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ENGINEERING MATHEMATICS

Text Book : Theory with worked out Examples and Practice Questions

Engineering Mathematics

(Solutions for Text Book Practice Questions)

01. Linear Algebra

01. Ans: 3

Sol: If rank of A is 1, then A has only one independent row.

The elements in R_1 and R_2 are proportional

$$\Rightarrow \frac{3}{P} = \frac{P}{3} = \frac{P}{P}$$

$$\Rightarrow P = 3$$

02. Ans: 0

Sol: Here A and B are symmetric matrices.

$\Rightarrow$ (AB – BA) is a skew symmetric matrix of order (3 × 3)

$\Rightarrow |AB - BA| = 0$

03. Ans: (c)

Sol: Now, $B = A^{-1} = \dfrac{adj\,(A)}{|A|}$

$$\Rightarrow B = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\therefore$ The element in the 2^{nd} row and 3^{rd} column of B is given by

$$\frac{1}{|A|}A_{32} = \frac{1}{|A|}(-1)^{3+2} M_{32}$$

$$= \frac{1}{2}(-1)(1-0) = \frac{-1}{2}$$

04. Ans: (a)

Sol: Here, A^n is a zero matrix.

$\therefore$ rank of $A^n = 0$

05. Ans: 46

Sol: Here, $|adj\ A| = |A|^2$ ($\because$ $|adj(A_{n\times n})| = |A|^{n-1}$)

$\Rightarrow 2116 = |A|^2$

$\Rightarrow |A| = \pm 46$

$\therefore$ Absolute value of $|A| = 46$

06. Ans: (b)

Sol: S_1) If A and B are symmetric then AB need not be equal to BA

for example, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

then A and B are symmetric but AB is not equal to BA.

$\therefore$ S_1 is false.

S_2) If A and B are symmetric then AB – BA is a skew-symmetric matrix of order 3.

$\therefore$ |AB – BA| = 0 ($\because$ determinant of a skew-symmetric matrix of odd order is 0)

Hence, S_2 is true.

07. Ans: (a)

Sol: Each element of the matrix in the principal diagonal and above the diagonal, we can chosen in q ways.

Number of elements in the principal diagonal = n

Number of elements above the principal diagonal = $n\left(\frac{n-1}{2}\right)$

By product rule,

number of ways we can choose these elements = $q^n . q^{n\left(\frac{n-1}{2}\right)}$

Required number of symmetric matrices

$= q^{n\left(\frac{n+1}{2}\right)}$

08. Ans: (b)

Sol: $A = \begin{bmatrix} n-1 & -1 & ..... & -1 \\ -1 & n-1 & ..... & -1 \\ .... & ..... & .... & .... \\ -1 & -1 & ..... & n-1 \end{bmatrix}$

$R_1 \rightarrow R_1 + R_2 + ... + R_{n-1}$

$A = \begin{bmatrix} 1 & 1 & ..... & 1 \\ -1 & n-1 & ..... & -1 \\ .... & ..... & .... & .... \\ -1 & -1 & ..... & n-1 \end{bmatrix}$

$R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 + R_1$, ...,

$R_{n-1} \rightarrow R_{n-1} + R_1$,

$A = \begin{bmatrix} 1 & 1 & ..... & 1 \\ 0 & n & ..... & 0 \\ .... & ..... & .... & .... \\ 0 & 0 & ..... & n \end{bmatrix}$

$\therefore |A| = n^{n-2}$

09. Ans: (a)

Sol: S1 is true because, any subset of linearly independent set of vectors is always linearly independent set.

S2 is not necessarily true,

for example, $\{x_1, x_2, x_3\}$ can be linearly independent set and x_4 is linear combination of x_1, x_2 and x_3.

10. Ans: (c)

Sol: The given matrix is skew-symmetric.

Determinant of a skew symmetric matrix of odd order is 0.

$\therefore$ Rank of A < 3.

Determinant of a non-zero skew symmetric matrix is ≥ 2

$\therefore$ Rank of A = 2

11. Ans: (a)

Sol: Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}$

For the system of linear equations to have a unique solution, det(A) $\neq$ 0.

$(0 - 2\alpha) + 2(2\alpha + 2\alpha) + (4 - 0) \neq 0$

$\Rightarrow -2\alpha + 8\alpha + 4 \neq 0$

$\Rightarrow 6\alpha + 4 \neq 0$

$\Rightarrow 6\alpha \neq -4$

$\Rightarrow \alpha \neq \frac{-2}{3}$

$\therefore$ Option (A) is correct.

12. Ans: (c)

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$

Applying $R_2 - 2R_1$, $R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & -5 & -2 \end{bmatrix}$$

Applying $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

which is an echlon matrix with two non-zero rows.

$\therefore$ Rank of A = 2

If rank of A is less than number of variables, then the system AX = O has infinitely many non-zero solutions.

If rank of A is less than number of variables, then the system AX = B cannot have unique solution.

Hence, option (C) is not true.

If rank of A is less than order of A, then the matrix A is singular.

$\therefore$ A^{-1} does not exist

13. Ans: (b)

Sol: Given, $A = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$

$$\Rightarrow |A| = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix}$$

$\Rightarrow |A| = k(k^2 - 1) - (k - 1) + (1 - k)$

$\Rightarrow |A| = (k - 1)[k^2 + k - 2]$

$\Rightarrow |A| = (k - 1)^2 (k + 2)$

Thus, the system has a unique solution when

$$(k - 1)^2 (k + 2) \neq 0$$

(or) $k \neq 1$ and $k \neq -2$

14. Ans: (c)

Sol: The augmented matrix is

$$(A \mid B) = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix} \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \to 5R_3 + R_2 \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho(A \mid B) = 2$ (< number of variables).

