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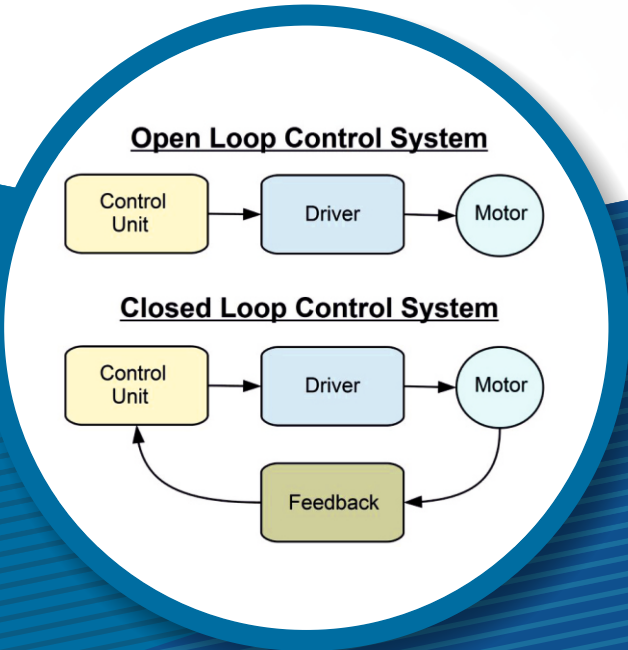


ELECTRONICS & TELECOMMUNICATION ENGINEERING

CONTROL SYSTEMS

Text Book : Theory with worked out Examples and Practice Questions

Chapter 1 Basics of Control Systems
(Solutions for Text Book Practice Questions)

Objective Practice Solutions

01. Ans: (c)

Sol: $2\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+4y(t) = r(t) + 2r(t-1)$

Apply LT on both sides

$2s^2\,Y(s)+3sY(s)+4Y(s) = R(s)+2e^{-s}R(s)$

$Y(s)(2s^2 + 3s+4) = R(s)(1+2e^{-s})$

$$\frac{Y(s)}{R(s)} = \frac{1+2e^{-s}}{2s^2+3s+4}$$

02. Ans: (b)

Sol: $I.R = 2.e^{-2t}u(t)$

Output response $c(t) = (1-e^{-2t})\,u(t)$

Input response $r(t)$ = ?

$$T.F = \frac{C(s)}{R(s)}$$

$$T.F = L(I.R) = \frac{2}{s+2}$$

$$R(s) = \frac{C(s)}{T.F} = \frac{\frac{1}{s}-\frac{1}{s+2}}{\frac{2}{s+2}} = \frac{1}{s}$$

$$R(s) = \frac{1}{s}$$

$r(t) = u(t)$

03. Ans: (b)

Sol: Unit impulse response of unit-feedback control system is given

$$c(t) = t.e^{-t}$$

$$T.F = L(I.R) = \frac{1}{(s+1)^2}$$

$$\text{Open Loop T.F} = \frac{\text{Closed Loop T.F}}{1-\text{Closed Loop T.F}}$$

$$= \frac{\frac{1}{(s+1)^2}}{1-\frac{1}{(s+1)^2}}$$

$$= \frac{1}{s^2+2s}$$

04. Ans: (a)

Sol: G changes by 10%

$\Rightarrow \frac{\Delta G}{G}\times 100 = 10\%$

$C_1 = 10\%$

[∵ open loop] whose sensitivity is 100%]

%G change = 10%

$$\frac{\%\text{ of change in M}}{\%\text{ of change in G}} = \frac{1}{1+GH}$$

$$\%\text{ of change in M} = \frac{10\%}{1+(10)1} = 1\%$$

% change in C_2 by 1%

05.

Sol: M = C/R

$$\frac{C}{R} = M = \frac{GK}{1+GH}$$

$$S_K^M = \frac{\partial M}{\partial K}\times\frac{K}{M} = 1$$

[∵K is not in the loop ⇒ sensitivity is 100%]

$$S_H^M = \frac{\partial M}{\partial H}\times\frac{H}{M} = \frac{\partial}{\partial H}\left(\frac{GK}{1+GH}\right)\frac{H}{M}$$

$$=\left(\frac{GK(-G)}{(1+GH)^2}\right)\left[\frac{H}{\frac{GK}{1+GH}}\right]$$

$$S_H^M = \frac{-GH}{(1+GH)}$$

06.

Sol: Given data

$$G = 2\times10^3, \partial G = 100$$

$$\%\text{ change in } G = \frac{\partial G}{G}\times100 = 5\%$$

$$\%\text{ change in } M = 0.5\%$$

$$\frac{\%\text{ of change in } M}{\%\text{ of change in } G} = \frac{1}{1+GH}$$

$$\frac{0.5\%}{5\%} = \frac{1}{1+2\times10^3 H}$$

$$1+2\times10^3 H = 10$$

$$H = 4.5\times10^{-3}$$

07. Ans: (b)

Sol: $K = \frac{\text{output}}{\text{input}}$

$$= \frac{c(t)}{r(t)}$$

$$= \frac{mm}{{}^0c}$$

08. Ans: (d)

Sol: Introducing negative feedback in an amplifier results, increases bandwidth.

Conventional Practice Solutions

01.

Sol: Transfer function gives the mathematical representation of a system and it gives system characteristics by relating the input and output.

Properties of Transfer Function

- The transfer function is defined only for a linear time-invariant system.
- The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response. Alternatively, the transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input, with all initial conditions of the system set to zero.
- The transfer function is independent of the input of the system.
- The transfer function of a continuous-data system is expressed only as a function of the complex variable ‘s’. It is not a function of the real variable, time or any other variable that is used as the independent variable. For discrete-data systems modeled by difference equations, the transfer function is a function of ‘z’ where the z-transform is used.

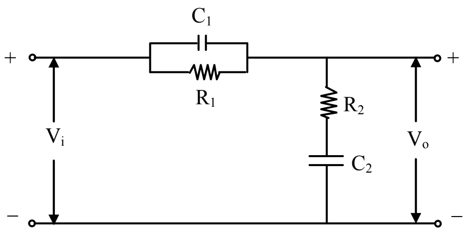


Fig: Lag-lead Network

$$Z_1 = R_1 \parallel \frac{1}{C_1 s} = \frac{R_1 \cdot \frac{1}{C_1 s}}{R_1 + \frac{1}{C_1 s}}$$

$$= \frac{\frac{R_1}{C_1 s}}{\frac{C_1 R_1 s + 1}{C_1 s}} = \frac{R_1}{C_1 R_1 s + 1}$$

$$Z_2 = R_2 + \frac{1}{C_2 s} = \frac{C_2 R_2 s + 1}{C_2 s}$$

Transfer function of lag-lead compensator is

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{\frac{C_2 R_2 s + 1}{C_2 s}}{\frac{R_1}{C_1 R_1 s + 1} + \frac{C_2 R_2 s + 1}{C_2 s}}$$

$$= \frac{\frac{C_2 R_2 s + 1}{C_2 s}}{\frac{R_1 C_2 s + (C_2 R_2 s + 1)(C_1 R_1 s + 1)}{(C_1 R_1 s + 1)(C_2 s)}}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{(C_2 R_2 s + 1)(1 + C_1 R_1 s)}{C_2 R_1 s + C_1 C_2 R_1 R_2 s^2 + (C_1 R_1 + C_2 R_2) s + 1}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{C_1 C_2 R_1 R_2 s^2 + (C_1 R_1 + C_2 R_2) s + 1}{C_1 C_2 R_1 R_2 s^2 + (C_1 R_1 + C_2 R_2 + C_2 R_1) s + 1}$$

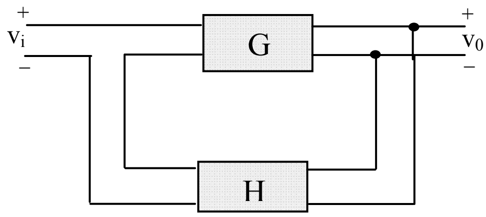
02.

Sol: Overall Transfer function

$$M = \frac{G}{1 + GH} \quad \ldots\ldots\ldots\ldots (1)$$

Effect of Feedback on overall Gain

- From Equation (1) feedback affects the gain G of non feedback system by the factor 1 + GH.



- The quantity GH may itself include minus sign, so the general effect of feedback is that it may increase or decrease the Gain G.
- In a practical control system G and H are functions of frequency, so the magnitude of 1+ GH may be greater than 1 in one frequency range but less than 1 in another range. Therefore feedback could increase the system gain in one frequency range but decrease it in another frequency range.

Effect of Feedback on Sensitivity:

- In general, a good control system should be insensitive to parameter variations due to disturbance/noise but sensitive to the input commands.
- Consider G to be a gain parameter that may vary. The sensitivity of the gain of the overall system, M to the variation in G is defined as,

$$s_G^M = \frac{\partial M / M}{\partial G / G} = \frac{\%\text{ change in } M}{\%\text{change in } G}$$

Where ∂M denotes the incremental change in M due to the incremental change in G (i.e ∂G).The sensitivity function is written as,

$$s_G^M = \frac{\partial M}{\partial G}\frac{G}{M} = \frac{1}{1+GH}$$

- Above relation shows that if GH is a positive constant, the magnitude of the sensitivity function can be made arbitrarily small by increasing GH, provided that the system remains stable.
- For open loop system, $s_G^M = 1$
- GH is a function of frequency, the magnitude of 1 + GH may be less than unity over some frequency ranges, So that feedback could be harmful to the sensitivity to parameter variations in certain cases.
- Feedback can increase or decrease the sensitivity of a system.

03.

Sol: $G(s) = \frac{25}{s(s+2)}$, $H(s) = \frac{1}{4}$

$$TF = \frac{G(s)}{1+G(s)H(s)}$$

(i) $$s_G^T = \frac{1}{1+GH} = \frac{1}{1+\frac{25}{s(s+2)}\times\frac{1}{4}}$$

$$= \frac{4s(s+2)}{4s(s+2)+25}$$

$$= \frac{4s^2+8s}{4s^2+8s+25}$$

$$s_G^T/_{s=j\omega=j1} = \frac{8j\omega-4\omega^2}{25-4\omega^2+8j\omega}$$

$$= \left|\frac{8j-4}{21+8j}\right| = 0.398$$

(ii) $$s_H^T = \frac{-GH}{1+GH} = \frac{\frac{-25}{s(s+2)}\times\frac{1}{4}}{1+\frac{25}{s(s+2)}\times\frac{1}{4}}$$

$$= \frac{-25}{4s(s+2)+25}$$

$$= \frac{-25}{4s^2+8s+25}$$

$$= \frac{-25}{-4+25+8j} = \left|\frac{-25}{21+8j}\right| = 1.11$$

04.

Sol: Given path transfer function $G(s) = \frac{K}{s(s+P)}$

Case (i): when feedback path H(s) =1

$$\text{Transfer function} = \frac{G(s)}{1+G(s).H(s)}$$

$$= \frac{G(s)}{1+G(s)}$$

$$M = \frac{K}{s^2+Ps+K}$$

Sensitivity with respect to K:

$$s_K^M = \frac{\%\text{changes in M}}{\%\text{changes in K}} = \frac{\left(\frac{\partial M}{M}\right)\times 100}{\left(\frac{\partial K}{K}\right)\times 100}$$

$$= \frac{\partial M}{\partial K}\times\frac{K}{M}$$

$$\left[\frac{(s^2+Ps+K)1-K.1}{(s^2+Ps+K)^2}\right]\frac{K}{\frac{K}{s^2+Ps+K}}$$

$$= \frac{s^2+Ps}{s^2+Ps+K} = \frac{s^2+3s}{s^2+3s+12}$$

Sensitivity with respect to P:

$$s_P^M = \frac{\partial M}{\partial P}.\frac{P}{M}$$

$$= \frac{-Ks}{\left(s^2+Ps+K\right)^2}.\frac{P}{\frac{K}{\left(s^2+Ps+K\right)}}$$

$$= \frac{-sP}{s^2+Ps+K} = \frac{-3s}{s^2+3s+12}$$

Case (ii): When H(s) = 1+ αs

Closed loop transfer function

$$M = \frac{G(s)}{1+G(s).H(s)}$$

$$= \frac{\frac{K}{s(s+P)}}{1+\frac{K}{s(s+P)}(1+\alpha s)}$$

$$= \frac{K}{s^2+Ps+K+K\alpha s}$$

$$M = \frac{K}{s^2+s(P+K\alpha)+K}$$

Sensitivity with respect to K:

$$s_K^M = \frac{\partial M}{\partial K}\times\frac{K}{M}$$

$$= \frac{\left[s^2+s(P+K\alpha)+K\right]1-K.(s\alpha+1)}{\left[s^2+s(P+K\alpha)+K\right]^2}\times\frac{K}{\frac{K}{s^2+s(P+K\alpha)+K}}$$

$$= \frac{s^2+sP}{s^2+s(P+K\alpha)+K}$$

$$= \frac{s^2+3s}{s^2+s[3+(0.14\times12)]+12}$$

$$s_K^M = \frac{s^2+3s}{s^2+4.68s+12}$$

Sensitivity with respect to P:

$$s_P^M = \frac{\partial M}{\partial P}.\frac{P}{M}$$

$$= \frac{-sK}{\left[s^2+s(P+K\alpha)+K\right]^2}\times\frac{P}{\frac{K}{s^2+s(P+K\alpha)+K}}$$

$$= \frac{-sP}{s^2+s(P+K\alpha)+K} = \frac{-3s}{s^2+4.68s+12}$$

Sensitivity with respect α:

$$s_\alpha^M = \frac{\partial M}{\partial \alpha}.\frac{\alpha}{M}$$

$$= \frac{-sK^2}{\left[s^2+s(P+K\alpha)+K\right]^2}.\frac{\alpha}{\frac{K}{s^2+s(P+K\alpha)+K}}$$

$$= \frac{-sK\alpha}{s^2+s(P+K\alpha)+K}$$

$$s_\alpha^M = \frac{-1.68s}{s^2+4.68s+12}$$

Chapter 2 Signal Flow Graph and Block Diagram

Objective Practice Solutions

01. Ans: (d)

Sol: No. of loops = 3

Loop1: $-G_1G_3G_4H_1H_2H_3$

Loop2: $-G_3G_4H_1H_2$

Loop3: $-G_4H_1$

No. of Forward paths = 3

Forward Path1: $G_1G_3G_4$

Forward Path 2: $G_2G_3G_4$

Forward Path 3: G_2G_4

$$= \frac{G_1G_3G_4 + G_2G_3G_4 + G_2G_4}{1 + G_1G_3G_4H_1H_2H_3 + G_3G_4H_1H_2 + G_4H_1}$$

02. Ans: (a)

Sol: Number of forward paths = 2

Number of loops = 3

$$\frac{Y(s)}{R(s)} = \frac{\frac{1}{s}\times\frac{1}{s}\times\frac{1}{s}[1-0]+\frac{1}{s}}{1-\left[\frac{1}{s}\times(-1)\left(\frac{1}{s}\right)(-1)+\frac{1}{s}\times\frac{1}{s}(-1)+\left(\frac{1}{s}\times\frac{1}{s}(-1)\right)\right]}$$

$$= \frac{\frac{1}{s^3}+\frac{1}{s}}{1-\left[\frac{1}{s^2}-\frac{1}{s^2}-\frac{1}{s^2}\right]}$$

$$= \frac{\frac{1+s^2}{s^3}}{1+\frac{1}{s^2}} = \frac{\frac{1+s^2}{s^3}}{\frac{s^2+1}{s^2}}$$

$$= \frac{1+s^2}{s}\times\frac{1}{s^2+1} = \frac{1}{s}$$

03.

Sol: Number of forward paths = 2

Number of loops = 5,

Two non touching loops=4

$$TF = \frac{24[1-(-0.5)]+10[1-(-3)]}{1-[-24-3-4+(5\times2\times(-1)+(-0.5))]+[30+1.5+2]+\left(\left(\frac{-1}{2}\right)\times(-24)\right)}$$

$$= \frac{76}{88} = \frac{19}{22}$$

04.

Sol: Number of forward paths = 2

Number of loops = 5

$$T.F = \frac{G_1G_2G_3 + G_1G_4}{1 + G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3 + G_4H_2 + G_1G_4}$$

05. Ans: (c)

Sol: From the network

$$V_o(s) = \frac{1}{sC}I(s) \quad ....(1)$$

$$-V_i(s) + RI(s) + V_o(s) = 0$$

$$I(s) = \frac{1}{R}V_i(s) + \left(\frac{-1}{R}\right)V_o(s) ....(2)$$

From SFG

$$V_o(s) = x.I(s) \quad ....(3)$$

$$I(s) = \frac{1}{R}V_i(s) + y\,V_o(s) \quad .... (4)$$

From equ(1) and (3)

$$x = \frac{1}{sC}$$

From equ(2) and (4)

$$y = -\frac{1}{R}$$

06. Ans: (a)

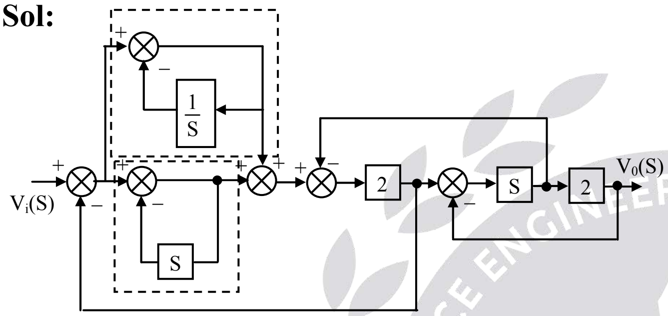
Sol: Use gain formula

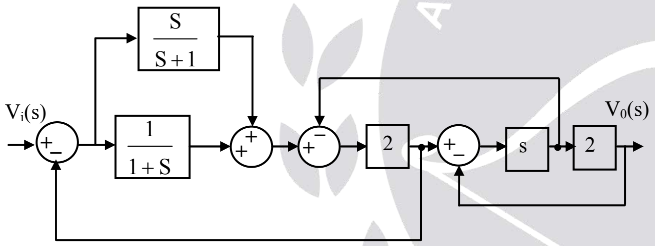
$$\text{transfer function} = \frac{G(s)}{1-\left(G(s)\dfrac{1}{G(s)}+G(s)\right)}$$

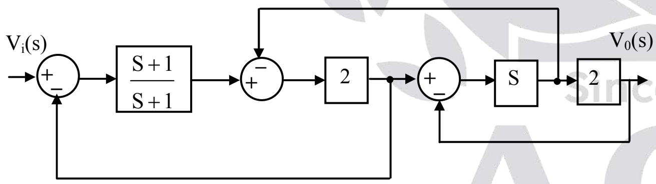
$$= \frac{G(s)}{1-1-G(s)} = -1$$

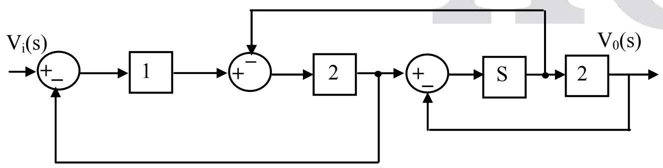
07.

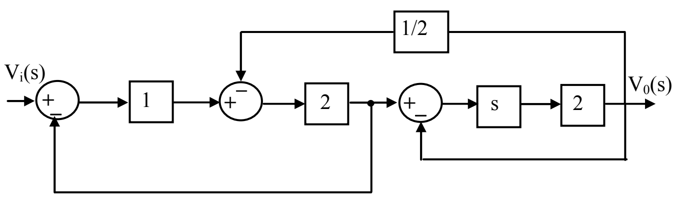
Sol:

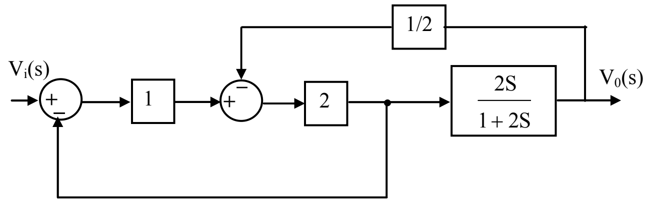


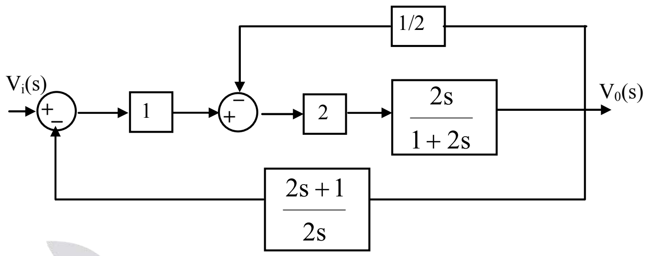


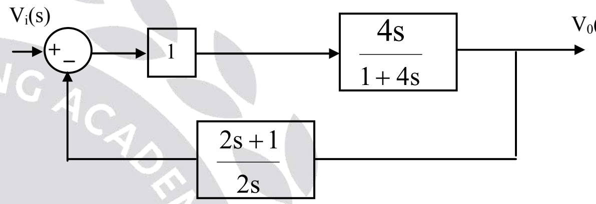












$$\Rightarrow \frac{V_0(s)}{V_i(s)} = \frac{\dfrac{4s}{1+4s}}{1+\dfrac{2(2s+1)}{1+4s}} = \frac{4s}{8s+3}$$

08.

Sol: Apply Mason's Gain formula

$$M = \frac{Y_{out}}{Y_{in}} = \frac{\sum_{k=1}^{N} M_k \Delta_k}{\Delta}$$

No. of forward paths = 2

First forward path gain = $G_1G_2G_3G_4$

Second forward path gain = $G_5G_6G_7G_8$

No. of loops = 4

First loop gain = $-G_2H_2$

Second loop gain = $-G_6H_6$

Third loop gain = $-G_3H_3$

Fourth loop gain = $-G_7H_7$

Non touching loops = 4

Loop gains $\rightarrow G_2H_2G_6H_6$

$\rightarrow G_2H_2G_7H_7$

$\rightarrow G_6H_6G_7H_7$

$\rightarrow G_2H_2G_3H_3$

Transfer function =

$$\frac{G_1G_2G_3G_4(1+G_6H_6+G_7H_7)+G_5G_6G_7G_8(1+G_2H_2+G_3H_3)}{1+G_2H_2+G_3H_3+G_6H_6+G_7H_7+G_2H_2G_6H_6+G_2H_2G_7H_7+G_3H_3G_6H_6+G_3H_3G_7H_7}$$

Conventional Practice Solutions

01.

