelm Kutta (1867-1944)

01. Ans: (c)

Sol: $f(x) = x^3 + x^2 + x + 7 = 0$

$f(-3) = -8$ and $f(-2) = 5$

A root lies in $(-3, -2)$

Let $x_1 = \dfrac{-2-3}{2} = -2.5$ is first approximation to the root

$\therefore f(x_1) = f(-2.5) < 0$

Now, Root lies in $[-2.5, -2]$

Let $x_2 = \dfrac{-2.5-2}{2} = -2.25$ is second approximation root.

02. Ans: 0.67

Sol: $f(x) = x^3 + x - 1 = 0$

Let $x_0 = 0.5$, $x_1 = 1$

$f(x_0) = f(0.5) = -0.375$

$f(x_1) = f(1) = 1$

$\therefore x_2 = \dfrac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$

is first approximation root

$$= \frac{1(0.5) - (-0.375)(1)}{1 - (-0.375)}$$

$$= \frac{0.5 + 0.375}{1.375} = \frac{0.875}{1.375}$$

$= 0.6363$

$f(x_2) = f(0.6363)$

$= (0.6363)^3 + (0.6363) - 1$

$= 0.2576 + 0.6363 - 1$

$= -0.1061 < 0$

Root lies in $(0.6363, 1)$

$$x_3 = \frac{f(x_1)x_2 - f(x_2).x_1}{f(x_1) - f(x_2)}$$

$$= \frac{1(0.6363) - (-0.1061)1}{1 + 0.1061}$$

$= 0.6711$

03. Ans: 2.33

Sol: Let $f(x) = (x^2 - 4x + 4)$

$f'(x) = (2x - 4)$

$x_o = 3$

$$\Rightarrow x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$$

$= 2.5$

$x_o = 3$, $x_1 = 2.5$

$\Rightarrow f(x_o) = f(3) = 1$ and

$f(x_1) = f(2.5) = 0.25$

$\therefore$ By Secant method

$$x_2 = \frac{x_o f(x_1) - x_1 f(x_o)}{f(x_1) - f(x_o)}$$

$= 2.33$

***C. Runge** and **M. W. Kutta** (German mathematicians) developed an important family of implicit and explicit iterative methods, which are used in temporal discretization for the approximation of solutions of ordinary differential equations. In numerical analysis, these techniques are known as Runge–Kutta methods.*

04. Ans: 4.3

Sol: $f(x) = (x^3 - 5x^2 + 6x - 8)$

$f'(x) = (3x^2 - 10x + 6)$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore\ x_1 = x_0 - \frac{f(x_o)}{f'(x_o)}$$

$$= 5 - \frac{22}{31} = 4.29$$

05. Ans: 1.67

Sol: Let $f(x) = x^3 - x - 3$, $f'(x) = 3x^2 - 1$ and $x_0 = 2$

First iteration:

Here, $f(2) = 2^3 - 2 - 3 = 3$

and $f'(2) = 3(4) - 1 = 11$

Now, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - \left(\dfrac{3}{11}\right)$

$\therefore\ x_1 = 1.73$

Second iteration:

Here, $f(x_1) = f(1.73)$

$= 1.73^3 - 1.73 - 3 = 0.45$

and $f'(x_1)$ and $f'(1.73) = 3\,(1.73)^2 - 1 = 7.98$

Now, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.73 - \left(\dfrac{0.45}{7.98}\right)$

$X_2 = 1.67$

06. Ans: 0.6861

Sol: Let $f(x) = x^3 + x - 1$, $x_0 = 1$

$f'(x) = 3x^2 + 1$

The first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4} = 0.75$$

The second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.75 - \frac{f(0.75)}{f'(0.75)}$$

$$= 0.75 - \frac{0.1718}{2.6875}$$

$$= 0.75 - 0.0639$$

$$x_2 = 0.6861$$

07. Ans: (c)

Sol: Putting n = 0 in the iteration formula of the above example

$$x_1 = \frac{4x_o^5 + N}{5x_o^4}$$

$$= \frac{4(2^5) + 30}{5(2^4)} = \frac{158}{80} = 1.975$$

08. Ans: (a)

Sol: **Given** $x_{n+1} = \dfrac{2x_n^3 + 1}{3x_n^2 + 1}$

Suppose the formula converges to the root after n iterations

$x_{n+1} = x_n = x$

$$x = \frac{2x^3 + 1}{3x^2 + 1}$$

$$\Rightarrow x^3 + x - 1 = 0$$

09. Ans: (c)

Sol: Let $\int_a^b f(x)dx = \int_0^1 e^x\,dx$ & $n = 4$

Then $a = 0$, $b=1$, $f(x) = e^x$ and

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x	0	0.25	0.50	0.75	1
$y = f(x) = e^x$	1	1.284	1.649	2.117	2.718