Sol: Number of forward paths from R to C = 6

$$\frac{C}{R}=\frac{M_1\Delta_1+M_2\Delta_2+M_3\Delta_3+M_4\Delta_4+M_5\Delta_5+M_6\Delta_6}{\Delta}$$

$M_1 = G_2G_4G_6$, $M_2 = G_1G_3G_5$

$M_3 = G_1G_6G_7$, $M_4 = G_2G_5G_8$

$M_5 = -G_2G_7H_1G_8G_6$

$M_6 = -G_5G_8H_2G_1G_7$

Number of individual loops = 3

$L_1 = -G_3H_1$, $L_2 = -G_4H_2$,

$L_3 = G_7G_8H_1H_2$

Gain product of two non-touching loops is

$L_1 L_2 = G_3G_4H_1H_2$

$\Delta = 1-(L_1+L_2+L_3)+(L_1 L_2)$

$\Delta = 1-(-G_3H_1-G_4H_2+G_7G_8H_1H_2)+G_3G_4H_1H_2$

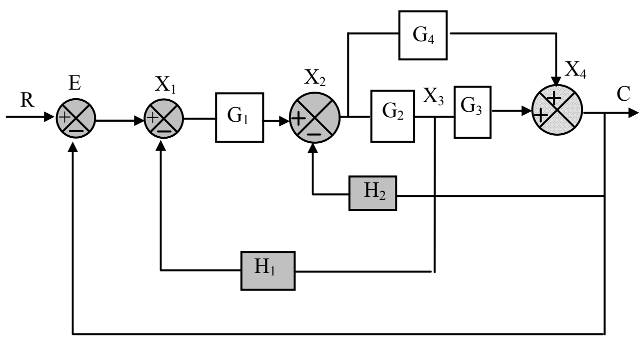
$\Delta_1 = 1+G_3H_1$, $\Delta_2 = 1+G_4H_2$, $\Delta_3 = 1$, $\Delta_4 = 1$

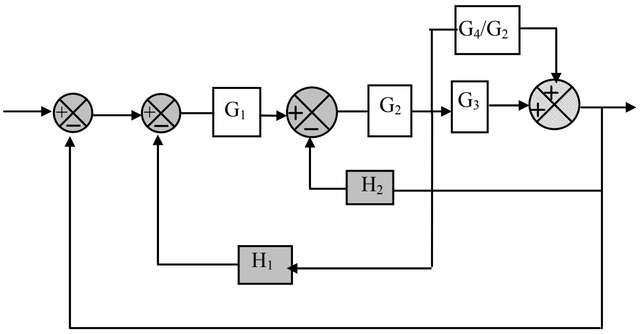
$\Delta_5 = 1$, $\Delta_6 = 1$

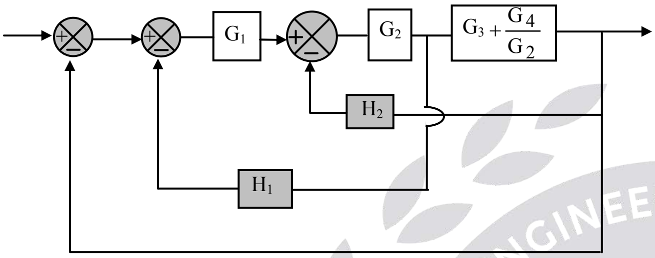
$$\frac{C}{R}=\frac{G_2G_4G_6(1+G_3H_1)+G_1G_3G_5(1+G_4H_2)+G_1G_6G_7+G_2G_5G_8-G_2G_8G_7G_6H_1-G_1G_5G_7G_8H_2}{1+G_3H_1+G_4H_2-G_7G_8H_1H_2+G_3G_4H_1H_2}$$

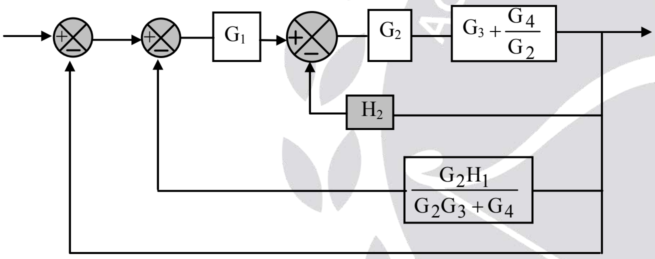
02.

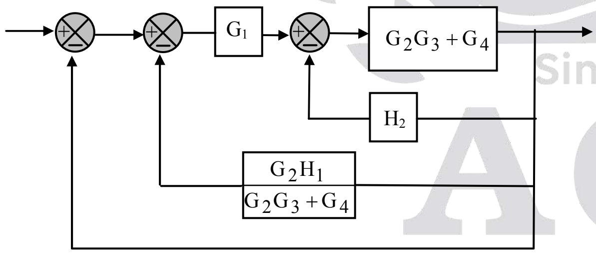
Sol: Block diagram Reduction:

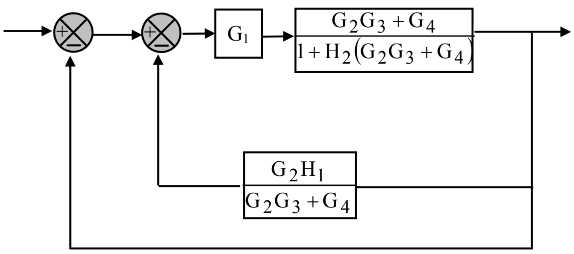


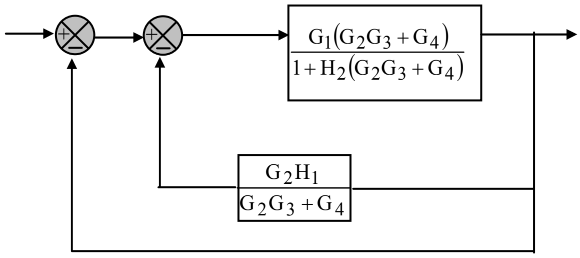


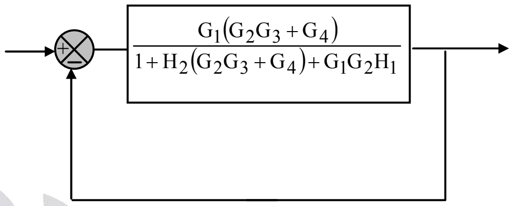


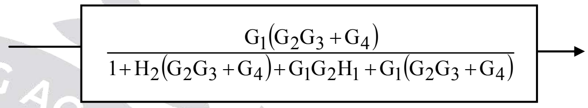












$$= \frac{G_1G_4 + G_1G_2G_3}{1 + G_2G_3H_2 + G_4H_2 + G_1G_2H_1 + G_1G_4 + G_1G_2G_3}$$

Signal flow graph is a graphical representation of set of linear algebraic equation between input and output.

Mason's Gain Formula: The relationship between an input variable and an output variable of a signal flow graph is given by

Transfer Function or Gain between X_{in} and $X_{out} = \frac{1}{\Delta}\sum_{k=1}^{n} M_k\Delta_k = \frac{X_{out}}{X_{in}}$

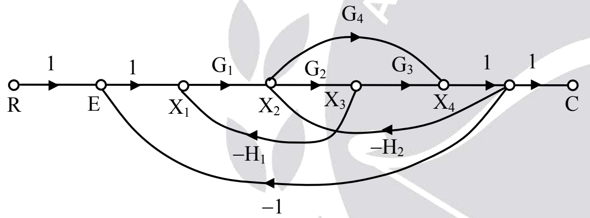
Where, n = total no. of forward paths

M_K = path gain of the k^{th} forward path

Δ_k = The value of Δ which is not touching the k^{th} forward path.

Δ = 1 – (sum of loop gains of all individual loops) + (sum of gain product of all possible combinations of two non-touching loops)- (sum of gain product of three non-touching loops) +..........

- While drawing a signal flow graph from a given block diagram the adjacent summing points and take off points (but not a take off point preceding a summing point in the direction of signal flow) are represented by a node and the block transfer function is represented by a line joining the respective nodes. The direction of signal flow is indicated by an arrow on the line.
- However, in the direction of signal flow if, a take off point precedes a summing point then such points are represented by two separate nodes with a transmittance of unity between them.



Number of forward paths from R to C = 2

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

$M_1 = G_1G_2G_3$, $M_2 = G_1G_4$

Number of individual loops = 5

$L_1 = -G_1G_2G_3$, $L_2 = -G_2G_3H_2$,

$L_3 = -G_1G_2H_1$, $L_4 = -G_1G_4$, $L_5 = -G_4H_2$,

Number of two non-touching loops = 0

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5)$$

$$\Delta = 1 - (-G_1G_2G_3 - G_2G_3H_2 - G_1G_2H_1 - G_1G_4 - G_4H_2)$$

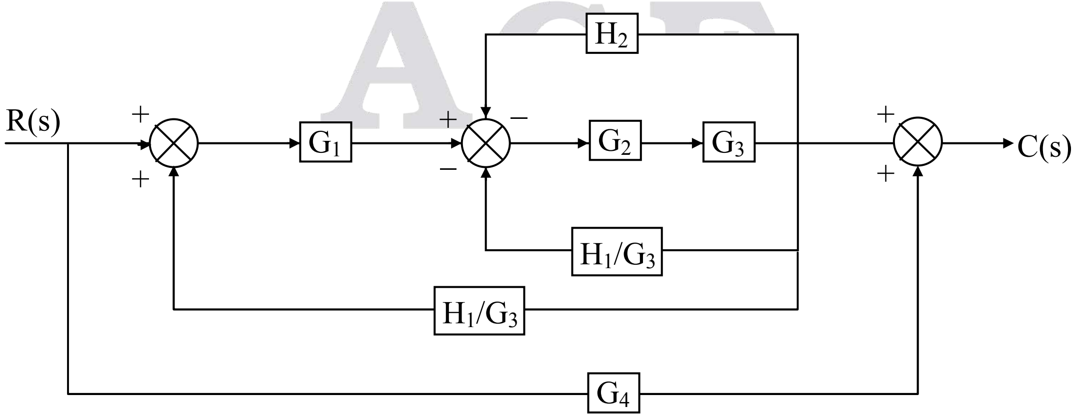
$\Delta_1 = 1$, $\Delta_2 = 1$ ($\because$ All the loops are touching forward path 1 and 2)

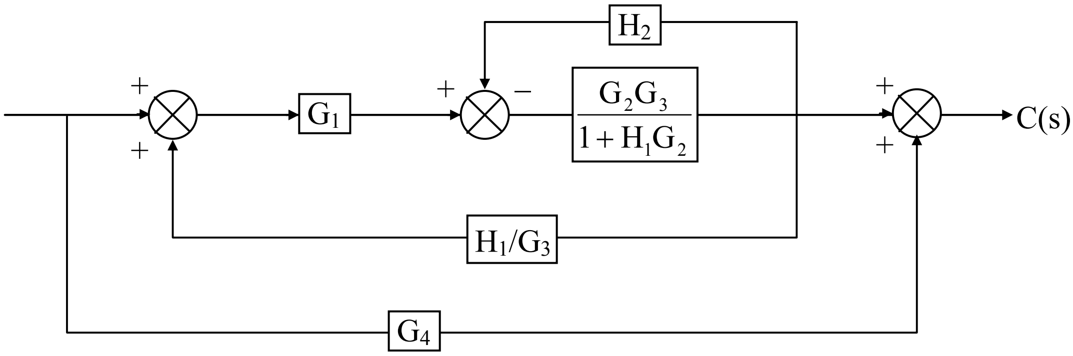
$$\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_4}{1-(-G_1G_2G_3 - G_2G_3H_2 - G_1G_2H_1 - G_1G_4 - G_4H_2)}$$

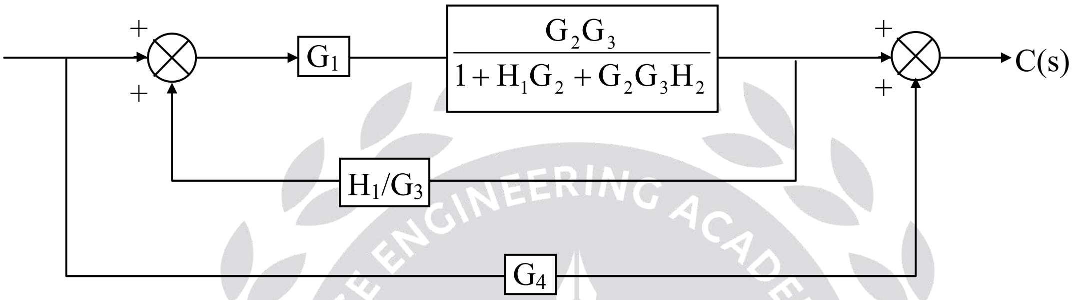
$$\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 + G_2G_3H_2 + G_1G_2H_1 + G_1G_4 + G_4H_2}$$

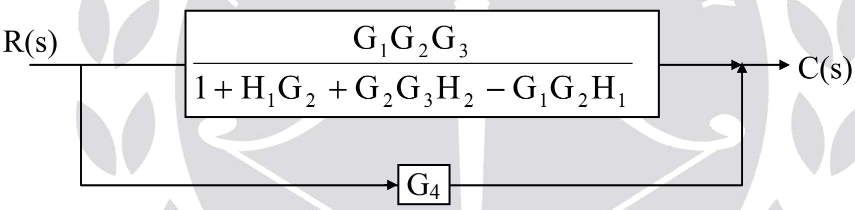
03.

Sol:





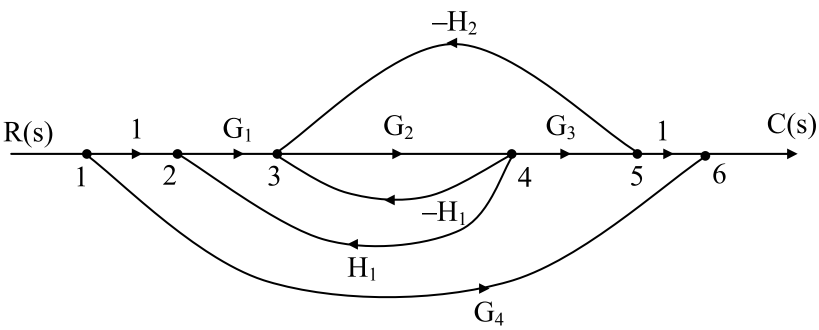




$$\frac{C(s)}{R(s)} = G_4 + \frac{G_1G_2G_3}{1+H_1G_2+G_2G_3H_2-G_1G_2H_1}$$

$$= \frac{G_1G_2G_3+G_4\left[1-G_1G_2H_1+G_2G_3H_2+G_2H_1\right]}{1-G_1G_2H_1+G_2H_1+G_2G_3H_2}$$

Signal flow graph:



From the signal flow graph

Number of forwarded paths = 2

$M_1 = G_1 G_2 G_3$

$M_2 = G_4$

$$\frac{C(s)}{R(s)} = \sum_{k=1}^{n} \frac{M_K \Delta_K}{\Delta} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

No of loops = 3

$L_1 = G_1 G_2 H_1$

$L_2 = -G_2 H_1$

$L_3 = -G_2 G_3 H_2$

No of non touching loops = 0

$\Delta = 1-(L_1 + L_2 + L_3)$

$= 1 - G_1 G_2 H_1 + G_2 H_1 + G_2 G_3 H_2$

$\Delta_1 = \Delta$ for that part of the graph which is not touching 1st forward path = 1

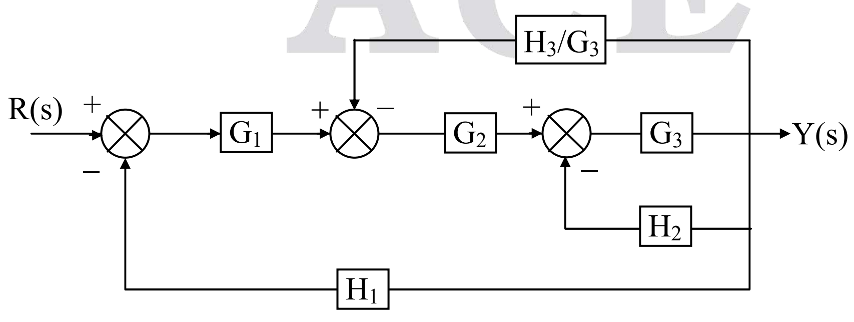
Similarly

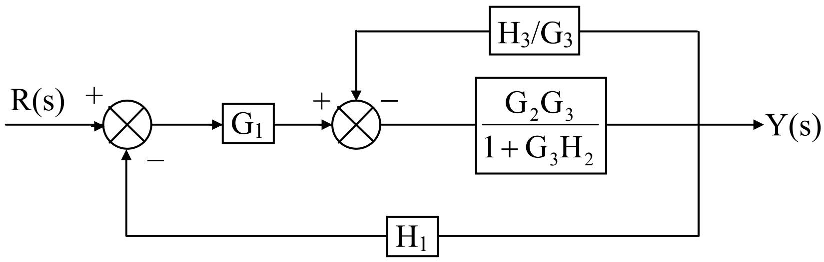
$\Delta_2 = 1 - G_1 G_2 H_1 + G_2 H_1 + G_2 G_3 H_2$

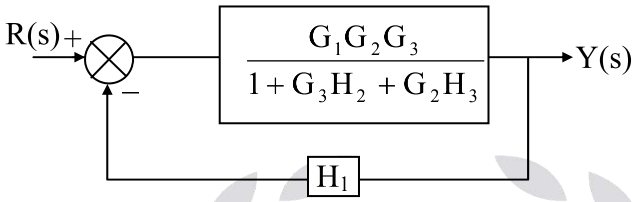
$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_4\left[1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1\right]}{1 - G_1G_2H_1 + G_2H_1 + G_2G_3H_2}$$

04.

Sol: When w(s) = 0

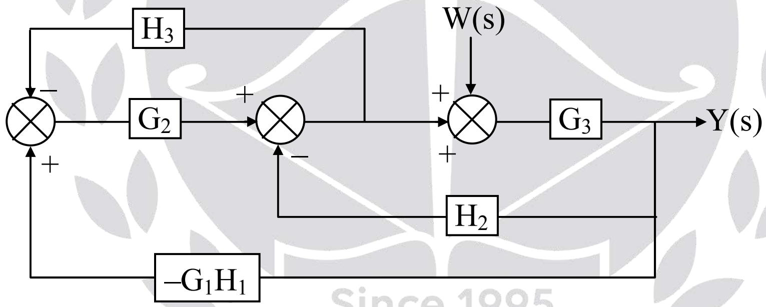




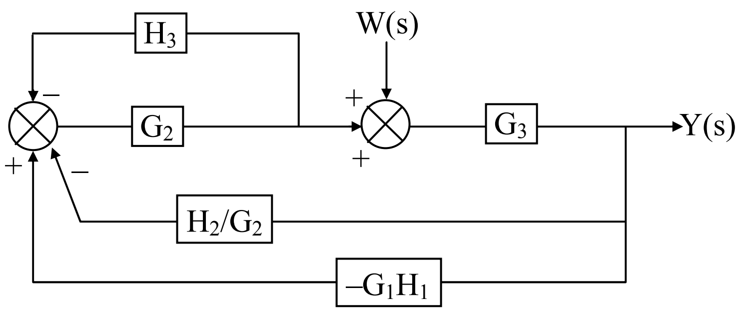


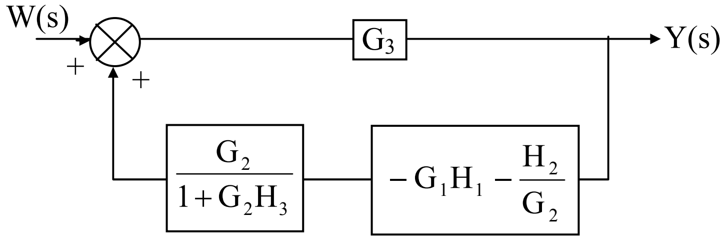
$$\therefore \left.\frac{Y(s)}{R(s)}\right|_{w=0} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1} = M(s)$$

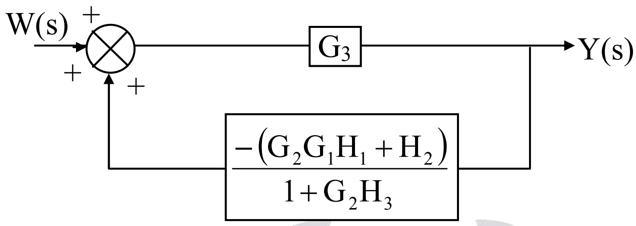
When R(s) = 0



By shifting summing point







$$M_w(s)=\left.\frac{Y(s)}{W(s)}\right|_{R=0}=\frac{G_3(1+G_2H_3)}{1+G_2H_3+G_2G_1G_3H_1+G_3H_2}$$

05.

Sol: $\frac{Y_5}{Y_1}$: no of forward paths = 2

$M_1 = G_1\,G_2\,G_3$ and $M_2 = G_4\,G_3$

No of loops = 5

$L_1 = -G_1\,H_1$, $L_2 = -G_3\,H_2$, $L_3 = -G_1\,G_2\,G_3\,H_3$, $L_4 = -G_4\,G_3\,H_3$ and $L_5 = -H_4$

Product of two non-touching loops = 2

$L_1L_5 = G_1H_1H_4$

$L_1L_2 = G_1H_1G_3H_2$

$$\therefore \frac{Y_5}{Y_1}=\frac{M_1\Delta_1+M_2\Delta_2}{1-(L_1+L_2+L_3+L_4+L_5)+[L_1L_5+L_1L_2]}$$

$$=\frac{G_1G_2G_3+G_3G_4}{1+G_1H_1+G_3H_2+G_1G_2G_3H_3+G_4G_3H_3+H_4+G_1H_1H_4+G_1H_1G_3H_2}$$

$\frac{Y_4}{Y_1}$:

Forward paths = 2

$M_1 = G_1\,G_2$

$M_2 = G_4$

And $\Delta_1 = (1+H_4)$

$\Delta_2 = (1+H_4)$

We know that $\Delta = (1-(L_1 + L_2 + L_3 + L_4 + L_5) + [L_1L_2 + L_1L_5]) = 0$

$$\therefore \frac{Y_4}{Y_1} = \frac{G_1G_2(1+H_4)+G_4(1+H_4)}{1+G_1H_1+G_3H_2+G_1G_2G_3H_3+G_4G_3H_3+H_4+G_1H_1H_4+G_1H_1G_3H_2}$$

$\dfrac{Y_2}{Y_1}$:

Forward paths = 1

$M_1 = 1$

$\Delta_1 = [1 + H_4 + G_3H_2]$

$$\therefore \frac{Y_2}{Y_1} = \frac{M_1\Delta_1}{\Delta} = \frac{[1+G_3H_2+H_4]}{1+G_1H_1+G_3H_2+G_1G_2G_3H_3+G_4G_3H_3+H_4+G_1H_1H_4+G_1H_1G_3H_2}$$

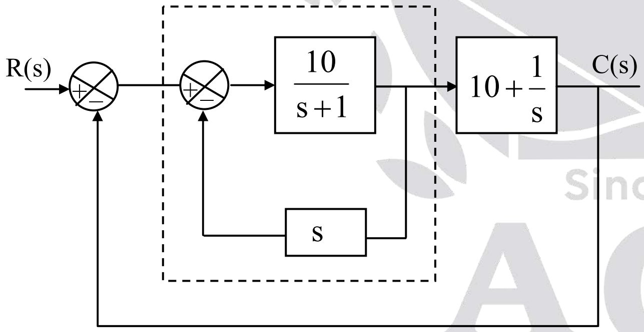
$\dfrac{Y_5}{Y_2}$:

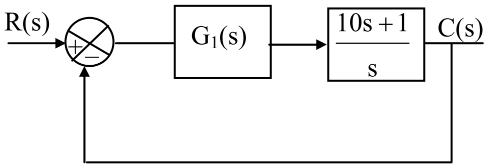
$$\frac{Y_5}{Y_2} = \frac{Y_5}{Y_1} \times \frac{Y_1}{Y_2}$$

$$\Rightarrow \frac{G_1G_2G_3+G_3G_4}{1+G_3H_2+H_4}$$

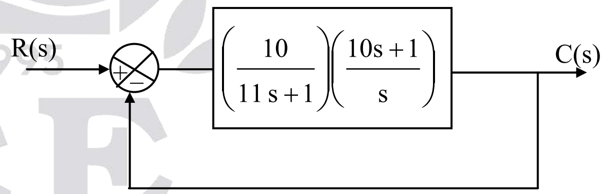
06.

Sol:





$$\text{Where } G_1(s) = \frac{\left(\frac{10}{s+1}\right)}{1+\left(\frac{10}{s+1}\right)(s)} = \frac{10}{11\,s+1}$$



$$\text{Where } G(s) = \frac{10(10S+1)}{S(11\,S+1)}$$

$H(s) = 1$

Chapter 3 Time Response Analysis

Objective Practice Solutions

01. Ans: (a)

Sol: $\dfrac{C(s)}{R(s)} = \dfrac{1}{1+sT}$, $R(s) = \dfrac{8}{s}$

$$C(s) = \frac{8}{s(1+sT)} \Rightarrow c(t) = 8\left(1-e^{-t/T}\right)$$

$$3.6 = 8\left(1-e^{\frac{-0.32}{T}}\right)$$

$$0.45 = 1 - e^{\frac{-0.32}{T}}$$

$$0.55 = e^{\frac{-0.32}{T}}$$

$$-0.59 = \frac{-0.32}{T}$$

T = 0.535 sec

02. Ans: (c)

Sol: $\cos\phi = \xi$

$\cos 60 = 0.5$

$\cos 45 = 0.707$

Poles left side $0.5 \le \xi \le 0.707$

Poles right side $-0.707 \le \xi \le -0.5$

$\therefore\ 0.5 \le |\xi| \le 0.707$

3 rad/s $\le \omega_n \le$ 5 rad/s

03. Ans: (c)

Sol: For R-L-C circuit:

$$T.F = \frac{V_o(s)}{V_i(s)}$$

$$V_o(s) = \frac{1}{Cs} I(s)$$

$$= \frac{1}{Cs}\frac{V_i(s)}{R + Ls + \frac{1}{Cs}}$$

$$T.F = \frac{V_o(s)}{V_i(s)} = \frac{1}{RCs + LCs^2 + 1}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\omega_n = \frac{1}{\sqrt{LC}} \qquad 2\xi\omega_n = \frac{R}{L}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\xi = \frac{10}{2}\sqrt{\frac{10\times10^{-6}}{1\times10^{-3}}} = 0.5$$

$$M.P = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

$= 16.3\% \approx 16\%$

04. Ans: (b)

Sol: $$TF = \frac{8/s(s+2)}{1-\left(\frac{-8\,as}{s(s+2)} - \frac{8}{s(s+2)}\right)}$$

$$= \frac{8}{s(s+2)+8as+8}$$

$$= \frac{8}{s^2+2s+8as+8}$$

$$= \frac{8}{s^2+(2+8a)s+8}$$

$\omega_n^2 = 8 \Rightarrow \omega_n = 2\sqrt{2}$

$2\xi\omega_n = 2 + 8a$

$\xi = \dfrac{1+4a}{2\sqrt{2}}$

$\dfrac{1}{\sqrt{2}} = \dfrac{1+4a}{2\sqrt{2}} \Rightarrow a = 0.25$

05. Ans: 4 sec

Sol: $\text{T.F} = \dfrac{100}{(s+1)(s+100)}$

$= \dfrac{100}{s^2 + 101s + 100}$

$\omega_n^2 = 100$

$\omega_n = 10$

$2\xi\omega_n = 101$

$\xi = \dfrac{101}{20}$

$\xi > 1 \rightarrow$system is over damped i.e., roots are real & unequal.

Using dominate pole concept,

$\text{T.F} = \dfrac{100}{100(s+1)} = \dfrac{1}{s+1}$, Here $\tau = 1$ sec

$\therefore$ Setting time for 2% criterion $= 4\tau$

$= 4$ sec

06.

Sol: $M_p = \dfrac{C(t_p) - C(\infty)}{C(\infty)}$

$= \dfrac{1.254 - 1.04}{1.04} = 0.2$

$\xi = \sqrt{\dfrac{(\ln M_P)^2}{(\ln M_P)^2 + \pi^2}}$

$M_p = 0.2$; $\xi = 0.46$

07. Ans: (d)

Sol: Given data**:** $\omega_n = 2$, $\zeta = 0.5$

Steady state gain =1

$\text{OLTF} = \dfrac{K_1}{s^2 + as + 2}$ and $H(s) = K_2$

$\text{CLTF} = \dfrac{G(s)}{1+G(s)}$

$\dfrac{C(s)}{R(s)} = \dfrac{K_1}{s^2 + as + 2 + K_1K_2}$

DC or steady state gain from the TF

$\dfrac{K_1}{2 + K_1K_2} = 1$

$K_1(1 - K_2) = 2$ ……… (1)

CE is $s^2 + as + 2 + K_1K_2 = 0$

$\omega_n = \sqrt{2 + K_1K_2} = 2$

$4 = (2 + K_1K_2)$

$K_1K_2 = 2$ ………. (2)

Solving equations (1) & (2) we get

$K_1 = 4$, $K_2 = 0.5$

$2\zeta\,\omega_n = a$

$2 \times \dfrac{1}{2} \times 2 = a$

$a = 2$

08. Ans: A – T, B – S, C- P, D – R, E – Q

Sol:

(A) If the poles are real & left side of s-plane, the step response approaches a steady state value without oscillations.

(B) If the poles are complex & left side of s-plane, the step response approaches a steady state value with the damped oscillations.

(C) If poles are non-repeated on the $j\omega$ axis, the step response will have fixed amplitude oscillations.

(D) If the poles are complex & right side of s-plane, response goes to '∞' with damped oscillations.

(E) If the poles are real & right side of s-plane, the step response goes to '∞' without any oscillations.

09. Ans: (c)

Sol: If R ↑ damping ↑

$$\Rightarrow \xi = \frac{R}{2}\sqrt{\frac{C}{L}}$$

(i) If R↑, steady state voltage across C will be reduced (wrong)

(Since steady state value does not depend on ξ)

If ξ ↑, C (∞) = remain same

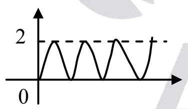
(ii) If ξ↑, ω_d ↓ $\left(\omega_d = \omega_n\sqrt{1-\xi^2}\right)$

(iii) If ξ ↓, t_s ↑ ⇒ 3rd

Statement is false

(iv) If ξ = 0

True



⇒ 2 and 4 are correct

10.

Sol: (i) Unstable system

∴ error = ∞

(ii) $G(s) = \frac{10(s+1)}{s^2}$

Step → $R(s) = \frac{1}{s}$

$k_p = \infty$

$$e_{ss} = \frac{A}{1+k_p} = \frac{1}{1+\infty} = 0$$

Parabolic ⇒ $k_a = 10$

$$e_{ss} = \frac{1}{10} = 0.1$$

11.

Sol: $G(s) = 10/s^2$ (marginally stable system)

∴ Error can't be determined

12.

Sol: $e_{ss} = \frac{1}{11}$, $R(s) = \frac{1}{s}$

$$e_{ss} = \frac{A}{1+k_p} = \frac{1}{1+k_p} = \frac{1}{11} = \frac{1}{1+10}$$

$$k_p = \underset{s\to 0}{\mathrm{Lt}}\ G(s)$$

$$10 = \underset{s\to 0}{\mathrm{Lt}}\ G(s)$$

$k = 10$

$R(s) = \frac{1}{s^2}$ (ramp)

$$e_{ss} = \frac{A}{k_v} = \frac{1}{k_v} = \frac{1}{10}$$

(System is increased by 1)

⇒ $e_{ss} = 0.1$

13. Ans: (a)

Sol: $T(s) = \frac{(s-2)}{(s-1)\ (s+2)^2}$ (unstable system)

14. Ans: (b)

Sol: Given data: r(t) = 400tu(t) rad/sec

Steady state error =10°

i.e., $e_{ss} = \dfrac{\pi}{180^o}(10^o)$ radians

$G(s) = \dfrac{20K}{s(1+0.1s)}$ and $H(s) = 1$

$r(t) = 400tu(t) \Rightarrow 400/s^2$

Error $(e_{ss}) = \dfrac{A}{K_V} = \dfrac{400}{K_V}$

$K_V = \lim\limits_{s\to 0} s\,G(s)$

$K_V = \lim\limits_{s\to 0} s\dfrac{20K}{s(1+0.1s)}$

$K_V = 20K$

$e_{ss} = \dfrac{400}{20K}$

$e_{ss} = \dfrac{20}{K} = \dfrac{\pi}{18}$

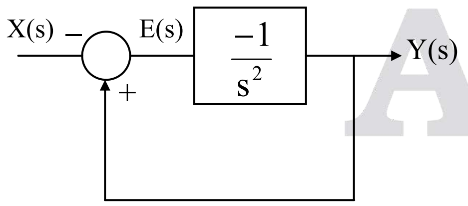
$K = 114.5$

15. Ans: (d)

Sol: $\dfrac{d^2y}{dt^2} = -e(t)$

$s^2 Y(s) = -E(s)$

$x(t) = t\,u(t) \Rightarrow X(s) = \dfrac{1}{s^2}$



$Y(s) = \dfrac{-1}{s^2}E(s)$

$\dfrac{Y(s)}{E(s)} = \dfrac{-1}{s^2}$

$\dfrac{E(s)}{X(s)} = \dfrac{-1}{1+\dfrac{1}{s^2}}$

$E(s) = \dfrac{-s^2}{1+s^2}X(s)$

$= \dfrac{-s^2}{1+s^2}\times\dfrac{1}{s^2} = \dfrac{-1}{1+s^2}$

$= L^{-1}\left[\dfrac{-1}{1+s^2}\right] = -\sin t$

16. Ans: (a)

Sol: $e_{ss} = 0.1$ for step input

For pulse input = 10

time = 1 sec

error is function of input

$t \to \infty$ input = 0

$\therefore$ Error = zero

17. Ans: (c)

Sol: $\dfrac{C(s)}{R(s)} = \dfrac{\dfrac{100}{(s+1)(s+5)}}{1+\dfrac{100\times 0.2}{(s+1)(s+5)}}$

$= \dfrac{100}{(s+1)(s+5)+20}$

$= \dfrac{100}{s^2+6s+5+20}$

$= \dfrac{100}{s^2+6s+25}$

$\omega_n^2 = 25$, $\omega_n = 5$

$2\xi\omega_n = 6$

$\xi = \dfrac{6}{10} = \dfrac{3}{5}$

$\omega_d = \omega_n\sqrt{1-\xi^2}$

$$= 5\sqrt{1-\left(\frac{3}{5}\right)^2}$$

$$= 5\times\frac{4}{5} = 4 \text{ rad/sec}$$

18. Ans: (c)

Sol: $f(t) = \dfrac{Md^2x}{dt^2} + B\dfrac{dx}{dt} + Kx(t)$

Applying Laplace transform on both sides, with zero initial conditions

$F(s) = Ms^2X(s)+BsX(s) + KX(s)$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}$$

Characteristic equation is $Ms^2 + Bs+K = 0$

$$s^2 + \frac{B}{M}s + \frac{K}{M} = 0$$

Compare with $s^2 + 2\zeta\,\omega_n s + \omega_n^2 = 0$

$$2\zeta\omega_n = \frac{B}{M}$$

$$\xi = \frac{B}{2\sqrt{MK}} \qquad \omega_n = \sqrt{\frac{K}{M}}$$

Time constant $T = \dfrac{1}{\zeta\omega_n} = \dfrac{1}{B}\times 2M$

$$T = \frac{2M}{B}$$

Hence, statements 2 & 3 are correct

19. Ans: (c)

Sol: type 1 system has a infinite positional error constant.

20. Ans: (a)

Sol: Given $G(s) = \dfrac{1}{s(1+s)(s+2)}, H(s) = 1$.

It is type-I system

Positional error constant $k_p = \underset{s\to 0}{Lt}\ G(s)H(s)$

$$k_p = \underset{s\to 0}{Lt}\ \frac{1}{s(1+s)(s+2)}$$

$$= \infty$$

Steady state error due to step input

$$= \frac{1}{1+k_p} = 0$$

21.

Sol: Open loop T/F $G(s) = \dfrac{A}{S(S+P)}$

$$\text{C.L T/F} = \frac{A}{S^2+SP+A}$$

$$\omega_n = \sqrt{A}$$

Setting time = $4/\xi\omega_n = 4$

$2\xi\omega_n = P \qquad \therefore \dfrac{4}{P/2} = 4$

$\xi\omega_n = P/2 \qquad \Rightarrow P = \dfrac{8}{4} = 2$

$$e^{\frac{-\pi\xi}{\sqrt{1+\xi^2}}} = 0.1 \Rightarrow \frac{\pi\xi}{\sqrt{1-\xi^2}} = \ell n10 = 2.3$$

$$\Rightarrow \frac{\xi^2}{1-\xi^2} = 0.5373$$

$$\Rightarrow 1.5373\ \xi^2 = 0.5373$$

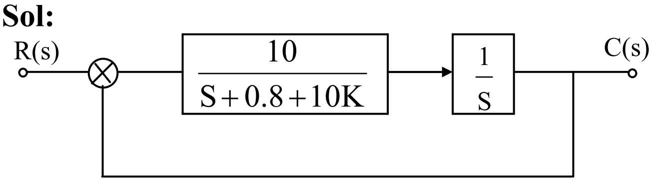
$$\xi = 0.59$$

$\xi\omega_n = 1$

$\Rightarrow \omega_n = 1.694 \Rightarrow A = \omega_n^2 = 2.87$

22.

Sol:



$$\frac{C(s)}{R(s)} = \frac{10}{s(s+0.8+10K)+10}$$

$$= \frac{10}{s^2+s(0.8+10K)10}$$

$\omega_n = \sqrt{10}$ $\quad\quad 2\xi\omega_n = 0.8 + 10\text{ K}$

$$\Rightarrow 2\times\frac{1}{2}\times\sqrt{10} = 0.8+10K$$

$$\Rightarrow K = 0.236$$

$$t_r = \frac{\pi-\phi}{\omega_d} = \frac{\pi-\cos^{-1}(\xi)}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi-\pi/3}{2.88} = 0.74\sec$$

$$t_p = \frac{\pi}{\omega_d} = 1.1\sec$$

$$\%Mp = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} = 0.163\times 100 = 16.3\%$$

$$t_s\ (\text{for } 2\%) = \frac{4}{\xi\omega_n} = \frac{4}{0.5\times\sqrt{10}} = 2.53\sec$$

Conventional Practice Solutions

01.

Sol: $\%\ M_p = 25\%$

$$t_p = \frac{\pi}{\omega_d} = 2\sec$$

$$T.F = \frac{k_1/s^2}{1+(1+k_2s)\frac{k_1}{s^2}} = \frac{k_1}{s^2+k_1+k_1k_2s}$$

$$e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.25$$

$\xi = 0.4037$

$$\omega_d = \frac{\pi}{2} = \omega_n\sqrt{1-\xi^2}$$

$\omega_n = 1.7169$ rad/sec

$\omega_n = \sqrt{k_1}$

$k_1 = \omega_n^2 = 1.7169^2 = 2.94$

$k_1 = 2.94$

$2\xi\omega_n = k_1k_2$

$$\frac{2\times 0.4\times 1.72}{2.94} = k_2 \Rightarrow k_2 = 0.472$$

02.

Sol: $T.F = \dfrac{A}{s^2+ks+A}$

Given $\xi = 0.6$

$\omega_d = 8$

$\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\omega_n = \frac{8}{\sqrt{1-(0.6)^2}} = 10 \text{ rad/sec}$$

Comparing with standard second order system, TF $= \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n = 10$rad/sec

$A = \omega_n^2 = 10^2 = 100$

A = 100

$2\xi\omega_n = k$

$2(0.6)(10) = k$

k = 12

input $R(s) = \dfrac{2}{S}$

$$\frac{C(s)}{R(s)} = \frac{100}{s^2 + 12s + 100}$$

$$\therefore \%M_P = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100 = 9.47\%$$

$$\frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 = 9.47$$

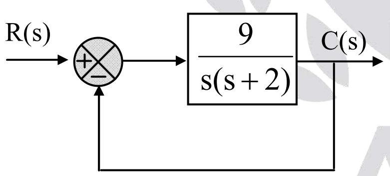
$$c(t_p) = \frac{9.47}{100} \times c(\infty) + c(\infty) = \frac{9.47}{100} \times (2) + 2$$

$c(t_p) = 2.189$

$\therefore$ Peak value of response = 2.189

03.

Sol: (i) In the absence of derivative feedback (a = 0), the system can be represented as



$$\frac{C(s)}{R(s)} = \frac{\frac{9}{s(s+2)}}{1 + \frac{9}{s(s+2)}} = \frac{9}{s^2 + 2s + 9}$$

$CE = s^2 + 2s + 9 = 0 \Rightarrow s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\omega_n^2 = 9$ and $2\zeta\omega_n = 2$

$\omega_n = \sqrt{9}$ $\zeta\omega_n = 1$

$\omega_n = 3$ rad/sec $\zeta.3 = 1 \Rightarrow \zeta = \dfrac{1}{3}$

Damping ratio $\xi = \dfrac{1}{3}$

Natural undamped frequency = 3 rad/sec

Calculation of steady state error for a given ramp input $r(t) = tu(t)$ implies $R(s) = \dfrac{1}{s^2}$

Steady state error for unit ramp input is

$$e_{ss} = \frac{A}{K_v} = \frac{1}{K_v} \quad (A=1)$$

$K_v = \lim_{s\to 0} s.G(s)$ = Velocity error constant

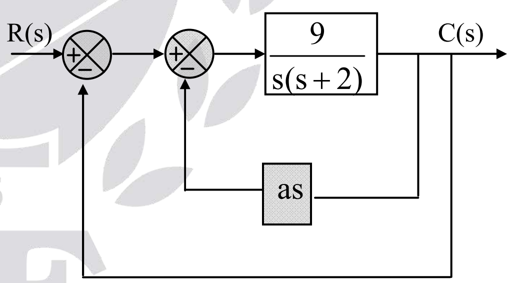
Given open loop transfer function is

$$G(s) = \frac{9}{s(s+2)}$$

$$K_v = \lim_{s\to 0} s.\frac{9}{s(s+2)} = \frac{9}{0+2} = \frac{9}{2}$$

$$\& \quad e_{ss} = \frac{1}{\frac{9}{2}} = \frac{2}{9} = 0.22$$

(ii) By Mason's gain formula



$$\frac{C(s)}{R(s)} = \frac{\frac{9}{s(s+2)}}{1 - \left(-\frac{9as}{s(s+2)} - \frac{9}{s(s+2)}\right)}$$

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + s(9a+2) + 9}$$

$CE = s^2 + (9a+2)s + 9 = 0$

$= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

Comparing, we get

$\omega_n = 3\,\text{rad/sec}$

$2\zeta\omega_n = (9a+2)$ (ζ=0.7 given)

$(2)(0.7)(3) = (9a+2)$

$4.2 = 9a+2$

$a = 0.24$

Steady state error for unit ramp input is

$$e_{ss} = \frac{A}{K_v} = \frac{1}{K_v} \quad (A=1)$$

$$G(s) = \frac{9}{s(s+2)+9as} = \frac{9}{s(s+2+9a)}$$

$$K_v = \lim_{S\to 0} sG(s) = \frac{9}{2+9(0.24)} = 2.16 \;\&$$

$$e_{ss} = \frac{1}{2.16} = 0.46$$

(iii) Given data ζ=0.7 and $e_{ss} = \frac{2}{9}$ as in

Case (i):

The gain of 9 in the forward loop be adjusted to higher value 'K' to reduce the steady state error

$$\therefore \frac{K}{s(s+2)}$$

$$\text{With this, } G(s) = \frac{K}{s(s+2)+Kas}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+2)+Kas+K}$$

CE is $s^2 + s(2+Ka) + K = 0$ ---------- (1)

and $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ ---------- (2)

Comparing equation (1) and (2), we get

$\omega_n^2 = K$ and $\omega_n = \sqrt{K}$

$2\zeta\omega_n = 2+aK \Rightarrow 2\zeta.\sqrt{K} = 2+aK$

$$\zeta = \frac{2+Ka}{2\sqrt{K}} = 0.7 \quad \text{---------- (3)}$$

Steady state error for unit ramp input is as in case (i)

$$e_{ss} = \frac{1}{K_v} = \frac{2}{9} \quad \text{implies } K_v = \frac{9}{2}$$

$$K_v = \lim_{S\to 0} sG(s)$$

$$= \mathop{Lt}_{s\to 0} s\frac{K}{s(s+2)+Kas} = \frac{K}{Ka+2} = \frac{9}{2}$$

$$\frac{K}{Ka+2} = \frac{9}{2} \quad \text{---------- (4)}$$

Substituting equation (3) in (4)

$$\frac{K}{2(0.7)\sqrt{K}} = \frac{9}{2} \quad \text{gives } K=39.69$$

Substituting the value of K in (4) gives a = 0.1718

Therefore, K = 39.69 and a = 0.1718

By increasing gain to 39.69 and maintain a = 0.1718, we will get the required response.

04.

Sol: Given $M_p = 0.2$, $C = 10^{-6}$F and L = 1H

R = ?

From given diagram,

$$\frac{V_0(s)}{V_i(s)} = \frac{1/sC}{R+Ls+1/sC}$$

$$TF = \frac{1}{LCs^2+sRC+1} = \frac{1/LC}{s^2+\frac{R}{L}s+1/LC}$$

By comparing above transfer function with standard second order equation.

$$2\,\xi\,\omega_n = \frac{R}{L} \text{ and } \omega_n = \frac{1}{\sqrt{LC}}$$

Where $\xi = \sqrt{\dfrac{(\ell n M_p)^2}{\pi^2 + (\ell n M_p)^2}}$

$$= \sqrt{\frac{(\ell n 0.2)^2}{\pi^2 + (\ell n 0.2)^2}}$$

$= 0.456$

$$2\xi \times \frac{1}{\sqrt{LC}} = \frac{R}{L}$$

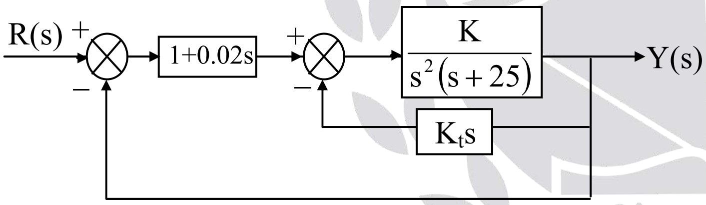
$$2\xi = R\sqrt{\frac{C}{L}}$$

$$R\sqrt{10^{-6}} = 0.912$$

$\Rightarrow R = 912\Omega$

05. (a)

Sol: Input is unit ramp and n(t) = 0



Inner block becomes, $\dfrac{K(1+0.02s)}{s^2(s+25)+K_t sK}$

Now error $e_{ss} = \dfrac{A}{K_V}$

Where

$$K_V = \underset{S\to 0}{Lt}\, sG(s) = \underset{S\to 0}{Lt}\, \frac{sK(1+0.02s)}{s\left[s(s+25)+K_t K\right]}$$

$$= \frac{K}{KK_t}$$

$= 1/K_t$

Error $e_{ss} = \dfrac{1}{K_V} = K_t$

To get the constraints on the value of K and K_t

C.E = $s^2(s+25) + KK_t s + K(1+0.025s) = 0$

$s^3 + 25s^2 + K(K_t + 0.02)s + K = 0$

$$\begin{array}{c|cc} s^3 & 1 & K(K_t+0.02) \\ s^2 & 25 & K \\ s^1 & \dfrac{25K(K_t+0.02)}{25} & 0 \\ s^0 & K & \end{array}$$

For the system to be stable, K> 0, and

$$\frac{25K(K_t+0.02)-K}{25} > 0$$

$\Rightarrow 25K(K_t + 0.02) > K$

$\Rightarrow K_t + 0.02 > 1/25$

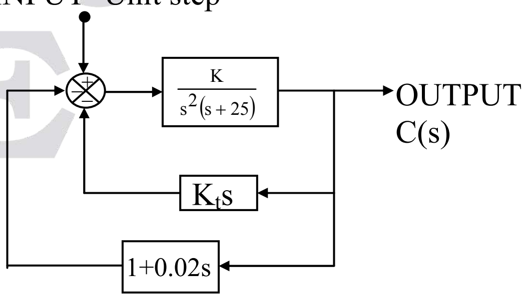
$\Rightarrow K_t > 0.04 - 0.02$

$\Rightarrow K_t > 0.02$

The constraints are $K_t > 0.02$ and $K > 0$

Now the block diagram is,

(b) INPUT=Unit step



By applying Masson's gain formula

$$\frac{C(s)}{R(s)} = \frac{K}{s^2(s+25)+KK_t s+K(1+0.02s)}$$

$R(s) = \dfrac{1}{s}$ (step input)

$$C(s) = \frac{K}{s\left(s^2(s+25) + KK_t s + K(1+0.02s)\right)}$$

$$C(\infty) = \lim_{s\to 0} sC(s)$$

$$= \lim_{s\to 0} s\frac{K}{s\left(s^2(s+25)+KK_t s+K(1+0.02s)\right)}$$

$$=1$$

06.