The formula of trapezoidal rule to the given data is given by

$$\int_a^b f(x)dx \cong \int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3)\right]$$

$$\Rightarrow \int_0^1 e^x dx \cong \int_0^1 p(x)dx = \frac{(0.25)}{2}\begin{bmatrix}(1+2.718)+\\ 2(1.284+1.649+2.117)\end{bmatrix}$$

$$\Rightarrow \int_0^1 e^x dx \cong \int_0^1 p(x)dx = \frac{(0.25)}{2}\left[(3.718)+2(5.05)\right]$$

$$\therefore \int_0^1 e^x dx \cong \int_0^1 p(x)dx = \frac{(0.25)}{2}\left[(3.718+10.1)\right] = 1.727$$

10. Ans: 58.66

Sol: Let $\int_a^b f(x)dx = \int_1^5 (-x^2 + 9x - 2)dx$

and $n = 2$

Then $a = 1$, $b = 5$,

$y = f(x) = -x^2 + 9x - 2$ &

$$h = \frac{b-a}{n} = \frac{5-1}{2} = 2$$

x	1	3	5
$y = f(x) = -x^2 + 9x - 2$	6	16	18

Here, $y_o = 6$, $y_1 = 16$ and $y_2 = 18$

The formula of simpson's 1/3[rd] rule for the given data is given by

$$\int_a^b f(x)dx \approx \int_a^b P(x)dx = \frac{h}{3}\begin{bmatrix}(y_o + y_n) + 2(y_2 + y_4 + ...)\\ +4(y_1 + y_3 + ...)\end{bmatrix}$$

$$\therefore \int_a^b f(x)dx \approx \int_1^5 P(x)dx = \frac{2}{3}\left[(6+18)+4(16)\right]$$

$$= 58.6666$$

11. Ans: 5.132

Sol: Let $\int_b^a f(x)dx = \int_0^{\frac{\pi}{2}} (8 + 4\cos x)dx$

Then $a = 0$, $b = \frac{\pi}{2}$ and $f(x) = 8 + 4\cos(x)$

(i) Approximate value:

x	0	π / 2
$y = f(x) = 8 + 4\cos(x)$	12	8

Here, $y_o = 12$ & $y_1 = 8$

The formula of trapezoidal rule is

$$\int_a^b f(x)dx \approx \int_a^b P(x)dx = \frac{h}{2}\left[(y_o + y_n) + 2(y_1 + y_2 + ...)\right]$$

$$\therefore \int_0^{\pi/2} f(x)dx \simeq \int_0^{\pi/2} P(x)dx = \frac{\left(\frac{\pi}{2}\right)}{2}[12+8]$$

$$= 15.70796$$

(ii) Exact value:

Now, $\int_a^b f(x)dx = \int_0^{\pi/2}(8+4\cos(x))dx$

$$\Rightarrow \int_a^b f(x)dx = (8x+4\sin x)_0^{\pi/2}$$

$$\therefore \int_0^b f(x)dx \simeq \left[8\left(\frac{\pi}{2}\right)+4\sin\left(\frac{\pi}{2}\right)\right]-(0+0)$$

$= 16.56637$

(iii) Error

Error = Exact value – Approximate value

$\Rightarrow$ Error = 16.56637 – 15.70796

= 0.8584

$$\therefore \% \text{ Error} = \frac{|\text{Error}|}{\text{Exact value}}\times 100$$

$$= \frac{0.8584}{16.56637}\times 100 = 5.1816\% = 5.2\%$$

12. Ans: (a)

Sol: Let $\int_a^b f(x)dx = \int_4^{5.2} \ell n\, x dx$ and h = 0.3

Then a = 4, b = 5.2, f(x) = $l$nx

X	4	4.3	4.6	4.9	5.2
Y=f(x) = $l$nx	1.3862	1.4586	1.5260	1.5892	1.6486
	y_0	y_1	y_2	y_3	y_4