Sol: The given second order system

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = F(t)$$

Taking Laplace transform on both side we will get

$as^2\theta(s) + bs\theta(s) + c\theta(s) = F(s)$

$\Rightarrow (as^2 + bs + c)\,\theta(s) = F(s)$

$$\text{Transfer function} = \frac{\theta(s)}{F(s)} = \frac{1}{as^2 + bs + c}$$

$$\therefore \text{ Step response } \theta(s) = \left(\frac{1}{as^2 + bs + c}\right)\frac{1}{s}$$

$$\frac{1}{c}\left(\frac{\frac{c}{a}}{s^2 + \frac{b}{a}s + \frac{c}{a}}\right)\frac{1}{s}$$

$$= \left(\frac{1/a}{s^2 + \frac{b}{a}s + \frac{c}{a}}\right)\frac{1}{s}$$

compare with standard second system, with unit step input applied, we get response as

$$\frac{\omega_n^2}{s\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}$$

$$\theta(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_d t + \phi\right)$$

$$\omega_n = \sqrt{\frac{c}{a}}$$

$$2\xi\omega_n = \frac{b}{a}$$

$$\Rightarrow \xi = \frac{1}{2}\frac{b}{\sqrt{ac}}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$= \sqrt{\frac{c}{a}}\sqrt{1-\frac{1}{4}\frac{b^2}{ac}}$$

$$= \sqrt{\frac{c}{a}}\sqrt{\frac{4ac-b^2}{4ac}}$$

$$\omega_d = \frac{1}{2a}\sqrt{4ac-b^2}$$

Let assume $\xi = \frac{1}{2}\frac{b}{\sqrt{ac}} < 1$

Then

$$\theta(t) = \frac{1}{c}\left[1 - \frac{e^{\frac{-bt}{2a}}}{\sqrt{\frac{4ac-b^2}{4ac}}}\sin\left\{\left(\frac{1}{2a}\sqrt{4ac-b^2}\right)t + \cos^{-1}\left(\frac{b}{2\sqrt{ac}}\right)\right\}\right]$$

Chapter 4 Stability

Objective Practice Solutions

01.

Sol: CE = $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

s^5	1	8	7	
s^4	4(1)	8(2)	4(1)	
s^3	6(1)	6(1)	0	
s^2	1	1	0	$\rightarrow$ Row of AE
s^1	0(2)	0	0	$\rightarrow$ Row of zero
s^0	1			

No. of AE roots = 2
No. of sign changes
Below AE = 0
No. of RHP = 0
No. of LHP = 0
No. of $j\omega p$ = 2

No. of CE roots = 5
No. of sign changes
in 1st column = 0
$\therefore$ No .of RHP = 0
No. of $j\omega p$ = 2
$\Rightarrow$ No .of LHP = 3

System is marginally stable.

(ii) $s^2 + 1 = 0$

$s = \pm 1\, j = \pm j\omega_n$

$\omega_n = 1$ rad/ sec

Oscillating frequency $\omega_n = 1$ rad/sec

02.

Sol: (i) $s^5 + s^4 + s^3 + s^2 + s + 1 = 0$

$+ s^5$	1	1	1
$+ s^4$	1	1	1
$+ s^3$	0(2)	0(1)	0
$+ s^2$	$\frac{1}{2}$	1	
(1) $- s^1$	-3	0	
(2) $+ s^0$	1		

AE (1) = $s^4 + s^2 + 1 = 0$

$$\frac{d(AE)}{ds} = 4s^3 + 2s = 0$$

$\Rightarrow 2s^3 + s = 0$

AE

No. of sign changes below AE = 2
No. of AE roots = 4
No .of RHP = 2
No .of LHP = 2
No. of $j\omega p$ = 0

CE

No. of sign changes in 1st column = 2
No. of CE roots = 5
No. of RHP = 2
No. of LHP = 3
No. of $j\omega p$ = 0

System is unstable

(ii) $s^6 + 2s^5 + 2s^4 + 0s^3 - s^2 - 2s - 2 = 0$

s^6	1	2	-1	-2
s^5	2(1)	0	$-2(-1)$	0
s^4	2(1)	+0	$-2(-1)$	0
s^3	0(4)	0	0	0
s^2	0(ε)	-1	0	0
s^1	4/ε			
$-s^0$	-1			

AE = $s^4 - 1 = 0$

$$\frac{dAE}{ds} = 4s^3 + 0 = 0$$

CE	AE
No. of CE roots = 6	No. of AE roots = 4
No. of sign changes in the 1st column= 1	No. of sign changes below AE = 1
No .of RHP = 1	No. of RHP = 1
No .of LHP = 3	No. of $j\omega$p = 2
No. of $j\omega$p = 2	No. of LHP = 1

03.

Sol: CE = $s^3 + 20\,s^2 + 16s + 16\,K = 0$

$$\begin{array}{c|cc} s^3 & 1 & 16 \\ s^2 & 20 & 16K \\ s^1 & \frac{20(16)-16K}{20} & 0 \\ s^0 & 16K & \end{array}$$

(i) For stability $\dfrac{20(16)-16K}{20} > 0$

$\Rightarrow 20\,(16) - 16\,K > 0$

$\Rightarrow K < 20$ and $16\,K > 0 \;\Rightarrow K > 0$

Range of K for stability $0 < K < 20$

(ii) For the system to oscillate with ω_n it must be marginally stable

i.e., s^1 row should be 0

s^2 row should be AE

$\boxed{\therefore \text{A.E roots} = \pm j\omega_n}$

$\therefore\ s^1$ row $\Rightarrow 20\,(16) - 16\,K = 0$

$\Rightarrow K = 20$

AE is $20s^2 + 16\,K = 0$

$20s^2 + 16\,(20) = 0$

$\Rightarrow s = \pm\, j4$

$\omega_n = 4$ rad/sec

04.

Sol: CE = $1 + \dfrac{K(s+1)}{s^3 + as^2 + 2s + 1} = 0$

$s^3 + as^2 + (K + 2)\,s + K + 1 = 0$

$s^3 + as^2 + (K+2)\,s + (K+1) = 0$

$$\begin{array}{c|cc} s^3 & 1 & K+2 \\ s^2 & a & K+1 \\ s^1 & \frac{a(K+2)-(K+1)}{a} & 0 \\ s^0 & K+1 & \end{array}$$

Given,

$\omega_n = 2$

$\Rightarrow s^1$ row = 0

s^2 row is A.E

$a\,(K + 2) - (K + 1) = 0$

$a = \dfrac{K+1}{K+2}$

AE = $as^2 + K + 1 = 0$

$= \dfrac{K+1}{K+2}s^2 + K + 1 = 0$

$(k+1)\left(\dfrac{s^2}{k+2} + 1\right) = 0$

$s^2 + k + 2 = 0$

$s = \pm\, j\sqrt{(k+2)}$

$\omega_n = \sqrt{k+2} = 2$

$k = 2$

$a = \dfrac{k+1}{k+2} = \dfrac{3}{4} = 0.75$

05.

Sol: $s^3 + ks^2 + 9s + 18$

$$\begin{array}{c|cc} s^3 & 1 & 9 \\ s^2 & K & 18 \\ s^1 & \frac{9K-18}{K} & 0 \\ s^0 & 18 & \end{array}$$

Given that system is marginally stable,

Hence, s^1 row = 0

$\frac{9K-18}{K} = 0$

$9K = 18 \Rightarrow K = 2$

A.E is $9s^2 + 18 = 0$

$Ks^2 + 18 = 0$,

$2s^2 + 18 = 0 \Rightarrow 2s^2 = -18$

$s = \pm j3$

$\therefore \omega_n = 3$ rad/sec.

06. Ans: (d)

Sol: Given transfer function $G(s) = \frac{k}{(s^2+1)^2}$

Characteristic equation $1 - G(s).H(s) = 0$

$1 - \frac{k}{(s^2+1)^2} = 0$

$s^4 + 2s^2 + 1 - k = 0$ ….. (1)

RH criteria

s^4	1	2	1−K
s^3	4	4	-
s^2	1	1−K	
s^1	4K		
s^o	1−K		

$AE = s^4 + 2s^2 + 1 - K$

$\frac{d}{ds}(AE) = 4s^3 + 4s$

$1-K > 0$ no poles are on RHS plane and LHS plane.

All poles are on $j\omega$- axis

$\therefore 0 < K < 1$ system marginally stable.

07. Ans: (d)

Sol: Assertion: FALSE

Let the TF= s. "s" is the differentiator

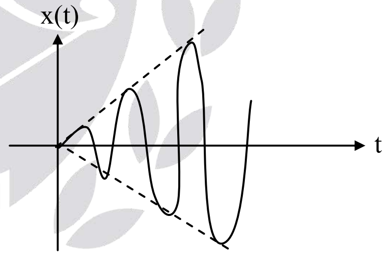
Impulse response $L^{-1}[TF] = L^{-1}[s] = \delta'(t)$

$\underset{t\to\infty}{Lt}\ \delta'(t) = 0$

$\therefore$ It is BIBO stable

Reason: True

$x(t) = t \sin t$



$\underset{t\to\infty}{Lt}\ x(t) = \underset{t\to\infty}{Lt}\ t \sin t$ is unbounded

08. Ans: (a)

Sol: Assertion: TRUE

If feedback is not properly utilized the closed loop system may become unstable.

Reason: True

Feedback changes the location of poles

Let $G(s) = \frac{-2}{s+1}$ $H(s) = 1$

Open loop pole s = –1 (stable)

$$\text{CLTF} = \frac{\frac{-2}{s+1}}{1+\frac{-2}{s+1}} = \frac{-2}{s-1}$$

Closed loop pole is at s = 1 (unstable)

$\therefore$ After applying the feedback no more system is open loop. It becomes closed loop system. Hence poles are affected.

Conventional Practice Solutions

01.

Sol: Given open loop Transfer function

$$G(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

$$\xrightarrow{CE} \quad s(s+4)(s^2+4s+20) + K = 0$$

$$(s^2+4s)(s^2+4s+20) + K = 0$$

$$s^4 + 4s^3 + 20s^2 + 4s^3 + 16s^2 + 80s + K = 0$$

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

$$\begin{array}{c|ccc} s^4 & 1 & 36 & K \\ s^3 & 8 & 80 & \\ s^2 & 26 & K & \\ s^1 & 80-\frac{8K}{26} & & \\ s^0 & K & & \end{array}$$

i) The system to be stable for

$$K > 0 \;\&\; \frac{26\times 80 - 8K}{26} > 0$$

$K < 260$

$\therefore\ 0 < K < 260$

ii) The system to cause sustained oscillations

$K = 260$

$26s^2 + 260 = 0$

$s = \pm j\sqrt{10}$

$\omega_n = \sqrt{10}\,\text{rad/sec}$

02.

Sol: C.E. equation

$s^6 + s^5 - 6s^4 + 0.s^3 - s^2 - s + 6 = 0$

s^6	1	-6	-1	6
s^5	1	0	-1	
s^4	-6	0	6	
s^3	$0(-24)$	0		
s^2	$0(\in)$	6		
s^1	$\frac{6}{\in}\times 24$			
s^0	6			

A.E equations $-6s^4 + 6 = 0$

$$\frac{dA}{ds} = -24\,s^3 = 0$$

Number of LHP = 2

Number of RHP = 2

Number of $j\omega p$ = 2

AE = $s^6 + 8s^4 + 32s^2 + 64 = 0$

$$\frac{d(AE)}{ds} = 6s^5 + 32s^3 + 64s = 0$$

$$3s^5 + 16s^3 + 32s = 0$$

No.of CE Roots = 8	No. of AE Roots = 6
• Number of sign changes = 2. • Number of Roots in RH of s-plane = 2 • Number of Roots on jω axis = 2. • Number of Roots in LH of s-plane = 4. • So, the system is unstable.	• Number of sign changes below the AE = 2 • Number of Roots in RH of s-plane = 2. • Number of Roots in LH of s-plane = 2 • Number of Roots on jω axis = 2. • AE roots are also CE roots and these roots are symmetrical with respect to the origin

03.

Sol: CE is 1+G(s) = 0

$$1 + \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s} = 0$$

$$s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128 = 0$$

R-H Tabulation

s^8	1	10	48	128	128
s^7	3(1)	24(8)	96(32)	192(64)	0
s^6	2(1)	16(8)	64(32)	128(64)	0
s^5	0(3)	0(16)	0(32)	0	0
s^4	$\frac{8}{3}(1)$	$\frac{64}{3}(8)$	64(24)	0	0
s^3	$-8(-1)$	$-40(-5)$	0	0	0
s^2	3(1)	24(8)	0	0	0
s^1	3	0	0	0	0
s^0	8	0	0	0	0

04.

Sol:

s^4	1`	5	10k
s^3	k	10	0
s^2	$\frac{5k-10}{k}$	$\frac{10k^2}{k}$	
s^1	$\dfrac{\frac{5k-10}{k}\times 10 - 10k^2}{\frac{5k-10}{k}}$	0	
s^0	10k		

As there should not be any sign changes in the first column of Routh's array k > 0,

$$\frac{5k-10}{k} > 0$$

$$\Rightarrow \frac{50k-100}{k} - 10k^2 > 0$$

$$\left(\text{As } \frac{5k-10}{k} > 0\right)$$

$50k - 100 - 10k^3 > 0$

$10k^3 - 50k + 100 < 0$ …….. (1)

$$\Rightarrow \frac{5k-10}{k} > 0$$

$5k - 10 > 0$ [As $k > 0$]

$5k > 10$

$k > 2$ …….(2)

$$\left.\begin{matrix}\text{put } x = 2 \\ x = 3\end{matrix}\right\}\text{......equation (1)}$$

$10(2)^3 - 50(2) + 100 > 0$

$10(3)^3 - 50(3) + 100 > 0$

For all k > 2, (1) is not satisfied

So, given system unstable for any value of k.

05.

Sol: The given open loop transfer function

$$A = \frac{\mu(s+\alpha)^2}{\alpha^2(1+s)s^3}$$

Closed loop transfer function

$$= \frac{A}{1+A} = \frac{\mu(s+\alpha)^2}{\alpha^2(1+s)s^3 + \mu(s+\alpha)^2}$$

Characteristic equation

$$\alpha^2(1+s)s^3 + \mu(s^2+\alpha^2+2\alpha s) = 0$$

$$\Rightarrow \alpha^2 s^4 + \alpha^2 s^3 + \mu s^2 + 2\alpha\mu s + \mu\alpha^2 = 0$$

The system stability can be calculated by applying Routh Hurwitz criteria.

Apply RH criteria

s^4	α^2	μ	$\mu\alpha^2$
s^3	α^2	$2\mu\alpha$	0
s^2	$\mu - 2\mu\alpha$	$\mu\alpha^2$	
s^1	$\frac{2\alpha\mu^2 - 4\mu^2\alpha^2 - \mu\alpha^4}{\mu - 2\mu\alpha}$	0	
s^0	$\mu\alpha^2$		

For stable system,

There should not be sign changes in the 1st column of Routh's array.

As α^2 is the positive for α,

$\mu(1-2\alpha) > 0$ ….(1)

From the last row $\mu\alpha^2 > 0$

As α^2 is the positive, μ also should be positive

from (1), as μ is the positive

$1 - 2\alpha > 0$

$1 > 2\alpha$

$\alpha < ½$

$\mu\alpha(2\mu - 4\mu\alpha - \alpha^3) > 0$

As μ is positive $\alpha(2\mu - 4\mu\alpha - \alpha^3) > 0$

Two cases will be possible

(i) $\alpha > 0$, $2\mu - 4\mu\alpha - \alpha^3 > 0$ &

(ii) $\alpha < 0$, $2\mu - 4\mu\alpha - \alpha^3 < 0$

(i) $\alpha > 0$, $\mu(2 - 4\alpha) > \alpha^3$

$$\alpha > 0,\ \mu > \frac{\alpha^3}{2-4\alpha}$$

(ii) $\alpha < 0$, $\mu(2 - 4\alpha) < \alpha^3$

$$\mu < \frac{\alpha^3}{2-4\alpha} \text{ ……. (2)}$$

$$\text{As } \alpha < 0, \frac{\alpha^3}{2-4\alpha} \text{ is} - \text{ve}$$

But μ is positive so this case is wrong

$$\therefore \alpha > 0,\ \alpha < \frac{1}{2},\ \mu > \frac{\alpha^3}{2-4\alpha}$$

Chapter 5 Root Locus Diagram

Objective Practice Solutions

01. Ans: (a)

Sol: $s_1 = -1 + j\sqrt{3}$

$s_2 = -3 - j\sqrt{3}$

$$G(s).H(s) = \frac{K}{(s+2)^3}$$

$s_1 = -1 + j\sqrt{3}$

$$G(s).H(s) = \frac{K}{\left(-1+j\sqrt{3}+2\right)^3}$$

$$= \frac{K}{\left(1+j\sqrt{3}\right)^3}$$

$$= -3\tan^{-1}(\sqrt{3})$$

$$= -180°$$

It is odd multiples of 180°, Hence s_1 lies on Root locus

$s_2 = -3 - j\sqrt{3}$

$$G(s).H(s) = \frac{K}{\left(-3-j\sqrt{3}+2\right)^3}$$

$$= \frac{K}{\left(-1-j\sqrt{3}\right)^3}$$

$$= -3\,[180° + 60°] = -720°$$

It is not odd multiples of 180°, Hence s_2 is not lies on Root locus.

02. Ans: (a)

Sol: Over damped – roots are real & unequal $\Rightarrow 0 < k < 4$

(b) k = 4 roots are real & equal

$\Rightarrow$ Critically damped $\xi = 1$

(c) $k > 4 \Rightarrow$ roots are complex

$0 < \xi < 1 \Rightarrow$ under damped

03. Ans: (a)

Sol: Asymptotes meeting point is nothing but centroid

$$\text{centroid } \sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{p - z}$$

$$= \frac{-3-0}{3-0} = -1$$

centroid = (–1, 0)

04. Ans: (b)

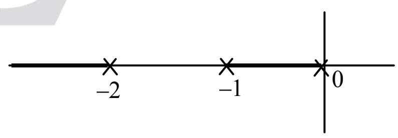
Sol: break point = $\frac{dK}{ds} = 0$

$$\frac{d}{ds}\left(G_1(s).H_1(s)\right) = 0$$

$$\frac{d}{ds}\left[s(s+1)(s+2)\right] = 0$$

$3s^2 + 6s + 2 = 0$

$s = -0.422, -1.57$



But s = – 1.57 do not lie on root locus

So, s = –0.422 is valid break point.

Point of intersection wrt jω axis

$$s^3 + 3s^2 + 2s + k = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \frac{6-k}{3} & 0 \\ s^0 & k & \end{array}$$

As s^1 Row = 0

$k = 6$

$3s^2 + 6 = 0$

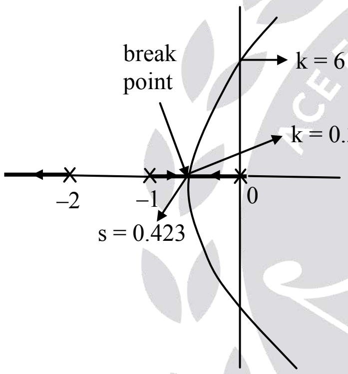
$s^2 = -2$

$s = \pm j\sqrt{2}$

point of inter section: $s = \pm\ j\sqrt{2}$

05. Ans: (b)

Sol:



$$\frac{K}{s(s+1)(s+2)}$$

substitute s = –0.423 and apply the magnitude criteria.

$$\left|\frac{K}{(-0.423)(-0.423+1)(-0.423+2)}\right| = 1$$

K = 0.354

when the roots are complex conjugate then the system response is under damped.

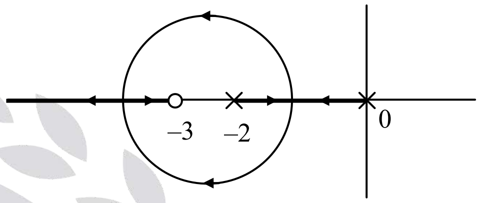
From K > 0.384 to K < 6 roots are complex conjugate then system to be under damped the values of k is 0.384 < K < 6.

06. Ans: (c)

Sol: If the roots are lies on the real axis then system exhibits the non-oscillatory response. from K ≥ 0 to K ≤ 0.384 roots lies on the real axis. Hence for 0 ≤ K ≤ 0.384 system exhibits the non-oscillatory response.

07. Ans: (a)

Sol:



$$\frac{d}{ds}[G(s).H(s)] = \frac{d}{ds}\left[\frac{k(s+3)}{s(s+2)}\right]$$

$s^2 + 6s + 6 = 0$

break points – 1.27, – 4.73

radius = $\frac{4.73-1.27}{2} = 1.73$

center = (–3, 0)

08. Ans: (c)

Sol: G(s).H(s) = $\frac{K(s+3)}{s(s+2)}$

$$k\big|_{s=-4} = \left|\frac{(-4)(-4+2)}{(-4+3)}\right|$$

$$= \left|\frac{(-4)(-2)}{(-1)}\right| = 8$$

09. Ans: (a)

Sol: $s^2-4s+8 = 0 \Rightarrow s = 2\pm2j$ are two zeroes

$s^2+4s+8 = 0 \Rightarrow s = -2\pm2j$ are two poles

$$\phi_A = 180 - \angle GH\Big|_{S=2\pm2j}$$

$$GH = \frac{k[s-(2+2j)][s-(2-2j)]]}{[s-(-2+2j)][s-(-2-2j)]]}$$

$$\angle GH\Big|_{S=2+2j} = \frac{\angle k\angle 4j}{\angle 4\angle 4+4j}$$

$$= 90^\circ - 45^\circ = 45^\circ$$

$$\phi_A = 180^\circ - 45^\circ = \pm 135^\circ$$

10. Ans: (b)

Sol: $s^2-4s+8 = 0 \Rightarrow s = 2\pm2j$ are two zeroes

$s^2+4s+8 = 0 \Rightarrow s = -2\pm2j$ are two poles

$$\phi_d = 180^0 + \angle GH\Big|_{S=-2\pm2j}$$

$$\angle GH\Big|_{S=-2\pm2j} = \angle \frac{k[s-(2+2j)][s-(2-2j)]}{[s-(-2+2j)][s-(-2-2j)]}\Bigg|_{S=-2\pm2j}$$

$$= \frac{\angle k(-4)(-4+4j)}{\angle 4j}$$

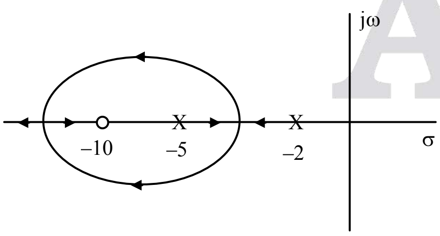
$$= 180^\circ + 180^\circ - 45^\circ - 90^\circ = 225^\circ$$

$$\phi_d = 180^\circ + 225^\circ = 405^\circ$$

$$\therefore \phi_d = \pm 45^\circ$$

11. Ans: (d)

Sol: Poles s = –2, –5 ; Zero s = –10



$\therefore$ Breakaway point exist between –2 and –5

12.

Sol: Refer Pg No: 84, Vol-1 Ex: 8

Conventional Practice Solutions

01.

Sol: $G(s)H(s) = \dfrac{K(s+3)}{s(s+2)}$

Poles s = 0, –2

Zeros s = –3

Number of asymptotes = |P –Z| = 1

(i) Break points:

$$\frac{d}{ds}G(s)H(s) = 0$$

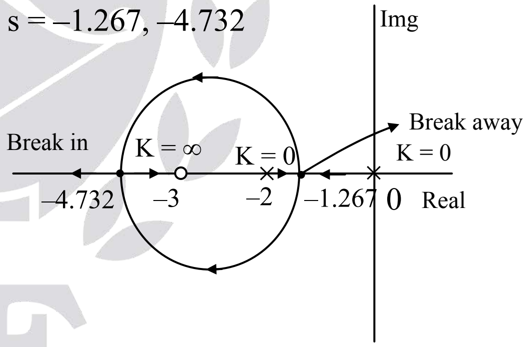
$$\frac{d}{ds}\left[\frac{s+3}{s(s+2)}\right] = 0$$

$$(s^2+2s) - (s+3)(2s+2) = 0$$

$$s^2+2s = 2s^2+2s+6s+6$$

$$s^2+6s+6 = 0$$

$$s = -1.267, -4.732$$



$\therefore$ Both points are valid break points.

(ii) $|G(s)H(s)| = 1$

$$\left|\frac{s(s+2)}{s+3}\right| = K$$

$$K\Big|_{s=-1} = \left|\frac{-1(1)}{2}\right| = \frac{1}{2} = 0.5$$

$$K\Big|_{s=-4} = \left|\frac{(-4)(-2)}{-1}\right| = 8$$

(iii) Proof of circle for complex plane

$$\angle\frac{k(s+3)}{s(s+2)} = -180°$$

at s = σ +jω…. angle criteria

$$\angle\frac{(\sigma+j\omega+3)}{(\sigma+j\omega)(\sigma+2+j\omega)} = -180°$$

$$\tan^{-1}\frac{\omega}{3+\sigma} - \tan^{-1}\frac{\omega}{\sigma} - \tan^{-1}\frac{\omega}{\sigma+2} = -180°$$

$$180 + \tan^{-1}\frac{\omega}{3+\sigma} = \tan^{-1}\frac{\omega}{\sigma} + \tan^{-1}\frac{\omega}{\sigma+2}$$

Taking tan on both sides, we get

$$\frac{\omega}{3+\sigma} = \frac{\frac{\omega}{\sigma}+\frac{\omega}{\sigma+2}}{1-\left(\frac{\omega}{\sigma}\right)\left(\frac{\omega}{\sigma+2}\right)}$$

$$\frac{\omega}{3+\sigma} = \frac{\omega(\sigma+2)+\omega\sigma}{\sigma(\sigma+2)-\omega^2}$$

$\sigma^2 + 2\sigma - \omega^2 = (2\sigma + 2)(3+\sigma)$

$\sigma^2 + 2\sigma - \omega^2 = 6\sigma + 2\sigma^2 + 6 + 2\sigma$

$\Rightarrow \sigma^2 + 6\sigma + \omega^2 + 6 = 0$

$\Rightarrow (\sigma+3)^2 + (\omega-0)^2 = (\sqrt{3})^2$

It is of the form,

$(x-a)^2 + (y-b)^2 = r^2$ which is a circle

Where, centre (a, b) = (–3, 0) and radius is $r = \sqrt{3}$

02.

Sol: $G(s)H(s) = \dfrac{K}{s(s+4)(s^2+4s+20)}$

Number of Poles s = 0, –4, –2 ± 4i

Number of Loci = 4(P > Z)

Centroid $\sigma = \dfrac{-8}{4} = -2$

Angle of asymptotes

$\theta = 45^o, 135^o, 225^o, 315^o$

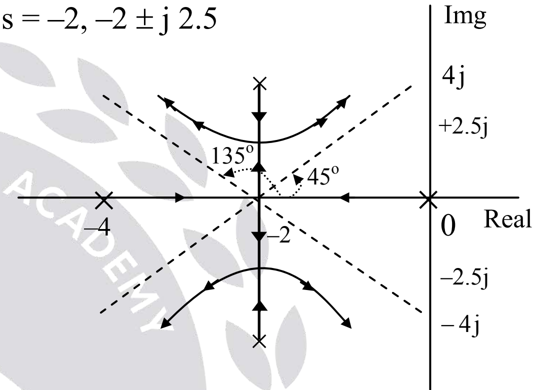
Break points:

$$\frac{d}{ds}G(s)H(s) = 0$$

$$\frac{d}{ds}\left(\frac{1}{(s^4+8s^3+36s^2+80s)}\right) = 0$$

$4s^3 + 24s^2 + 72s + 80) = 0$

s = –2, –2 ± j 2.5



All the 3 points are valid break points

CE

$\rightarrow s(s+4)(s^2+4s+20) + K = 0$

$s^4 + 8s^3 + 36s^2 + 80s + K = 0$

s^4	1	36	K
s^3	8	80	0
s^2	26	K	
s^1	$\frac{26\times80-8K}{26}$		
s^0	K		

(i) The value of K for the system to be stable

$K > 0$ & $\dfrac{26\times80-8K}{26} > 0$

$K < 260$

$\therefore 0 < K < 260$

(ii) For K = 260

$$26s^2 + 260 = 0$$

$$s^2 = -10$$

$$s = \pm j\sqrt{10}$$

$$\omega_n = \sqrt{10} \text{ rad/sec}$$

$\therefore$ Natural frequency of sustained oscillations = $\sqrt{10}$ rad/sec

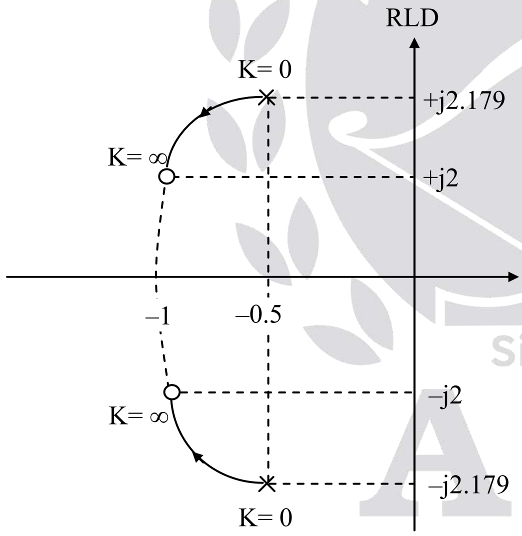
03.

Sol: $G(s) = \dfrac{K(s^2+2s+5)}{(s^2+s+5)}; H(s) = 1$

$$G(s)H(s) = \frac{K(s^2+2s+5)}{(s^2+s+5)}$$

Pole = – 0.5 ± j2.179

Zeros = –1 ± j2



Proof of circle for complex plane

$$\angle\frac{K(s^2+2s+5)}{s^2+s+6} = -180° \text{ at } s = \sigma + j\omega$$

[Angle criteria should be satisfied]

$$\angle\frac{\{(\sigma+j\omega)^2+2(\sigma+j\omega)+5\}}{(\sigma+j\omega)^2+(\sigma+j\omega)+5} = -180^o$$

$$\angle\frac{[\{\sigma^2-\omega^2+2\sigma+5\}+j\{2\omega+2\sigma\omega\}]}{(\sigma^2-\omega^2+\sigma+5)+j(2\omega\sigma+\omega)} = -180^o$$

$$\tan^{-1}\left(\frac{2\omega+2\sigma\omega}{\sigma^2-\omega^2+2\sigma+5}\right) - \tan^{-1}\left(\frac{2\sigma\omega+\omega}{\sigma^2-\omega^2+\sigma+5}\right) = -180^o$$

$$\tan^{-1}\left(\frac{2\sigma\omega+2\omega}{\sigma^2-\omega^2+2\sigma+5}\right) = -180^o + \tan^{-1}\left\{\frac{2\sigma\omega+\omega}{\sigma^2-\omega^2+\sigma+5}\right\}$$

Apply tan on both sides

$$\frac{2\sigma\omega+2\omega}{\sigma^2-\omega^2+2\sigma+5} = \frac{2\sigma\omega+\omega}{\sigma^2-\omega^2+\sigma+5}$$

$(2\sigma\omega+2\omega)(\sigma^2 - \omega^2 + \sigma + 5) = (2\sigma\omega+\omega)(\sigma^2 - \omega^2 + 2\sigma + 5)$

$(2\sigma + 2)(\sigma^2 - \omega^2 + \sigma + 5) = (2\sigma + 1)(\sigma^2 - \omega^2 + 2\sigma + 5)$

$-\sigma^2 - \omega^2 + 5 = 0$

$\sigma^2 + \omega^2 = 5$

Comparing with $(x-a)^2 + (y-b)^2 = r^2$

Which is a circle with centre (a, b) and radius 'r'

Center (0,0)

radius $\sqrt{5}$

$\therefore$ Root loci is a circle.

04.

Sol: Poles at (–2, 0), (–2, 0)

Zeros at (–1, 0), (–1, 0)

i.e., double poles, double zeros at same point

$$G(s)H(s) = \frac{K(S+1)^2}{(S+2)^2}$$

Put s = σ + jω, for complex plane angle criteria should be satisfied

For any point on root loci

$$2\tan^{-1}\left(\frac{\omega}{\sigma+1}\right) - 2\tan^{-1}\left(\frac{\omega}{\sigma+2}\right) = -180^o$$

$$\angle \frac{K(\sigma + j\omega + 1)^2}{(\sigma + j\omega + 2)^2} = -180^o$$

$$\tan^{-1}\left(\frac{\omega}{\sigma+1}\right) - \tan^{-1}\left(\frac{\omega}{\sigma+2}\right) = -90^o$$

$$\tan^{-1}\left\{\frac{\frac{\omega}{\sigma+1} - \frac{\omega}{\sigma+2}}{1+\frac{\omega}{\sigma+1}\frac{\omega}{\sigma+2}}\right\} = -90^o$$

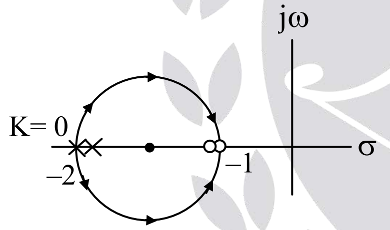
$$\frac{\frac{\omega}{\sigma+1} - \frac{\omega}{\sigma+2}}{1+\frac{\omega}{\sigma+1}\frac{\omega}{\sigma+2}} = \infty \Rightarrow 1 + \frac{\omega}{\sigma+1} \times \frac{\omega}{\sigma+2} = 0$$

$\sigma^2 + 3\sigma + 2 + \omega^2 = 0$

$(\sigma+1.5)^2 + \omega^2 = 0.25$

Comparing with $(x - a)^2 + (y - b)^2 = r^2$ which is a equation of circle.

$\therefore$ Plot of root loci is a circle.



05.

Sol: Necessary conditions for stability:

Let us consider the characteristic equation

$a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0$

For stability necessary condition are

- All the coefficients (a_0, a_1 etc..) in the CE should be of the same sign.
- There should not be any missing coefficients in the CE.

  The given open loop transfer function

  $$G(s) = \frac{k}{s(1+s\tau_1)(1+s\tau_2)}$$

  Hence characteristic equation

(i) $CE = 1 + \dfrac{K}{s(1+s\tau_1)(1+s\tau_2)} = 0$

$s(1 + s\tau_1)(1 + s\tau_2) + K = 0$

$s^3\tau_1\tau_2 + s^2(\tau_1 + \tau_2) + s + K = 0$

R-H tabulation

s^3	$\tau_1\tau_2$	1
s^2	$\tau_1+\tau_2$	K
s^1	$\dfrac{(\tau_1+\tau_2)-K\tau_1\tau_2}{\tau_1+\tau_2}$	0
s^0	K	0

For stability $\tau_1\tau_2 > 0$, $\tau_1 + \tau_2 > 0$, $K > 0$ and

$$\frac{(\tau_1+\tau_2)-K\tau_1\tau_2}{\tau_1+\tau_2} > 0$$

$\Rightarrow (\tau_1+\tau_2) > K\tau_1\tau_2$

$$\frac{(\tau_1+\tau_2)}{\tau_1\tau_2} > K \Rightarrow K < \frac{(\tau_1+\tau_2)}{(\tau_1.\tau_2)}$$

So condition for stability

$$0 < K < \frac{\tau_1+\tau_2}{\tau_1\tau_2}$$

(ii) Given open loop transfer function

$$G(s) = \frac{K}{s(1+s\tau_1)(1+s\tau_2)}$$

Ex: Let assume $G(s) = \dfrac{K}{s(s+2)(s+5)}$

No. of root locus branches=3(3rd order system)

No. of Asymptotes N = P – Z = 3 – 0 = 3

Angle of Asymptotes $= \dfrac{(2\ell+1)180^0}{P-Z}$

$\ell = 0, 1, 2$

$= 60^0, 180^0, 300^0$

Centroid σ = $\dfrac{\left(\sum \text{real part poles} - \sum \text{real part of zeros}\right) \text{of } G(s)H(s)}{P-Z}$

$$= \left(\frac{-2-5}{3}\right) = \left(\frac{-7}{3}\right) = -2.33$$

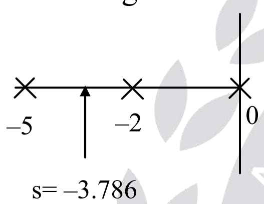
Break away point $G_1(s)H_1(s) = \dfrac{1}{s(s+2)(s+5)}$

$$\frac{dK}{ds} = \frac{d}{ds}\left(G_1(s)H_1(s)\right) = 0$$

$$\frac{d}{ds}\left[s\left(s^2+7s+10\right)\right] = 0$$

$$\frac{d}{ds}\left[s^3+7s^2+10s\right] = 0$$

$3s^2+14s+10=0$ gives $s = -0.88, -3.786$



As two poles exist on right side of s = – 3.786 root loci does not exist.

$\therefore$ It is not a valid break point.

s = –0.88 is a valid break away point.

Intersection of the RLD with respect to the imaginary axis is calculated from RH criteria

$CE = s^3+7s^2+10s+K=0$

Routh tabulation

s^3	1	10
s^2	7	K
s^1	$\dfrac{70-K}{7}$	0
s^0	K	0

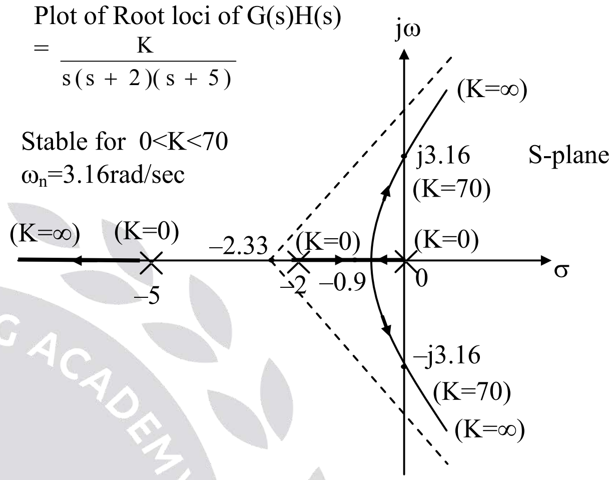
$\dfrac{70-K}{7} = 0$ gives K = 70

AE is $7s^2+K=0$

$7s^2 = -K$

$7s^2 = -70 \quad (s = j\omega)$

s = ±j3.16 rad/sec. is the point of intersection of root loci with imaginary axis



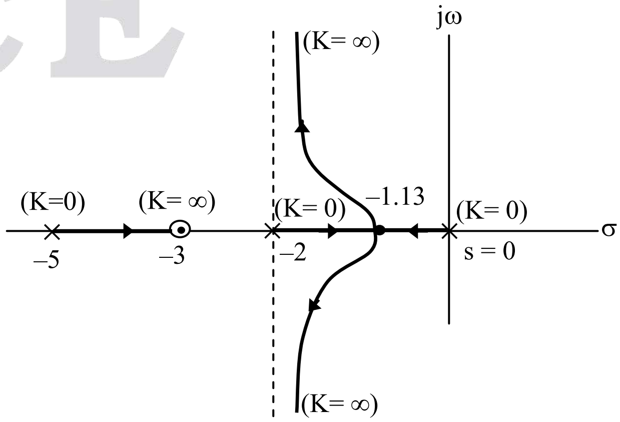
Root locus diagram after addition of a Zero to the OLTF

Let the zero be added to the OLTF at s = –3

i.e; $G(s)=\dfrac{K(s+3)}{s(s+2)(s+5)}$

Centroid (σ) = $\sigma = \dfrac{(0-2-5+3)}{3-1} = -2$

Angle of asymptotes = ± 90°



Chapter 6 Frequency Response Analysis

Objective Practice Solutions

01. Ans: (c)

Sol: $G(s).H(s) = \dfrac{100}{s(s+4)(s+16)}$

Phase crossover frequency (ω_{pc}):

$\angle G(j\omega).H(j\omega)/_{\omega=\omega_{pc}} = -180°$

$-90° - \tan^{-1}(\omega_{pc}/4) - \tan^{-1}(\omega_{pc}/16) = -180°$

$-\tan^{-1}(\omega_{pc}/4) - \tan^{-1}(\omega_{pc}/16) = -90°$

$\tan[\tan^{-1}(\omega_{pc}/4) + \tan^{-1}(\omega_{pc}/16)] = \tan(90°)$

$$\frac{\frac{\omega_{pc}}{4}+\frac{\omega_{pc}}{16}}{1-\frac{\omega_{pc}}{4}.\frac{\omega_{pc}}{16}} = \frac{1}{0}$$

$\omega_{pc}^2 = 16 \times 4 \Rightarrow \omega_{pc} = 8$ rad/sec

02. Ans: (d)

Sol: $G(s).H(s) = \dfrac{100}{s(s+2)(s+16)}$

Gain margin (G.M) = $\dfrac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}$

$$|G(j\omega).H(j\omega)|_{\omega=\omega_{pc}} = \frac{100}{\omega_{pc}\sqrt{\omega_{pc}^2+16}\sqrt{\omega_{pc}^2+16^2}}$$

$$= \frac{5}{64}$$

$$G.M = \frac{64}{5} = 12.8$$

03. Ans: (c)

Sol: $G(s).H(s) = \dfrac{2e^{-0.5s}}{(s+1)}$

gain crossover frequency,

$\omega_{gc} = |G(j\omega).H(j\omega)|_{\omega=\omega_{gc}} = 1$

$$\frac{2}{\sqrt{\omega_{gc}^2+1}} = 1$$

$\omega_{gc}^2 + 1 = 4 \Rightarrow \omega_{gc} = \sqrt{3}$ rad / sec

04. Ans: (b)

Sol: $\omega_{gc} = \sqrt{3}$rad / sec

$P.M = 180° + \angle G(j\omega).H(j\omega) / \omega = \omega_{gc}$

$\angle G(j\omega).H(j\omega)/_{\omega=\omega_{gc}} = -0.5\ \omega_{gc} - \tan^{-1}(\omega_{gc})$

$= -109.62°$

$P.M = 70.39°$

05. Ans: (a)

Sol: $M_r = 2.5 = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

$$2\xi\sqrt{1-\xi^2} = \frac{1}{2.5}$$

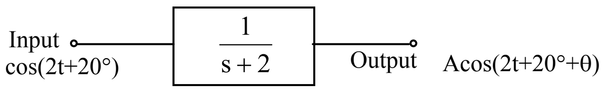
$\xi^4 - \xi^2 + 0.04 = 0$

$\xi^2 = 0.958 \qquad \xi^2 = 0.0417$

$\xi = 0.204 \qquad (M_r > 1)$

06. Ans: (a)

Sol: Closed loop T.F = $\dfrac{1}{s+2}$



$$A = \frac{1}{\sqrt{\omega^2+4}} = \frac{1}{\sqrt{4+4}} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

$\phi = -\tan^{-1}\omega/2$

$= -\tan^{-1}2/2$

$\Rightarrow \phi = -\tan^{-1}(1) = -45°$

$$\text{output} = \frac{1}{2\sqrt{2}}\cos\left(2t + 20^{o} - 45^{o}\right)$$

$$= \frac{1}{2\sqrt{2}}\cos(2t - 25^{o})$$

07. Ans: (c)

Sol: Initial slope = –40 dB/dec

Two integral terms $\left(\frac{1}{s^2}\right)$

$\therefore$ Part of TF $= G(s)H(s) = \dfrac{K}{s^2}$

at ω = 0.1

Change in slope = –20 – (–40)

= 20°

Part of TF = G(s) H(s) $= \dfrac{K\left(1+\frac{s}{0.1}\right)}{s^2}$

At ω = 10 slope changed to −60 dB/dec

Change in slope = –60–(–20)

= – 40dB/dec

$$\text{TF } (G(s)H(s)) = \frac{K\left(1+\frac{s}{0.1}\right)}{s^2\left(\frac{s}{10}+1\right)^2}$$

20 log K− 2 (20 log 0.1) = 20 dB

20 log K = 20–40

20 log K = –20

K = 0.1

$$G(s)H(s) = \frac{(0.1)\left(1+\frac{s}{0.1}\right)}{s^2\left(1+\frac{s}{10}\right)^2}$$

$$= \frac{(0.1)\times 10^2(s+0.1)}{(0.1)s^2(s+10)^2}$$

$$G(s)H(s) = \frac{100(s+0.1)}{s^2(s+10)^2}$$

08. Ans: (b)

Sol: $G(s)H(s) = \dfrac{Ks}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)}$

12 = 20 log K+ 20 log 0.5

12 = 20logK + (−6)

20 log K = 18 dB = 20 log 2^3

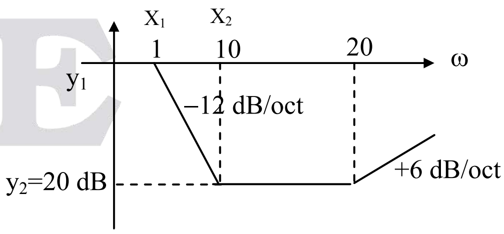
K = 8

$$G(s)H(s) = \frac{8s\times 2\times 10}{(2+s)(10+s)}$$

$$G(s)H(s) = \frac{160s}{(2+s)(10+s)}$$

09. Ans: (b)

Sol:



$$G(s)H(s) = \frac{K\left(1+\frac{s}{10}\right)^2\left(1+\frac{s}{20}\right)}{(1+s)^2}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = -40\,\text{dB/dec}$$

$$\frac{20 - y_1}{\log 10 - \log 1} = -40$$

$$y_1 = +60\,\text{dB}\big|_{\omega \le 1}$$

$\Rightarrow 20 \log K = 60$

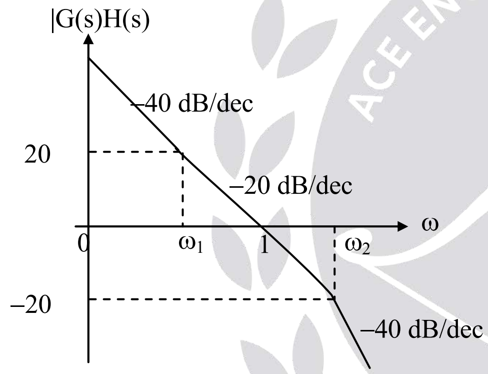
$$K = 10^3$$

$$G(s)H(s) = \frac{10^3 (s+10)^2 (s+20)}{10^2 \times 20 \times (s+1)^2}$$

$$= \frac{(s+10)^2 (s+20)}{2(s+1)^2}$$

10. Ans: (d)

Sol:



ω_1 calculation:

$$\frac{0 - 20}{\log 1 - \log \omega_1}$$

$= -20$ dB/dec

$\omega_1 = 0.1$

ω_2 calculation:

$$\frac{-20 - 0}{\log \omega_2 - \log 1}$$

$= -20$dB/dec

$\omega_2 = 10$

$$G(s)H(s) = \frac{K\left(1 + \frac{s}{0.1}\right)}{s^2\left(1 + \frac{s}{10}\right)}$$

$20\log K - 2\,(20 \log 0.1) = 20$

$$20 \log K = 20 - 40$$

$$K = 0.1$$

$$G(s)H(s) = \frac{0.1 \times \frac{1}{0.1}(0.1 + s)}{s^2 \frac{1}{10}(10 + s)}$$

$$= \frac{10(0.1 + s)}{s^2 (10 + s)}$$

11.