The formula of Simpson's 1/3 rd rule t the given data is given by

$$\int_a^b f(x)dx \cong \int_a^b p(x)dx$$

$$= \frac{h}{3}[(y_0+y_4)+2(y_2)+4(y_1+y_3)]$$

$$\Rightarrow \int_4^{5.2} f(x)dx \cong \int_4^{5.2} p(x)dx$$

$$= \frac{0.3}{3}[(1.3862+1.6486)+2(1.5260)+4(1.4586+1.5892)]$$

$$= \left(\frac{0.3}{3}\right)[(3.0348)+(3.052)+(12.1912)]$$

$$= \left(\frac{0.3}{3}\right)[18.278]$$

$$\therefore \int_4^{5.2} f(x)dx \cong \int_4^{5.2} p(x) = 1.8278 \cong 1.83$$

13. Ans: 0.4

Sol: f(x) = $10x - 20x^2$ = y

$$\text{Step size} = \frac{b-a}{n} = \frac{0.5}{5} = 0.1$$

x	0	0.1	0.2	0.3	0.4	0.5
y	0	0.8	1.2	1.2	0.8	0

Required answer

$$= \frac{h}{2}[(y_0+y_5)+2(y_1+y_2+y_3+y_4)]$$

$$= \frac{0.1}{2}[0+2(0.8+1.2+1.2+0.8)]$$

= 0.4

14. Ans: (c)

Sol: Given $f(x)=\frac{1}{x}$

The value of table for x and y

x	1	1.25	1.5	1.75	2
y	1	0.8	0.66667	0.57143	0.5

Using Simpsons $\frac{1}{3}$ rule

$$\int \frac{1}{x}dx=\frac{h}{3}[(y_0+y_4)+4(y_1+y_3)+2(y_2)]$$

$$\int \frac{1}{x}dx=\frac{0.25}{3}[((1+0.5)+4\times(0.8+0.57143)+2\times(0.66667))]$$

$$\int \frac{1}{x}dx=\frac{0.25}{3}[(1+05)+4\times(1.37143)+2\times(0.66667)]$$

$$\int \frac{1}{x}dx=0.69325$$

Solution by Simpsons $\frac{1}{3}$ rule is 0.69325

15. Ans: 0.6452

Sol: Let $\int_a^b f(x)dx=\int_0^1 e^{-x}dx$ & h = 0.5

Then f(x) = e^{-x}, a = 0, b = 1

x	0	0.5	1
y= f(x) = e^{-x}	1	0.6065	0.3675
	y_0	y_1	y_2

By Trapezoidal rule,

$$\int_0^1 e^{-x}dx=\frac{h}{2}[(y_0+y_2)+2(y_1)]$$

$$=\frac{(0.5)}{2}[(1+0.3678)+2(0.6065)]$$

= 0.6452

16. Ans: (c)

Sol: Let $\int_a^b f(x)\,dx=\int_0^1 e^x\,dx$

Then f(x) = e^x, a = 0 and b = 1

x	0.0	0.5	1
y = f(x) = e^x	1	1.6487	2.7182

Let x_0 = 0.0, x_1 = 0.5 and x_2 = 1.0

Then y_0 = 1, y_1 = 1.6487, y_2 = 2.7182

and h = 0.5

(1) Approximate value:

$$\int_0^1 e^x\,dx\approx\int_0^1 p(x)\,dx=\frac{h}{3}[(y_0+y_2)+4(y_1)]$$

$$=\frac{0.5}{3}[(1+2.7182)+4(1.6487)]$$

$$=\frac{0.5}{3}[3.7182+6.5948]$$

$$=\frac{0.5}{3}[10.313]=\frac{5.1565}{3}=1.7188$$

(2) Exact value:

$$\int_0^1 e^x\,dx=(e^x)_0^1=(e-1)=2.71828-1$$

= 1.71828

(3) Error:

∴ Error = Exact value – Approximate value

= 1.71828 – 1.71828

= –0.00052

Hence, Absolute error = |Error|

= |–0.00052|

= 0.00052

17. Ans: 0.992

Sol: $y^1 = f(x, y) = 4 - 2xy$

$x_0 = 0$, $y_0 = 0.2$, $h = 0.2$

By Taylor's theorem,

$y(x) = y(x_0 + h)$

$= y(x_0) + h\, y^1(x_0) + \frac{h^2}{2!} y^{11}(x_0)$

$= 0.2 + 0.24) + \frac{(0.2)^2}{2!}(-0.4)$

$= 0.992$

18. Ans: 1

Sol: $f(x, y) = 4 - 2xy$

$x_0 = 0$, $y_0 = 0.2$, $f_1 = 0.2$

By Euler's formula

$y_1 = y_0 + h\, f(x_0, y_0) = 0.2 + 0.2(4 - 0) = 1$

19. Ans: 1.1

Sol: By Euler's formula,

$y_1 = y_0 + h\, f(x_0, y_0)$

$y_1 = 1 + (0.1)(1 - 0) = 1.1$

20. Ans: 0.02

Sol: $f(x, y) = x + y$

$x_0 = 0$, $y_0 = 0$, $h = 0.2$

$k_1 = h(f_0, y_0)$

$= 0.2(0+0) = 0$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.2(0.2 + (0.0))$

$= 0.04$

$y_1 = 0 + \frac{1}{2}(0 + 0.04) = 0.02$

21. Ans: 0.96

Sol: Let $\frac{dy}{dx} = f(x, y) = 4 - 2xy$

$x_o = 0$, $y_o = 0.2$, $4 = 0.2$

$k_1 = h.f(x_o, y_o) = 0.2(4 - x_o y_o) = 0.8$

$k_2 = h\, f\left(x_o + \frac{h}{2}, y_o + \frac{k_1}{2}\right)$

$= (0.2)(4 - 2(0.1)(0.6))$

$= (0.2)(3.88) = 0.776$

$k_3 = h\, f(x_o + h, y_o + \frac{k_2}{2})$

$= (0.2)(4 - 2(0.2)(0.976)) = 0.7219$

$k_4 = h.\, f(x_o + h, y_o + k_3)$

$= (0.2)(4 - 2(0.2)(0.9219)) = 0.7262$

$y(0.2) = y_1 = y_o + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.2 + \frac{1}{6}(0.8 + 2(0.776 + 0.7219) + 0.7262)$

$= 0.97$

22. Ans: 1.1103

Sol: Forth order R-K method

$k_1 = hf(x_0, y_0) = (0.1)f(0,1) = (0.1).(1) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.1)f(0.05, 1.05)$

$= (0.1).(1.1) = 0.11$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= (0.1)f(0.05, 1.055)$$
$$= (0.1).(1.105) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(0.1, 1.1105)$$
$$= (0.1).(1.2105)$$
$$= 0.1211$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = 1 + \frac{1}{6}[0.1 + 2(0.11) + 2(0.1105) + (0.1211)]$$

$y_1 = 1.1103$

$\therefore$ y(0.1) = 1.1103

23. Ans: i. $8x^2 - 19x + 12$ ii. 6 iii. 13

Sol: $f(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$

$f(x) = 8x^2 - 19x + 12$

f(2) = 6

$f^1(2) = 13$

$f(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$

$= 1 + (x - 1)\, 13 + (x - 1)(x - 3)\, 8$

$= 8x^2 - 19x + 12$

f(2) = 6

$f^1(2) = 13$

24. Ans: $8x^2 - 19x + 12$, 6, 13

Sol:

x	P(x)	Δp	$\Delta^2 p$
1	1		
		$\frac{27-1}{3-1} = 13$	
3	27		$\frac{37-13}{4-1} = 8$
		$\frac{64-27}{4-3} = 37$	
4	64		

By Newton's divided difference formula

$P(x) = P(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$

$= 1 + (x - 1)13 + (x - 1)(x - 3).8$

$= 8x^2 - 19x + 12$

$P^1(x) = 16x - 19$

P(2) = 6

$P^1(2) = 13$

25. Ans: $x^2 + 2x + 3$, 4.25, 3

Sol: Since the given observations are at equal interval of width unity.

Construct the following difference table.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

Therefore f(x)

$f(x) = f(0) + C(x,1)\,\Delta f(0) + C(x, 2)\, f(0)$

$$= 3 + (x \times 3) + \left(\frac{x(x-1)}{2!} \times 2\right)$$

$f(x) = x^2 + 2x + 3$

$f^1(x) = 2x + 2$

$f(0.5) = 4.25$

$f'(0.5) = 3$