Sol: $\dfrac{200}{s(s+2)} = \dfrac{100}{s\left(1 + \frac{s}{2}\right)}$

$$x = -KT \Rightarrow -(100) \times \frac{1}{2} = x = -50$$

12. Ans: (c)

Sol: For stability (–1, j0) should not be enclosed by the polar plot.

For stability

$1 > 0.01\,K$

$\Rightarrow K < 100$

13.

Sol: GM = –40 dB

$$20 \log \frac{1}{a} = -40 \Rightarrow a = 10^2$$

POI = 100

14.

Sol: (i) $GM = \frac{1}{0.1} = +10 = 20\,dB$

$PM = 180° - 140° = 40°$

(ii) $PM = 180 - 150° = 30°$

$GM = \frac{1}{0} = \infty \qquad POI = 0$

(iii) ω_{PC} does not exist

$GM = \frac{1}{0} = \infty \quad PM = 180° + 0° = 180°$

(iv) ω_{gc} not exist

$\omega_{pc} = \infty$

$GM = \frac{1}{0} = \infty$

$PM = \infty$

(v) $GM = \frac{1}{0.5} = 2$

$PM = 180 - 90 = 90^0$

15. Ans: (d)

Sol: For stability (–1, j0) should not be enclosed by the polar plot. In figures (1) & (2) (–1, j0) is not enclosed.

$\therefore$ Systems represented by (1) & (2) are stable.

16. Ans: (b)

Sol: Open loop system is stable, since the open loop poles are lies in the left half of s-plane

$\therefore P = 0$.

From the plot N = –2 .

No.of encirclements N = P – Z

N = –2, P = 0 (Given)

$\therefore N = P - Z$

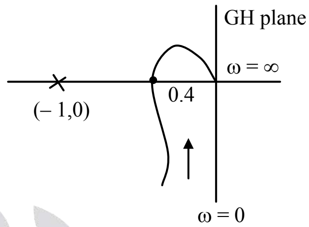
$-2 = 0 - Z$

Z = 2

Two closed loop poles are lies on RH of s-plane and hence the closed loop system is unstable.

17. Ans: (c)

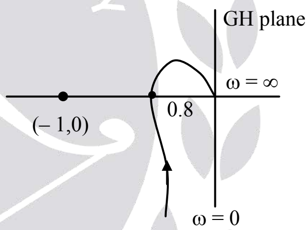
Sol:



$\frac{K_c}{K} = 0.4 \qquad$ When K = 1

Now, K double, $\frac{K_c}{K} = 0.4$

$K_c = 0.4 \times 2 = 0.8$



Even though the value of K is double, the system is stable (negative real axis magnitude is less than one)

Oscillations depends on ‘ ξ’

$\xi \propto \frac{1}{\sqrt{K}}$ as K is increased ξ reduced, then more oscillations.

18. Ans: (a)

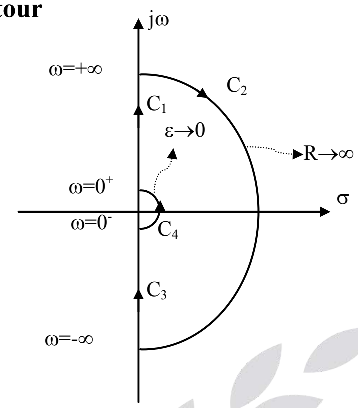
Sol: Given system $G(s) = \frac{10(s-12)}{s(s+2)(s+3)}$

It is a non minimum phase system since s = 12 is a zero on the right half of s-plane.

19.

Sol: Given that $G(s)H(s) = \dfrac{10(s+3)}{s(s-1)}$

s-plane
Nyquist Contour



- Nyquist plot is the mapping of Nyquist contour(s-plane) into G(s)H(s) plane.
- The Nyquist contour in the s-plane enclosing the entire right half of S-plane is shown figure.

The Nyquist Contour has four sections C_1, C_2, C_3 and C_4. These sections are mapped into G(s)H(s) plane

Mapping of section C_1: It is the positive imaginary axis, therefore sub s = jω, $(0 \le \omega \le \infty)$ in the TF G(s) H(s), which gives the polar plot

$$G(s)H(s) = \frac{10(s+3)}{s(s-1)}$$

Let s = jω

$$G(j\omega)H(j\omega) = \frac{10(j\omega+3)}{j\omega(j\omega-1)}$$

$$G(j\omega)H(j\omega) = \frac{10\sqrt{\omega^2+9}}{\omega\sqrt{\omega^2+1}} \angle \left\{ \tan^{-1}\left(\frac{\omega}{3}\right) - [90^0 + 180^0 - \tan^{-1}(\omega)] \right\}$$

At $\omega = 0 \Rightarrow \infty\angle{-270^0}$

At $\omega = \omega_{pc} = \sqrt{3} \Rightarrow 10\angle{-180^0}$

At $\omega = \infty \Rightarrow 0\angle{-90^0}$

point of intersection of the Nyquist plot with respect to negative real axis is calculated below

$$\text{Arg}G(j\omega)H(j\omega) = \arg\frac{10(j\omega+3)}{j\omega(j\omega-1)}$$

$= -180^0$ will give the 'ω_{pc}'

Magnitude of G(jω)H(jω) gives the point of intersection

$$\angle\tan^{-1}\left(\frac{\omega}{3}\right) - [90^0 + 180^0 - \tan^{-1}(\omega))$$

$$= -180^0\Big|_{\omega = \omega_{pc}}$$

$$\angle\tan^{-1}\left(\frac{\omega_{pc}}{3}\right) - [90^0 + 180^0 - \tan^{-1}(\omega_{pc})) = -180^0$$

$$\tan^{-1}\left(\frac{\omega_{pc}}{3}\right) + \tan^{-1}(\omega_{pc}) = 90^0$$

Taking "tan" both the sides

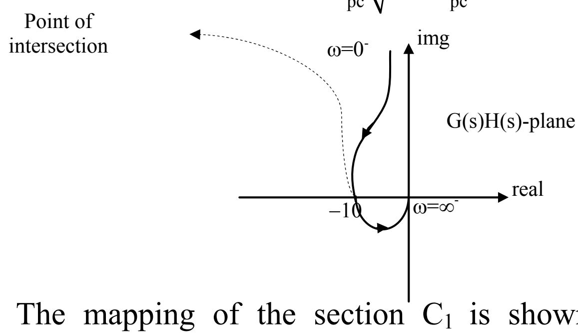
$$\frac{\dfrac{\omega_{pc}}{3} + \omega_{pc}}{1 - \dfrac{(\omega_{pc})^2}{3}} = \tan 90^0 = \infty$$

$$1 - \frac{\omega^2_{pc}}{3} = 0$$

$$\omega_{pc} = \sqrt{3}\ \text{rad/sec}$$

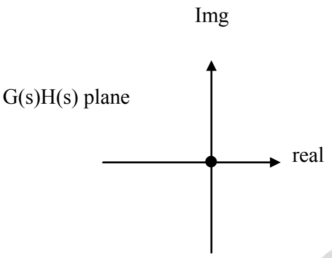
Therefore the point of intersection is

$$|G(j\omega)H(j\omega)| \text{ at } \omega_{pc} = \frac{10\sqrt{\omega_{pc}^{\ 2} + 3^2}}{\omega_{pc}\sqrt{1+\omega_{pc}^{\ 2}}} = 10$$



The mapping of the section C_1 is shown figure.

Mapping of section C_2: It is the radius 'R' semicircle, therefore sub $s = \lim_{R\to\infty} Re^{j\theta}$ (θ is from 90^0 to 0^0 to -90^0) in the TF G(s)H(s), which merges to the origin in G(s)H(s) plane.

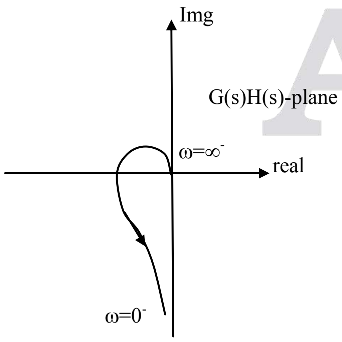


$$G(s)H(s) = \frac{10(s+3)}{s(s-1)}$$

$$G(Re^{j\theta})H(Re^{j\theta}) = \frac{2(Re^{j\theta}+3)}{Re^{j\theta}(Re^{j\theta}-1)} \approx 0$$

The plot is shown in figure.

Mapping of section C_3: It is the negative imaginary axis, therefore sub s = jω, ($-\infty \le \omega \le 0$) in the TF G(s)H(s), which gives the mirror image of the polar plot and is symmetrical with respect to the real axis, The plot is shown in figure.



Mapping of section C_4: It is the radius 'ε' semicircle, therefore sub $s = \lim_{\varepsilon\to 0} \varepsilon e^{j\theta}$ ($-90^o \le \theta \le 90^o$) in the TF G(s)H(s), which gives clockwise infinite radius semicircle in G(s)H(s) plane.

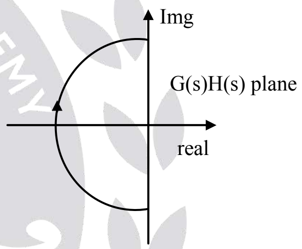
The plot is shown below

$$G(\varepsilon e^{j\theta})H(\varepsilon e^{j\theta}) = \frac{10(\varepsilon e^{j\theta}+3)}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}-1)}$$

$$G(\varepsilon e^{j\theta})H(\varepsilon e^{j\theta}) \approx \frac{10\times 3}{-\varepsilon e^{j\theta}} = \infty\angle 180^0 - \theta$$

When, $\theta = -90^0$ $\infty\angle 270^0$
$\theta = -40^0$ $\infty\angle 220^0$
$\theta = 0^0$ $\infty\angle 0^0$
$\theta = 40^0$ $\infty\angle 140^0$
$\theta = 90^0$ $\infty\angle 90^0$

It is clear that the plot is clockwise '∞' radius semicircle centred at the origin.



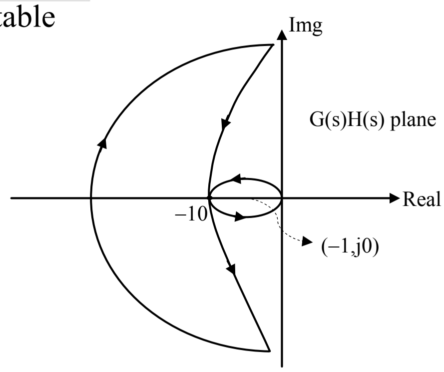
Combining all the above four sections, the Nyquist plot of $G(s)H(s) = \frac{10(s+3)}{s(s-1)}$ is shown in figure below

From the plot N=1

Given that P =1

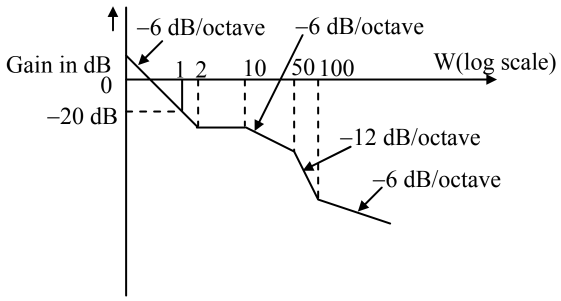
N = P–Z

Z = P – N = 1 – 1 = 0, therefore system is stable



20.

Sol: The given bode plot is shown below.



Initial slope = – 6 db/octave.
i.e., there is one pole at origin (or) one integral term.
portion of transfer function

$$G(s) = \frac{K}{s}$$

At ω = 2 rad/sec, slope is changed to 0dB/ octave.

$\therefore$ Change in slope = present slope – previous slope

= 0 – (–6) = 6 dB/octave

$\therefore$ There is a real zero at corner frequency $\omega_1 = 2$.

$$(1+sT_1) = \left(1+\frac{s}{\omega_1}\right) = \left(1+\frac{s}{Z}\right)$$

At ω = 10 rad/sec, slope is changed to –6dB/octave.

$\therefore$ change in slope = – 6 – 0
= – 6 dB/octave.

$\therefore$ There is a real pole at corner frequency $\omega_2 = 2$.

$$\frac{1}{1+sT_2} = \frac{1}{\left(1+\frac{s}{\omega_2}\right)} = \frac{1}{\left(1+\frac{s}{10}\right)}$$

At ω = 50 rad/sec, slope is changed to –12dB/octave.

$\therefore$ change in slope = – 12 – (–6)
= – 6 dB/octave

$\therefore$ There is a real pole at corner frequency ω_3 = 50 rad/sec.

$$\frac{1}{1+ST_3} = \frac{1}{\left(1+\frac{S}{\omega_3}\right)} = \frac{1}{\left(1+\frac{S}{50}\right)}$$

At ω = 100 rad/sec, the slope changed to –6 dB/octave.

$\therefore$ Change in slope = – 6 – (–12) = 6 dB/octave.

$\therefore$ There is a real zero at corner frequency ω_4 = 100 rad/sec.

$$\therefore (1+sT_4) = \left(1+\frac{s}{\omega_4}\right) = \left(1+\frac{s}{100}\right)$$

$$\therefore \text{ Transfer function } = \frac{K\left(1+\frac{s}{2}\right)\left(1+\frac{s}{100}\right)}{s\left(1+\frac{s}{50}\right)\left(1+\frac{s}{10}\right)}$$

$$= \frac{K(s+2)(s+100)}{s(s+50)(s+10)} \frac{\frac{1}{2}\cdot\frac{1}{100}}{\frac{1}{50}\cdot\frac{1}{10}}$$

$$= \frac{2.5K(s+2)(s+100)}{s(s+10)(s+50)}$$

In the given bode plot,
at ω = 1rad/sec, Magnitude = –20dB.

$$-20\text{dB} = 20\log K - 20\log\omega + 20\sqrt{1+\left(\frac{\omega}{2}\right)^2} + 20\sqrt{1+\left(\frac{\omega}{100}\right)^2}$$

$$-20\log\sqrt{1+\left(\frac{\omega}{50}\right)^2} - 20\log\sqrt{1+\left(\frac{\omega}{10}\right)^2}$$

At ω = 1 rad/sec,

$-20 = 20\log K - 20\log\omega/\omega = 1$ rad/sec

[∵ Remaining values eliminated]

$-\ \ 20 = 20\log K$

$\Rightarrow K = 0.1$

∴ Transfer function

$$\frac{C(s)}{R(s)} = \frac{0.25(s+2)(s+100)}{s(s+10)(s+50)}$$

Conventional Practice Solutions

01.

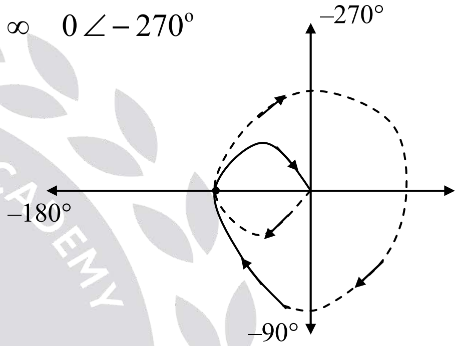
Sol: $G(s)H(s) = \dfrac{K}{s(1+sT_1)(1+sT_2)}$

$$|G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}$$

$$\angle G(j\omega)H(j\omega) = -90 - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

$\omega = 0 \quad \infty\angle-90^o$

$\omega = \infty \quad 0\angle-270^o$



$$-180 = -90 - \tan^{-1}\left(\frac{\omega T_1 + \omega T_2}{1-\omega^2T_1T_2}\right)$$

$$1 - \omega_{pc}^2 T_1T_2 = 0$$

$$\omega_{pc} = \frac{1}{\sqrt{T_1T_2}}$$

$$|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}} = \frac{K}{\frac{1}{\sqrt{T_1T_2}}\sqrt{\left(1+\frac{T_1}{T_2}\right)\left(1+\frac{T_2}{T_1}\right)}}$$

$$= \frac{K\sqrt{T_1T_2}}{\sqrt{\frac{(T_1+T_2)^2}{T_1T_2}}} = \frac{K(T_1T_2)}{T_1+T_2}$$

$$GM = \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}$$

$$= \frac{1}{\left(\frac{KT_1T_2}{T_1+T_2}\right)} = \frac{T_1+T_2}{KT_1T_2}$$

RH criteria:

The characteristic equation is

$s^3T_1T_2+s^2(T_1+T_2) + s + K = 0$

For stability the condition is

$$0 < K < \frac{T_1+T_2}{T_1T_2}$$

02.

Sol: $G(s) = \frac{10^4}{s(s+10)^2}$

$$|G(j\omega)| = \frac{10^4}{|j\omega(j\omega+10)^2|} = \frac{10^4}{\omega\left(\sqrt{\omega^2+10^2}\right)^2}$$

At $\omega = \omega_{gc}$, $|G(j\omega)| = 1$

$$\frac{10^4}{\omega(\omega^2+100)} = 1$$

$\omega^3+100\omega - 10^4 = 0$

Solve the above equation

$\omega_{gc} = 20$ as ω is real

$$\angle G(j\omega) = -90 - 2\tan^{-1}\left(\frac{\omega}{10}\right)$$

$$\angle G(s)\big|_{\omega=\omega_{gc}} = -90 - 2\tan^{-1}(2)$$

$$= -216.869^o$$

$PM = 180 + \angle G(s)\big|_{\omega=\omega_{gc}}$

$PM = -36.869$

03.

Sol: Given that

The steady state gain = 5

$M_r = \frac{10}{\sqrt{3}}$ at $\omega_r = 5\sqrt{2}$ rad/sec

$$TF = \frac{5\times\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Since the steady state gain is 5 so the transfer function is multiplied by it's magnitude i.e, steady state gain 5.

$$M_r = \frac{1\times 5}{2\xi\sqrt{1-\xi^2}} = \frac{10}{\sqrt{3}}$$

$\xi = 0.5$, 0.88

But $\xi < \frac{1}{\sqrt{2}}$, $\therefore$ $\xi = 0.5$

$\omega_r = \omega_n\sqrt{1-2\xi^2} = 5\sqrt{2}$

$\omega_n = 10$

$$TF = \frac{500}{s^2+10s+100}$$

04. **Refer Page No. 109 Example 25.**

05.

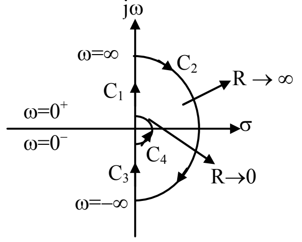
Sol: $G(s)H(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$|G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{1+(\omega^2T_1^2)}\sqrt{1+\omega^2T_2^2}},$$

$\angle G(j\omega)H(j\omega) = -90 - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$

Nyquist contour in s-plane



The Nyquist contour is having 4 sections C_1, C_2, C_3, C_4

Mapping of section C_1

$0 \le \omega \le \infty$

$|G(j\omega)H(j\omega)|_{\omega=0} = \infty$

$\angle G(j\omega)H(j\omega)|_{\omega=0} = -90° - 0 = -90°$

$\angle G(j\omega)H(j\omega)|_{\omega=\infty} = \dfrac{K}{\infty} = 0$

$\angle G(j\omega)H(j\omega)|_{\omega=\infty} = -90° - \tan^{-1}\infty - \tan^{-1}\infty$

$= -270°$

Nyquist plot intersect –ve real axis at ω_{pc}

$\angle G(j\omega)H(j\omega)|_{\omega=\omega_{pc}} = -180°$

$-180° = -90° - \tan^{1}\omega T_1 - \tan^{-1}\omega T_2$

$\tan^{-1}\omega T_1 + \tan^{-1}\omega T_2 = 90°$

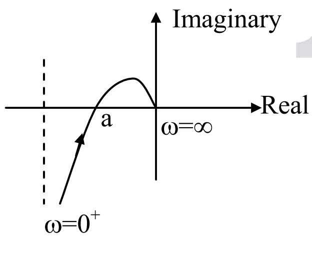
$$\frac{\omega T_1 + \omega T_2}{1-\omega^2 T_1 T_2} = \infty$$

$$\omega = \frac{1}{\sqrt{T_1T_2}}$$

$$|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}} = \frac{K}{\sqrt{T_1T_2}\sqrt{1+\frac{T_1}{T_2}}\sqrt{1+\frac{T_2}{T_1}}}$$

$$= \frac{KT_1T_2}{T_1+T_2}$$

Nyquist plot in G(s)H(s) plane

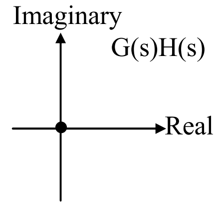


Mapping of section C_2

$s = Re^{j\theta}$, $R \to \infty$, $90° \ge \theta \ge -90°$

$$|G(j\omega)H(j\omega)| = \left|\frac{K}{s(1+sT_1)(1+sT_2)}\right|_{s=Re^{j\theta}, R\to\infty}$$

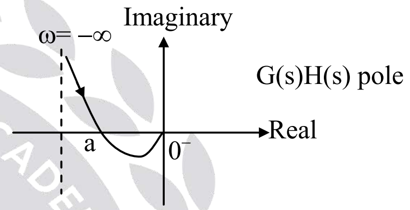
$$= \frac{K}{s^3} = 0$$



Mapping of section C_3:

It is mirror image of C_1, w.r.t –ve Real axis

$-\infty \le \omega \le 0^-$



Mapping of section C_4:

$S = \in e^{j\theta}$ $\in \to 0$, $-90° \le \theta \le 90°$

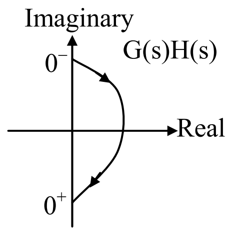
$$G(j\omega)H(j\omega) = \frac{K}{\in e^{j\theta}(1+\in e^{j\theta}T_1)(1+\in e^{j\theta}T_2)}$$

$$= \frac{K}{\in}\angle -\theta$$

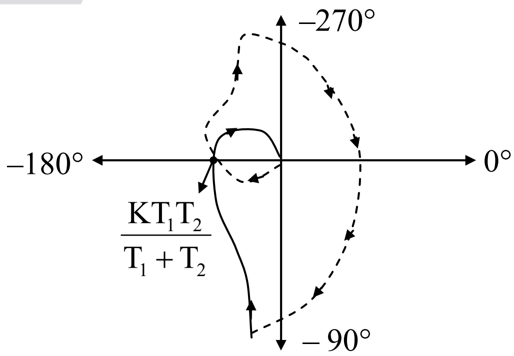
$\theta = -90°$, $\infty\angle 90°$

$\theta = 0°$, $\infty\angle 0°$

$\theta = 90°$, $\infty\angle -90°$



By combining all four sections, we get



(i) Point of intersection = 0.1

Then critical point (–1, + j0) lies outside of the unit circle.

Number of right hand open loop poles

P = 0.

Number of encirclements about the critical point N = 0.

N = P – Z

Z = 0

Z represents the number of right half of s - plane poles.

$\therefore$ The system is stable

(ii) Point of intersection = 10

then critical point lies inside the unit circle.

Number of encirclements about the critical point N = – 2

($\because$ clock wise direction)

$\therefore$ N = P – Z

–2 = 0 – Z

Z = 2

$\therefore$ Two poles on right half s-plane

$\therefore$ System is unstable

06.

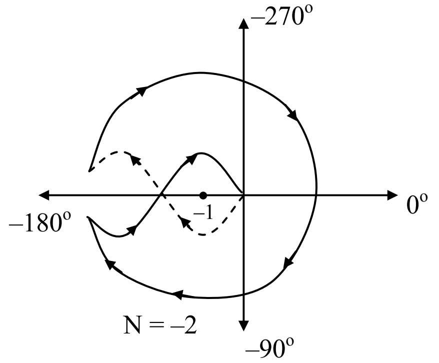
Sol: $G(s)H(s) = \dfrac{4s+1}{s^2(s+1)(2s+1)}$

$$|G(s)H(s)| = \frac{\sqrt{16\omega^2+1}}{\omega^2\sqrt{(\omega^2+1)(4\omega^2+1)}}$$

$$\angle G(s)H(s) = -180 - \tan^{-1}(\omega) + \tan^{-1}(4\omega) - \tan^{-1}(2\omega)$$

$\omega = 0 \quad \infty\angle{-180^o}$

$\omega = \infty \quad 0\angle{-270^o}$



By equating phase to – 180° then we can get phase cross - over frequency.

$-180 = -180 - \tan^{-1}(\omega) + \tan^{-1}(4\omega) - \tan^{-1}(2\omega)$

$$\Rightarrow \frac{\omega+2\omega}{1-2\omega^2} = 4\omega$$

$$\frac{3}{4} = 1 - 2\omega^2$$

$$2\omega^2 = \frac{1}{4} \Rightarrow \omega^2 = \frac{1}{8}$$

$$\omega_{pc} = \frac{1}{2\sqrt{2}} \text{ rad/sec}$$

$$|G(s)H(s)| = \frac{\sqrt{16\omega^2+1}}{\omega^2\sqrt{(\omega^2+1)(4\omega^2+1)}}$$

$$|G(s)H(s)|\Big|_{\omega=\omega_{pc}} = \frac{8\sqrt{16\frac{1}{8}+1}}{\sqrt{\left(\frac{1}{8}+1\right)\left(4\frac{1}{8}+1\right)}}$$

$$|G(s)H(s)|\Big|_{\omega=\omega_{pc}} = 10.67$$

$\therefore$ Critical point lies inside the unit circle.

Number of encirclement about the critical point is N = –2.

Number of right hand open loop poles

P = 0.

$\therefore$ N = P – Z

Z = 2

$\therefore$ Two poles are lies in right half s-plane.

$\therefore$ System is unstable.

RH criteria:

Characteristic equation is

1 + G(s) H(s) = 0

$\Rightarrow s^2 (s+1)(2s+1) + 4s + 1 = 0$

$\Rightarrow s^2(2s^2 + 3s + 1) + 4s + 1 = 0$

$\Rightarrow 2s^4 + 3s^3 + s^2 + 4s + 1 = 0$

$$\begin{array}{cc|ccc} + & S^4 & 2 & 1 & 1 \\ + & S^3 & 3 & 4 & \\ - & S^2 & \frac{-5}{3} & 1 & \\ + & S^1 & \frac{29}{5} & & \\ + & S^0 & 1 & & \end{array}$$

$\therefore$ Number of sign changes = 2

$\therefore$ System is unstable with two right half s-plane poles.

$\therefore$ The system is unstable.

07.

Sol: K = 0.9

$$M_r = 1.4 = \frac{K = 0.9}{2\xi\sqrt{1-\xi^2}}$$

$$\frac{2.8}{0.9} = \frac{1}{\xi\sqrt{1-\xi^2}}$$

$$\xi^2(1-\xi^2) = \left(\frac{9}{28}\right)^2$$

$\xi = 0.34, 0.94$

But $\xi < \frac{1}{\sqrt{2}}$ $\therefore \xi = 0.34$

$$\rightarrow \omega_r = \omega_n\sqrt{1-2\xi^2}$$

$$3 = \omega_n\sqrt{1-2(0.34)^2}$$

$\omega_n = 3.42$ rad/sec

$$\text{CLTF} = \frac{0.9\times 11.69}{s^2 + 2.32s + 11.69}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.975\,\text{sec}$$

$$\%\,M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 32.11\%$$

$$C(s) = \frac{0.9\times 11.69}{s^2 + 2.32s + 11.69}R(s)$$

$$C(\infty) = \underset{s\to 0}{Lt}\frac{0.9\times 11.69}{11.69} = 0.9$$

Steady state error = input – output

= 1 – 0.9

= 0.1

08.

Sol: $G(s)H(s) = \dfrac{40}{s(1+0.1s)} = \dfrac{40}{s(1+s/10)}$

Magnitude Plot

Factor	Slope of Factor	Slope of Mag plot	Frequency Range
$\frac{1}{s}$	–20 dB/dec	–20dB/dec	$0\rightarrow 10$
$\frac{1}{[1+s/10]}$	–20dB/dec	–40dB/dec	$10\rightarrow\infty$

$|G(j\omega)H(j\omega)|_{\omega=1} = 20\log 40 - 20\log 1 = 32$ dB

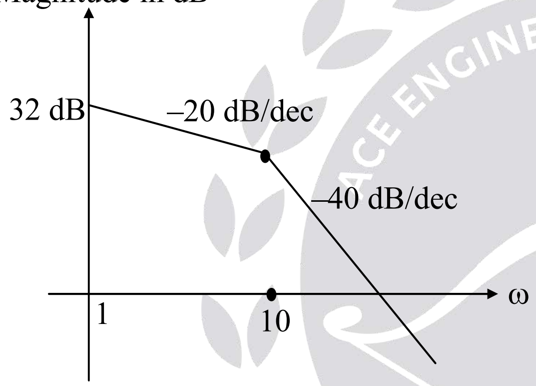
Phase plot:

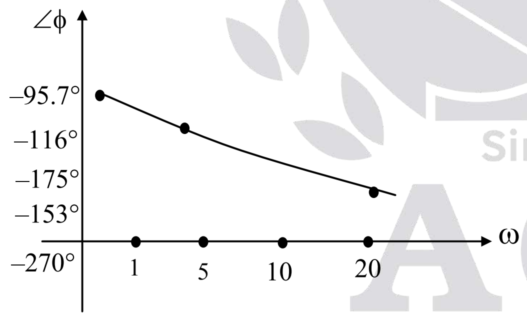
Phase $\phi = -90 - \tan^{-1}\left(\frac{\omega}{10}\right)$

ω	ϕ
1	–95.7°
5	–116°
10	–135°
20	–153°
.	
.	
∞	–180°

Magnitude plot:

Magnitude in dB





Calculation for ω_{pc}:-

at $\omega = \omega_{pc} \Rightarrow \angle G(j\omega)H(j\omega) = -180°$

$$-90° - \tan^{-1}\left(\frac{\omega_{pc}}{10}\right) = -180$$

$$\tan^{-1}\left(\frac{\omega_{pc}}{10}\right) = 90°$$

$\omega_{pc} = \infty$

Calculation for ω_{gc} :-

(a) $\omega = \omega_{gc}$, $|G(j\omega)H(j\omega)| = 1$

$$\left|\frac{40}{(j\omega)\left(10 + \frac{j\omega}{10}\right)}\right| = 1$$

$400 = 10\ \sqrt{\omega^2 + 10}$

$(400)^2 = \omega^2\ (\omega^2 + 100)$

Let $\omega^2 = x$

$x^2 + 100x - 160000 = 0$

$x_1 = 400\ \angle 97.18$

$x_2 = 400\ \angle -97.18$

$\omega_{gc}^2 = 400\ \angle -97$

$\omega_{gc} = 20\ \angle 48.5$

$\omega_{gc} = 20$ rad/sec

$$\frac{40}{|\omega||1 + 0.1j\omega|} = 1$$

$$\frac{40}{\omega\sqrt{1 + (0.1\omega)^2}} = 1$$

$(40)^2 = \omega^2\ (1 + 0.01\omega^2)$

$(40)^2 = \omega^2 + 0.01\ \omega^4$

$1600 = x + 0.01\ x^2$

$x_1 = 400\ \angle 97$

$x_2 = 400\ \angle -97$

$\omega_{pc} = \infty$ rad/sed

$GM = \infty$

$M|_{\omega=\omega_{pc}} = -\infty dB$

$GM = -M_{\omega=\omega_{pc}}$

$= -(-\infty) = \infty$

$\omega_{gc} = 20$ rad/sec

$\phi|_{\omega_{gc}} = -153°$

$PM = 180 - 153 = 27°$

09.

Sol: G.M = 20 log $\dfrac{1}{\left|G(j\omega)H(j\omega)\right|_{\omega=\omega_{pc}}}$

ω_{pc} : $\angle G(j\omega)\,H(j\omega) / \omega = \omega_{pc} = -180°$

$-90° - \tan^{-1}(\omega_{pc}) - \tan^{-1}(0.05\,\omega_{pc}) = -180°$

$\tan^{-1}(\omega_{pc}) + \tan^{-1}(0.05\,\omega_{pc}) = 90°$

$$\frac{\omega_{pc} + 0.05\,\omega_{pc}}{1-\omega_{pc}^2 \times 0.05} = \frac{1}{0}$$

$\omega_{pc} = = \sqrt{20}$ rad/sec

$$\left|G(j\omega)H(j\omega)\right| = \frac{K}{\sqrt{20}\sqrt{21}\sqrt{1.05}} = \frac{K}{21}$$

G.M = 20 log $\dfrac{21}{K} = 15$

K = 3.73

Phase margin = 180° + $\angle G(j\omega)H(j\omega)|_{\omega = \omega_{gc}}$

$\angle G(j\omega)H(j\omega) = 90° - \tan^{-1}(\omega) - \tan^{-1}(0.05\,\omega)$

$40° = 90° - [\tan^{-1}(\omega) + \tan^{-1}(0.05\omega)]$

$50° = \tan^{-1}(\omega) + \tan^{-1}(0.05\,\omega)$

$1.5\,\omega = 1.19 - 0.595\,\omega^2$

$0.595\,\omega^2 + 1.05\,\omega - 1.19 = 0$

$\omega_{gc} = 1.06$ rad/sec

At ω_{gc}, $|G(j\omega)H(j\omega)| = 1$

$$\frac{K}{1.06\sqrt{1+(1.06)^2}\sqrt{1+(1.06\times 0.5)^2}} = 1$$

K = 1.54

Chapter 7 Controllers & Compensators

Objective Practice Solutions

01. Ans: (a)

Sol: $G_C(s) = (-1)\left(-\frac{Z_2}{Z_1}\right) = (-1)(-1)\left(\frac{R_2 + \frac{1}{sC}}{R_1}\right)$

$$G_c(s) = \frac{(100\times 10^3) + \frac{1}{s\times 10^{-6}}}{10^6}$$

$$G_c(s) = \frac{1+0.1s}{s}$$

02. Ans: (c)

Sol: CE $\Rightarrow 1 + G_c(s)\,G_p(s) = 0$

$$= 1 + \frac{1+0.1s}{s}\times\frac{1}{(s+1)(1+0.1s)}$$

$$= 1 + \frac{1+0.1s}{s(s+1)(1+0.1s)} = 0$$

$\Rightarrow s^2 + s + 1 = 0 \Rightarrow \omega_n = 1,$

$$e^{\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]_{\xi=0.5}} = 0.163$$

$M_p = 16.3\%$

03. Ans: (b)

Sol: T.F $= \frac{k(1+0.3s)}{1+0.17s}$

$T = 0.17,\ aT = 0.3 \Rightarrow a = \frac{0.3}{0.17}$

$C = 1\ \mu F$

$$T = \frac{R_1R_2}{R_1+R_2}C,\quad a = \frac{R_1+R_2}{R_2}$$

$$\frac{R_1R_2}{R_1+R_2} = \frac{0.17}{1\times 10^{-6}} = 170000$$

$$\frac{R_1+R_2}{R_2} = 1.764$$

$aT = R_1 C$

$R_1 = \frac{aT}{C} = \frac{0.3}{C} = (0.3)(10^6) = 300\ k\Omega$

Bv

$300\ k + R_2 - 1.76\ R_2 = 0$

$R_2 = \frac{300}{0.70} = 394.736$

$= 400\ k\Omega$

04. Ans: (d)

Sol: PD controller improves transient stability and PI controller improves steady state stability. PID controller combines the advantages of the above two controllers.

05.

Sol: For $K_I = 0 \Rightarrow$

$$\frac{C(s)}{R(s)} = \frac{(K_P + K_D s)}{s(s+1) + (K_P + K_D s)}$$

$$= \frac{K_P + K_D s}{s^2 + (1+K_D)s + K_P}$$

$\omega_n = \sqrt{K_P}$

$2\xi\omega_n = 1 + K_D$

$\Rightarrow 2(0.9)\sqrt{K_P} = 1 + K_D$

$\Rightarrow 1.8\sqrt{K_P} = 1 + K_D$ ……… (1)

Dominant time constant $\frac{1}{\xi\omega_n} = 1$

$$\Rightarrow \omega_n = \frac{1}{0.9} = 1.111$$

$$K_P = \omega_n^2 = 1.11^2$$
$$= 1.234$$

From eq. (1),

$$\Rightarrow 1.8 \times \frac{1}{0.9} = 1 + K_D$$

$$\Rightarrow K_D = 1$$

Conventional Practice Solutions

01.

Sol: Given that

$$\theta_c(t) = 20\left[e(t) + \frac{1}{T_r}\int_0^t e(t)\,dt + T_d\frac{de(t)}{dt}\right]$$

Applying Laplace transform on both sides with zero initial conditions

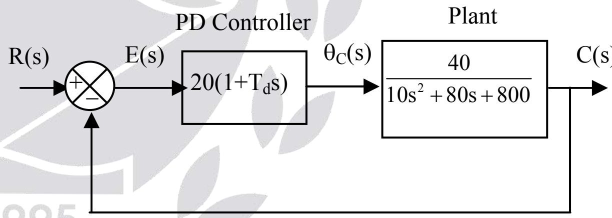
$$\theta_c(s) = 20\left[E(s) + \frac{E(s)}{T_r s} + T_d sE(s)\right]$$

$$\theta_c(s) = 20\left(1 + \frac{1}{T_r s} + T_d s\right)E(s) \quad \text{……….} (1)$$

i) If Integral action is not used then,

$$\theta_c(s) = 20(1+T_d s)E(s)$$

$$\frac{\theta_c(s)}{E(s)} = 20(1+T_d s)$$



Characteristic equation is 1+G(s)H(s) = 0

$$1 + \frac{40\times 20(1+T_d s)}{10(s^2+8s+80)} = 0$$

$$s^2 + s\left(\frac{80+800T_d}{10}\right) + 160 = 0 \text{ ……. } (2)$$

2nd order prototype characteristic equation is, $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ then we get,

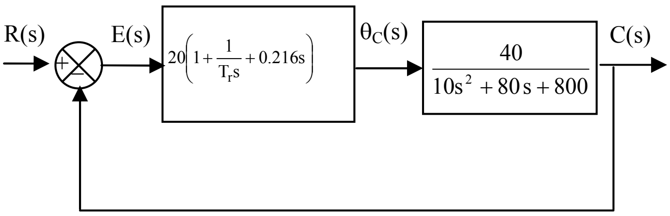
$$\omega_n = \sqrt{160} \Rightarrow \omega_n = 12.649 \text{ rad/sec}$$

and $2\xi\omega_n = \dfrac{80+800T_d}{10}$ for $\xi = 1$

$2 \times 1 \times 12.649 = 8\,(1+10\,T_d)$

$T_d = 0.216$ sec.

ii) If the value of derivative time is maintain then,



Characteristic equation is, $1+G(s)H(s) = 0$

$$1+20\left(1+\frac{1}{T_r s}+0.216s\right)\left(\frac{40}{10s^2+80s+800}\right)=0$$

$$10\,T_r s^3 + 252.8\,T_r s^2 + 1600\,T_r s + 800 = 0$$

R-H Tabulation

s^3	$10T_r$	$1600\ T_r$
s^2	$252.8\ T_r$	800
s^1	$\frac{404480T_r^2 - 8000\,T_r}{252.8\ T_r}$	0
s^0	800	0

For the system to be stable

s^1 row ≥ 0,

$404480T_r^2 - 8000\ T_r \geq 0$

$T_r \geq 0.019$ sec

$\therefore$ Minimum value of $T_r = 0.0197$sec.

02.

Sol: $G_C(s) = ?$

Ramp error constant $= K_V = 5$

$$\text{CLTF} = \frac{Y(s)}{R(s)} = \frac{k}{(s^2+20s+200)(s+a)}$$

$K_V = 5$

$$\underset{s\to 0}{\text{Lt}}\ s.G(s) = 5$$

$$\frac{G(s)}{1+G(s)} = \frac{k}{(s+a)(s^2+20s+200)}$$

$$\frac{1+G(s)}{G(s)} = \frac{s^3+s^2(20+a)+s(200+20a)+200a}{k}$$

$$\frac{1}{G} = \frac{s^3+s^2(20+a)+s(200+20a)+200a}{k} - 1$$

$$\frac{1}{G} = \frac{s^3+s^2(20+a)+s(200+20a)+200a-k}{k}$$

OLTF = G = G(s)

$$= \frac{k}{s^3+s^2(20+a)+s[200+20a]+200a-k};\ H(s)=1$$

$K_V = 5$

$$\underset{s\to 0}{\text{Lt}}\ sG(s) = 5$$

$\therefore$ G(s) should be type – 1

Here $200a - k = 0 \Rightarrow k = 200a$

$$G(s) = \frac{k}{s[s^2+(20+a)s+(200+20a)]}$$

$$\underset{s\to 0}{\text{lt}}\ s\frac{200a}{s[s^2+(20+a)s+(200+20a)]} = 5$$

$$\frac{200a}{200+20a} = 5$$

$200a - 100a = 5 \times 200$

$a = 10$

$K = 200a = 2000$

$$G(s) = \frac{2000}{s[s^2+(20+10)s+400]};\ H(s)=1$$

$$G(s) = \frac{2000}{s(s^2+30s+400)} = \frac{G_C(s)100}{s(s^2+10s+100)}$$

$$G_C(s) = \frac{20(s^2+10s+100)}{s^2+30s+400}$$

03.

Sol: $G(s) = \dfrac{100}{(s+4)(s+5)(s+7)}$

PM = 50° at ω = 2rad/sec

TF of PI controller

$= k_p + \dfrac{k_I}{s} \Rightarrow \dfrac{s(k_p) + (k_I)}{s}$

G(s) with controller = $G_C(s)$

$$= \frac{100(sk_p + k_I)}{(s+4)(s+5)(s+7)s}$$

PM = 180° + $\angle G(s)H(s)\big|_{\omega=\omega_{gc}}$

50 = 180° + $\angle G_C(s)H(s)$

$\therefore \angle G(s)H(s)\big|_{\omega=\omega_{gc}} = -130°$

$\angle G_C(s)H(s) = -130°$

Given ω_{gc} = 2 rad/s

$$\tan^{-1}\left(\frac{2k_p}{k_I}\right) - 90 - \tan^{-1}\left(\frac{2}{4}\right) - \tan^{-1}\left(\frac{2}{5}\right) - \tan^{-1}\left(\frac{2}{7}\right) = -130°$$

$$\tan^{-1}\left(\frac{2k_p}{k_I}\right) = -40^o + \tan^{-1}(0.5) + \tan^{-1}(0.4) + \tan^{-1}(2/7)$$

$$\tan^{-1}\left(\frac{2k_p}{k_I}\right) = 24.31$$

$$0.449 = \frac{2(k_p)}{k_I}$$

$$\frac{k_p}{k_I} = 0.224$$

$k_p = 0.224\, k_I$

At $\omega_{gc} \Rightarrow |G(s)H(s)| = 1$

$$\frac{100\sqrt{(\omega k_p)^2 + k_I^2}}{\sqrt{\omega^2+4^2}\sqrt{\omega^2+25}\sqrt{\omega^2+49}\ \ \omega} = 1$$

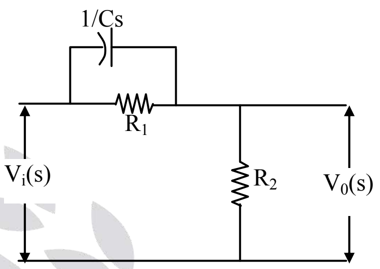
$$\frac{100\sqrt{[(2)k_p]^2 + k_I^2}}{2\sqrt{4+16}\sqrt{29}\sqrt{53}} = 1$$

$k_p = 0.72;\ k_I = 3.196$

TF of PI controller = $0.72 + \dfrac{3.196}{s}$

04.

Sol: Given lead network can be drawn as,



$$\frac{V_0(s)}{V_i(s)} = \frac{R_2}{R_2 + R_1 \,||\, \frac{1}{sC}}$$

$$= \frac{R_2}{R_1+R_2}\left[\frac{1+R_1Cs}{1+\frac{R_1R_2Cs}{R_1+R_2}}\right]$$

$T = \left(\dfrac{R_1R_2}{R_1+R_2}\right)C$ and $a = \dfrac{R_1+R_2}{R_2}$

Transfer function $= \dfrac{1}{a}\left(\dfrac{1+aTs}{1+Ts}\right)$

The sinusoidal transfer function is

$$G_c(j\omega) = \frac{1+jaT\omega}{1+j\omega T}$$

Phase angle at any frequency ω is given by

$\phi = \angle G_c(j\omega) = \tan^{-1}(a\omega T) - \tan^{-1}(\omega T)$

Differentiating with respect to 'ω'

$$\frac{d\phi}{d\omega} = \frac{aT}{1+(a\omega T)^2} - \frac{T}{1+(\omega T)^2}$$

When $\frac{d\phi}{d\omega} = 0$

$$\frac{aT}{1+(a\omega_m T)^2} - \frac{T}{1+(\omega_m T)^2} = 0$$

$$\Rightarrow \frac{aT}{1+(a\omega_m T)^2} = \frac{T}{1+(\omega_m T)^2}$$

$$\Rightarrow a\left(1+(\omega_m T)^2\right) = 1+(a\omega_m T)^2$$

$$\Rightarrow a + a\omega_m^2 T^2 = 1 + a^2\omega_m^2 T^2$$

$$\Rightarrow \omega_m^2 T^2 (a - a^2) = 1 - a$$

$$\Rightarrow \omega_m^2 T^2 a(1-a) = (1-a)$$

$$\Rightarrow \omega_m^2 T^2 = \frac{1}{a} \Rightarrow$$

$$\Rightarrow \omega_m^2 = \frac{1}{T^2 a} \Rightarrow \boxed{\omega_m = \frac{1}{T\sqrt{a}}}$$

At $\omega = \omega_m$ the maximum phase lead ϕ_m is

$$\phi\Big|_{\omega=\omega_m} = \tan^{-1}(a\omega_m T) - \tan^{-1}(\omega_m T)$$

$$= \tan^{-1}\left(a.\frac{1}{T\sqrt{a}}.T\right) - \tan^{-1}\left(\frac{1}{T\sqrt{a}}.T\right)$$

$$\phi_m = \tan^{-1}(\sqrt{a}) - \tan^{-1}\left(\frac{1}{\sqrt{a}}\right)$$

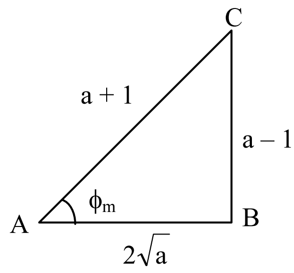
Apply 'tan' on both sides, then we get

$$\tan\phi_m = \frac{\sqrt{a} - \frac{1}{\sqrt{a}}}{1+\sqrt{a}.\frac{1}{\sqrt{a}}} = \frac{\frac{a-1}{\sqrt{a}}}{2}$$

$$\tan\phi_m = \frac{a-1}{2\sqrt{a}}$$

$$\phi_m = \tan^{-1}\left(\frac{a-1}{2\sqrt{a}}\right)$$

$\therefore$ Maximum value of phase lead ϕ_m is $\tan^{-1}\left(\frac{a-1}{2\sqrt{a}}\right)$



$$AC = \sqrt{(a-1)^2 + 4a}$$

$$= a^2 + 1 - 2a + 4a$$

$$= \sqrt{(a+1)^2}$$

$$= a + 1$$

$$\boxed{\therefore \sin\phi_m = \frac{a-1}{a+1}}$$

Chapter 8 State Space Analysis

Objective Practice Solutions

01. Ans: (a)

Sol: $TF = \dfrac{1}{s^2+5s+6}$

$= \dfrac{1}{(s+2)(s+3)}$

$= \dfrac{1}{s+2} + \dfrac{-1}{s+3}$

$\therefore\ A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$C = [1 \quad 1]$

02. Ans: (c)

Sol: Given problem is Controllable canonical form.

(or)

$TF = C[sI - A]^{-1}B + D$

$= [6 \ \ 5 \ \ 1] \begin{bmatrix} s & 1 & 0 \\ 0 & s & 1 \\ -5 & -3 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

$= \dfrac{3s^2+15s+18}{s^3+6s^2+3s+5}$

03. Ans: (d)

Sol: $\dfrac{d^2y}{dt^2} + \dfrac{3dy}{dt} + 2y = u(t)$

2^{nd} order system hence two state variables are chosen

Let x_1 (t), x_2 (t) are the state variables

CCF − SSR

Let x_1 (t) = y (t) ……….. (1)

x_2 (t) = $\dot{y}$(t) ……….. (2)

Differentiating (1)

$\dot{x}_1$(t) = $\dot{y}$(t) = x_2 (t) ……….. (3)

$\dot{x}_2$(t) = $\ddot{y}$(t) = u (t) − $3y^1$(t) − 2y(t)

= u(t) − $3x_2$ (t) − $2x_1$(t) ……. (4)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underset{A}{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u(t)$$

From equation 1. The output equation in matrix form

$y(t) = [1 \ \ 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, D = 0$

04. Ans: (b)

Sol: OCF − SSR

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

$y(t) = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

05. Ans: (c)

Sol: Normal form − SSR

$TF = \dfrac{Y(s)}{G(s)} = \dfrac{1}{s^2+3s+2} = \dfrac{1}{(s+1)(s+2)}$

⇒ Diagonal canonical form

The eigen values are distinct i.e., − 1 & − 2.

∴ Corresponding normal form is called as diagonal canonical form

DCF − SSR

$$\frac{Y(s)}{U(s)} = \frac{b_1}{s+1} + \frac{b_2}{s+2}$$

$b_1 = 1$, $b_2 = -1$

$$Y(s) = \underbrace{\frac{b_1}{s+1}}_{x_1} U(s) + \underbrace{\frac{b_2}{s+2}}_{x_2} U(s)$$

Let $Y(s) = X_1(s) + X_2(s)$

Where $y(t) = x_1(t) + x_2(t)$ ………. (1)

Where $X_1(s) = \frac{b_1}{s+1} U(s)$

$s X_1(s) + X_1(s) = b_1 U(s)$

Take Laplace Inverse

$\dot{x}_1 + x_1 = b_1 u(t)$ ……… (2)

$X_2(s) = \frac{b_2}{s+2} U(s)$

$s X_2(s) + 2 X_2(s) = b_2 U(s)$

Laplace Inverse

$\dot{x}_2 + 2x_2 = b_2 u(t)$

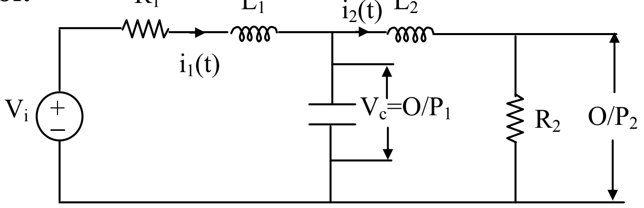
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t)$$

From (1) output equation.

$$y(t) = [1 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

06. Ans: (c)

Sol:



$O/P_1 \Rightarrow y_1 = V_c$

$O/P_2 \Rightarrow y_2 = R_2 i_2$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & R_2 \end{bmatrix} \begin{bmatrix} V_c \\ i_1 \\ i_2 \end{bmatrix}$$

$y = C X$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & R_2 \end{bmatrix}$$

07. Ans: (a)

Sol: T.F $= C[sI-A]^{-1}B + D$

$$= [1 \;\; 0] \begin{bmatrix} s+4 & 1 \\ 3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [1 \;\; 0] \frac{1}{s^2+5s+1} \begin{bmatrix} s+1 & -1 \\ -3 & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+1} [1 \;\; 0]_{1\times2} \begin{bmatrix} s+1 & -1 \\ -3 & s+4 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+1} [s+1 \;\; -1]_{1\times2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times1}$$

$$= \frac{1}{s^2+5s+1} [s+1-1]$$

$$= \frac{s}{s^2+5s+1}$$

08. Ans: (c)

Sol: State transition matrix $\phi(t) = L^{-1}[(sI-A)^{-1}]$

$$sI - A = \begin{bmatrix} s+3 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{(s+2)(s+3)} \begin{bmatrix} s+2 & 1 \\ 0 & s+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$L^{-1}[[sI-A]^{-1}] = \begin{bmatrix} e^{-3t} & e^{-2t} - e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$$

09. Ans: (b)

Sol: Controllability

$$[M]=[B\quad AB\quad A^2B..\quad A^{n-1}B]$$

$$AB=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\-3\end{bmatrix}$$

$$M=\begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix}$$

$$|M|=-1\neq 0 \text{ (Controllable)}$$

Observability

$$[N]=[C^T\ A^TC^T\ ....\ (A^T)^{n-1}C^T]$$

$$A^TC^T=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-2\\-2\end{bmatrix}$$

$$N=\begin{bmatrix}1 & -2\\ 1 & -2\end{bmatrix}$$

$$|N|=0 \text{ (Not observable)}$$

10. Ans: (c)

Sol: According to Gilberts test the system is controllable and observable.

11. Ans: (c)

Sol: $$\frac{Y(s)}{U(s)}=\frac{b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}$$

at node $\dot{x}_1$

$$\dot{x}_1=-a_1x_1-a_2x_2-a_3x_3$$

at $\dot{x}_2=x_1$ & $\dot{x}_3=x_2$

$$\therefore\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-a_1 & -a_2 & -a_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\therefore A=\begin{bmatrix}-a_1 & -a_2 & -a_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

12.

Sol: The given state space equations:

$$\dot{X}=X_2$$

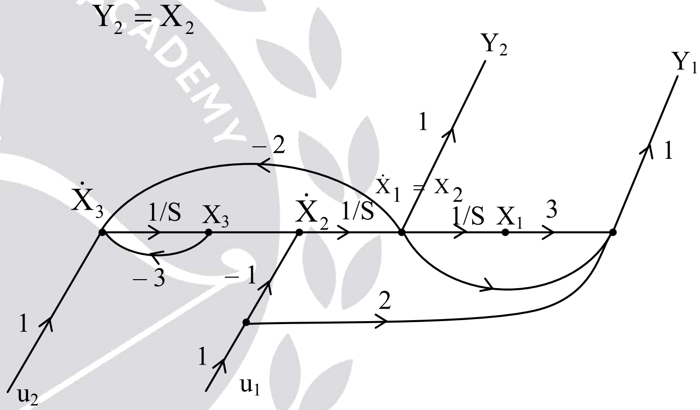
$$\dot{X}_2=X_3-u_1$$

$$\dot{X}_3=-2X_2-3X_3+u_2$$

and output equations are :

$$Y_1=X_1+3X_2+2u_1$$

$$Y_2=X_2$$



The given state space equations in matrix for

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}_{3\times3}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix}_{3\times1}+\begin{bmatrix}0 & 0\\ -1 & 0\\ 0 & 1\end{bmatrix}_{3\times2}\begin{bmatrix}u_1\\ u_2\end{bmatrix}_{2\times1}$$

$$\begin{bmatrix}Y_1\\ Y_2\end{bmatrix}=\begin{bmatrix}1 & 3 & 0\\ 0 & 1 & 0\end{bmatrix}_{2\times3}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix}_{3\times1}+\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}_{2\times2}\begin{bmatrix}u_1\\ u_2\end{bmatrix}_{2\times1}$$

Where A: State matrix

B: Input matrix

C: Output matrix

D: Transition matrix

Characteristic equation

$|sI - A| = 0$

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} S & -1 & 0 \\ 0 & S & -1 \\ 0 & 2 & S+3 \end{vmatrix} = 0$$

$\Rightarrow s[s(s+3)+2]+1(0) = 0$

$\Rightarrow s(s^2+3s+2) = 0$

$\Rightarrow s(s+1)(s+2) = 0$

The roots are 0, –1,–2.

Conventional Practice Solutions

01.

Sol: $\dfrac{Y(s)}{U(s)} = \dfrac{10(2s+3)}{(s+1)(s+2)(s+3)}$

i) Phase variable form

$$\frac{Y(s)}{U(s)} = \frac{10(2s+3)}{(s^2+3s+2)(s+3)}$$

$$= \frac{10(2s+3)}{s^3+3s^2+2s+3s^2+9s+6}$$

$$\frac{Y(s)}{U(s)} = \frac{10(2s+3)}{s^3+6s^2+11s+6}$$

$$\left[\dot{x}(t)\right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} [x(t)] + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} V$$

$y(t) = [3 \quad 2 \quad 0] \; x(t)$

ii) Normal forms

$$\frac{Y(s)}{U(s)} = \frac{10(2s+3)}{(s+1)(s+2)(s+3)}$$

Apply the partial fractions

$$\frac{10(2s+3)}{(s+1)(s+2)(s+3)} = \frac{5}{s+1} + \frac{10}{s+2} - \frac{15}{s+3}$$

$$\left[\dot{x}(t)\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 5 \\ 10 \\ -15 \end{bmatrix}$$

$y(t) = [1 \quad 1 \quad 1] \; x(t)$

02.

Sol: $\dfrac{Y(s)}{U(s)} = \dfrac{(s^2-3s-1)}{(s+1)^2(s+4)}$

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+4}$$

$$= \frac{1}{(s+1)^2} - \frac{2}{(s+1)} + \frac{3}{s+4}$$

$$Y(s) = \frac{U(s)}{(s+1)^2} - \frac{2U(s)}{s+1} + \frac{3U(s)}{s+4}$$

$Y(s) = x_1 - 2x_2 + 3x_3$

$$x_1 = \frac{U(s)}{(s+1)^2} = \frac{U(s)}{s+1} \times \frac{1}{s+1} = \frac{x_2}{s+1}$$

$\Rightarrow sx_1 + x_1 = x_2$

$\dot{x}_1 = -x_1 + x_2$ ------ (1)

$$x_2 = \frac{U(s)}{s+1}$$

$sx_2 + x_2 = U$

$\dot{x}_2 = U - x_2$ ------- (2)

$$x_3 = \frac{U}{s+4}$$

$sx_3 + 4x_3 = U$

$\dot{x}_3 = U - 4x_3$ ------ (3)

From equations 1, 2 and 3

Normal form is

$$[\dot{x}(t)] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$y(t) = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} x(t)$

03.

Sol: a) $A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$T.F = C(sI - A)^{-1}B + D$

$$(sI - A) = \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s+1 & -1 \\ +3 & s+5 \end{bmatrix}}{(s+1)(s+5)+3}$$

$$T.F = \frac{\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}}{s^2+6s+8}$$

$$= \frac{\begin{bmatrix} s+1+6 & -1+2s+10 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}}{s^2+6s+8}$$

$$= \frac{\begin{bmatrix} s+7 & 2s+9 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}}{s^2+6s+8}$$

$$= \frac{2s+14+10s+45}{s^2+6s+8} = \frac{12s+59}{s^2+6s+8}$$

b) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$e^{At} = \phi(t) = L^{-1}\{(sI-A)^{-1}\}$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s(s+3)+2}$$

Apply ILT to the above matrix

$$= L^{-1}\{(sI-A)^{-1}\} = L^{-1}\begin{Bmatrix} \frac{s+3}{s^2+3s+2} & \frac{+1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{Bmatrix}$$

$$= L^{-1}\begin{Bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{Bmatrix}$$

$$\phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$L^{-1}[(sI-A)^{-1}]X(0)$

$$=\begin{bmatrix} 2.e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$=\begin{bmatrix} e^{-t}-e^{-2t} \\ -e^{-t}+2e^{-2t}\end{bmatrix} \ldots\ldots (1)$$

$L^{-1}[(sI-A)^{-1}BU(s)]$

$$=\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}.\frac{1}{s+1}$$

$$=\begin{bmatrix} \dfrac{2(s+3)}{(s+1)^2(s+2)} \\ \dfrac{-4}{(s+1)^2(s+2)}\end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{-2}{(s+1)}+\dfrac{4}{(s+1)^2}+\dfrac{2}{s+2} \\ \dfrac{4}{(s+1)}-\dfrac{4}{(s+1)^2}-\dfrac{4}{(s+2)}\end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{-2}{(s+1)}+\dfrac{4}{(s+1)^2}+\dfrac{2}{s+2} \\ \dfrac{4}{(s+1)}-\dfrac{4}{(s+1)^2}-\dfrac{4}{s+2}\end{bmatrix}$$

$$=\begin{bmatrix} -2e^{-t}+4t.e^{-t}+2e^{-2t} \\ 4e^{-t}-4t.e^{-t}-4.e^{-2t}\end{bmatrix} \ldots\ldots\ldots (2)$$

Solution to state transition equation

$$=\begin{bmatrix} e^{-t}-e^{-2t} \\ -e^{-t}+2e^{-2t}\end{bmatrix}+\begin{bmatrix} -2e^{-t}+4t.e^{-t}+2e^{-2t} \\ 4e^{-t}-4t.e^{-t}-4.e^{-2t}\end{bmatrix}$$

$$=\begin{bmatrix} 4te^{-t}+e^{-2t}-e^{-t} \\ 3e^{-t}-4t.e^{-t}-2e^{-2t}\end{bmatrix}$$

04.

Sol: Gilbert test is valid for normal form and Jordon form

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3\end{bmatrix} \qquad B=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$C=\begin{bmatrix}1 & 0 & 2\end{bmatrix}$

For Controllability:

The third element in matrix A corresponding element in matrix B is zero.

$\therefore$ the given system is not controllable.

For Observability:

The second row and second column element in matrix A corresponding element in matrix C is zero.

$\therefore$ The given system is not observable.

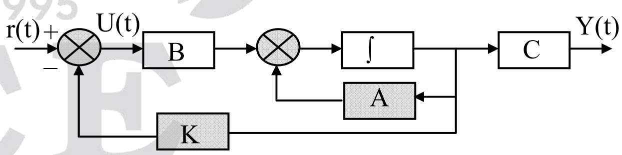
05.

Sol: $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ $B=\begin{bmatrix}0\\1\end{bmatrix}$

The system uses the state feedback control $u = -Kx$.

The desired closed loop poles at $S = -3+j3$ and $S = -3-j3$

Block diagram of state variable feedback



State Variable Feedback

$u = -K\,x$

u is the control signal

$\therefore\ \dot{x}(t) = (A - BK)\,x(t) + Br(t)$

Necessary and sufficient condition for arbitrary pole placement is that the system should be completely state controllable.

The rank of the controllability matrix is given by

$$|M| = [B \quad AB \quad A^2B \ldots\ldots A^{n-1}\; B]$$

$|M| = 0$ Not controllable

$\neq 0$ controllable

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad |M| = -1 \;\neq 0$$

So, the system is completely state controllable and arbitrary pole placement is possible. Find characteristic equation from a given matrix A.

Characteristic equation is $|sI - (A - Bk)| = 0$

$$A - Bk = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \quad k_2]$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & k_2 \end{bmatrix}$$

$$sI - (A - Bk) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ -k & s + k_2 \end{bmatrix}$$

$$|sI - (A - Bk)| = 0$$

$s(s+k_2) + k_1 = 0$

$s^2 + k_2 s + k_1 = 0$ ----------- (1)

From the given poles $(-2 \pm j4)$

$\therefore$ Characteristic equation is $(s + 2)^2 + 4^2 = 0$

$s^2 + 4s + 20 = 0$ ------------ (2)

Equate 1 & 2

$s^2 + 4s + 20 = s^2 + k_2 s + k_1$

$k_1 = 20, \; k_2 = 4$

SVFB controller,

$\therefore k = [20 \quad 4]$

06.

Sol: State transition matrix $\phi(t) = L^{-1}(sI-A)^{-1}$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}$$

$$= \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{1}{s^2+3s+2} \\ \dfrac{-2}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix}$$

$\phi(t) = L^{-1}[(sI-A)^{-1}]$

$$= L^{-1}\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

State equation $x(t) = L^{-1}[(sI-A)^{-1}]\, x(0) + L^{-1}[(sI-A)^{-1} BU(s)]$

$L^{-1}[(sI-A)^{-1}]\, x(0)$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix} \text{-------- (1)}$$

$L^{-1}[(sI-A)^{-1} BU(s)]$

$\therefore \; (sI-A)^{-1} BU(s)$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \frac{1}{s}$$

$$= \begin{bmatrix} \frac{5}{s(s+1)(s+2)} \\ \frac{5}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} 5/2 - 5e^{-t} + 5/2e^{-2t} \\ 5e^{-t} - 5e^{-2t} \end{bmatrix}$$

……… (2)

Adding (1) and (2) we get

State equation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - 3e^{-t} + \frac{3}{2}e^{-2t} \\ 3e^{-t} - 3e^{-2t} \end{bmatrix}$

$Y(t) = [0 \quad 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$Y = X_2$

$y(t) = 3e^{-t} - 3e^{-2t}$

07.

Sol: $\dot{x}_1 = -x_1(t) + 5x_2(t) \rightarrow (1)$

$\dot{x}_2 = -6x_1(t) + u(t) \rightarrow (2)$

$u(t) = -k_1 x_1(t) - k_2 x_2(t) + r(t) \rightarrow (3)$

From equation (2) and (3)

$\dot{x}_2 = -6x_1(t) - k_1 x_1(t) - k_2 x_2(t) + r(t)$

$= [-6-k_1] x_1(t) - k_2 x_2(t) + r(t) \rightarrow (4)$

From (1) and (4)

$A = \begin{bmatrix} -1 & 5 \\ -[6+k_1] & -k_2 \end{bmatrix} B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

C.E. equation

$|sI-A|=0$

$\begin{bmatrix} s+1 & -5 \\ 6+k_1 & s+k_2 \end{bmatrix} = 0$

$(s+1)(s+k_2) + 5(6+k_1)=0$

$s^2 + (1+k_2)s + 30 + 5k_1 + k_2 = 0$

Compared with standard 2nd order equation

$2\xi\omega_n = 1 + k_2$

$2(0.707)(10) = 1 + k_2$

$k_2 = 13.14$

$\omega_n^2 = 100$

$5k_1 + k_2 + 30 = 100$

$k_1 = 11.372$

$\therefore k_1 = 11.372, k_2 = 13.14$

08.

Sol: The given state equations

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} U$$

$$[Y] = [1 \quad 1 \quad 1] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

The output response

$y(t) = x_1(t) + x_2(t) + x_3(t)$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = e^{At}.X(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

Where $e^{At} = L^{-1}(sI-A)^{-1}$ called as state transition matrix

$$(sI-A) = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}$$

$$\therefore (sI-A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}$$

Since in problem, not given initial conditions so no zero input response.

$\therefore$ Zero state response only exist and it is equal to $= L^{-1}\left[\varphi(s)BU(s)\right]$

Where $\varphi(s) = (sI - A)^{-1}$

$$B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

U(s) = unit step input

$$= L^{-1}\left[\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\left[\frac{1}{s}\right]\right]$$

$$= L^{-1}\left[\begin{bmatrix} \frac{1}{s+1} \\ \frac{-2}{s+2} \\ \frac{1}{s+3} \end{bmatrix}\left(\frac{1}{s}\right)\right]$$

$$= L^{-1}\begin{bmatrix} \frac{1}{s(s+1)} \\ \frac{-2}{s(s+2)} \\ \frac{1}{s(s+3)} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ \frac{-1}{s} + \frac{1}{s+2} \\ \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 - e^{-t} \\ -1 + e^{-2t} \\ \frac{1}{3}\left(1 - e^{-3t}\right) \end{bmatrix}$$

$\therefore$ The output response

$$y(t) = x_1(t) + x_2(t) + x_3(t)$$

$$= 1 - e^{-t} - 1 + e^{-2t} + \frac{1}{3}\left(1 - e^{-3t}\right)$$

$$y(t) = -e^{-t} + e^{-2t} - \frac{1}{3}e^{-3t} + \frac{1}{3}$$