$\therefore$ The system has infinitely many solutions.

15. Ans: (c)

Sol: Given AX = B

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & k-1 & 3 \end{array}\right]$$

$R_3 \rightarrow R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-7 & 0 \end{array}\right]$$

If $k - 7 \neq 0$ then the system will have unique solution.

If k = 7, then rank of A = rank of [A | B] = 2 (< number of variables)

$\therefore$ The system has infinitely many solutions if k = 7.

16. Ans: (a)

Sol: Given n – r = 1, where r = $\rho(A)$ and n = order of the matrix

$\Rightarrow 3 - r = 1$

$\Rightarrow r = \rho(A) = 2$ = number of non-zero rows in an echelon form

$\therefore$ To have rank 2 form matrix A, k must be either –1 or 0.

17. Ans: (a)

Sol: Here Rank of A = Rank of [A|B] = 3

$\therefore$ The given system has a unique solution.

18. Ans: (c)

Sol: The augmented matrix of the given system is

$$(AB) = \begin{pmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$R_2 - 4R_1$, $R_3 - 3R_1$, $R_4 - R_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_3 - 4R_2$, $R_4 - 3R_2$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 15 & 21 \end{pmatrix}$$

$R_4 - R_3$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore \rho(A) = \rho(AB) = 3$ = no. of variables

Hence, there exists only one solution.

19. Ans: (d)

Sol: If $A_{n\times n}$ has n distinct eigen values, then A has n linearly independent eigen vectors.

If zero is one of the eigen values of A, then A is singular and A^{-1} does not exist.

If A is singular then rank of A < 3 and A cannot have 3 linearly independent rows.

$\therefore$ Only option (d) is correct.

20. Ans: (b)

Sol: If λ_1, λ_2, λ_3 are the eigen values of a 3×3 matrix A, then

(i) $\lambda_1 + \lambda_2 + \lambda_3$ = trace of A

(ii) $\lambda_1 \lambda_2 \lambda_3 = |A|$

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 18$ .... (1)

and $\lambda_1 \lambda_2 \lambda_3 = 0$ .... (2)

Here, the equations (1) & (2) satisfy with option (b) only.

$\therefore$ Option (b) is correct.

21. Ans: (a)

Sol: Since, A is singular, $\lambda = 0$ is an eigen value.

Also, rank of A = 1.

The root $\lambda = 0$ is repeated n – 1 times.

trace of A = n = 0 + 0 + ... + λ_n.

$\Rightarrow \lambda_n = n$

$\therefore$ The distinct eigen values are 0 and n.

22. Ans: (c)

Sol: The characteristic equation is

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

By Caley Hamilton's theorem,

$A^3 - 6A^2 + 11A - 6I = 0$

Multiplying by A^{-1},

$(A^2 - 6A + 11I) = 6A^{-1}$

23. Ans: (b)

Sol: Let $A = \begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$

Consider $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - (-2)\lambda + (-120 + 72) = 0$

$\Rightarrow \lambda^2 + 2\lambda - 48 = 0$

$\therefore \lambda = 6, -8$ are eigen values of A.

For $\lambda = 6$, the eigen vectors are given by

$[A - 6I]X = O$

$$\Rightarrow \begin{bmatrix} 4 & -4 \\ 18 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

The eigen vectors are of the form

$$X_1 = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -8$, the eigen vectors are given by

$[A + 8I]X = O$

$$\Rightarrow \begin{bmatrix} 18 & -4 \\ 18 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 18x - 4y = 0$

$\Rightarrow 9x - 2y = 0$

The eigen vectors are of the form

$$X_1 = k_2\begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

24. Ans: (c)

Sol: The given matrix is upper triangular. The eigen values are same as the diagonal elements 1, 2, –1 and 0.

The smallest eigen value is $\lambda = -1$. The eigen vectors for $\lambda = -1$ is given by

$(A - \lambda I)\,X = 0$

$\Rightarrow (A + I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$\Rightarrow w = 0,\ y = 0,\ 2x - z = 0$

$\therefore\ X = k[1\ \ 0\ \ 2\ \ 0]^T$

25. Ans: (b)

Sol: Let λ be the third eigen value.

Sum of the eigen values of A = Trace (A)

$\Rightarrow (-3) + (-3) + \lambda = -2 + 1 + 0$

$\Rightarrow \lambda = 5$

The eigen vector for $\lambda = 5$ is given by

$[A - 5I]X = O$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$$\therefore \text{ The third eigen vector} = k\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

26. Ans: 7

Sol: Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}$

eigen vector $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

We know that $AX = \lambda X$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ -16 - 2x \\ 15 \end{bmatrix} = \begin{bmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{bmatrix}$$

Clearly eigen value $\lambda = 15$

$\Rightarrow -16 - 2x = -30$

$\therefore\ -2x = -14$

$x = 7$

27. Ans: 2

Sol: If λ is an Eigen values of A, then

$\lambda^4 - 3\lambda^3$ is an Eigen value of $(A^4 - 3A^3)$

Putting $\lambda = 1, -1$, and 3 in $(\lambda^4 - 3\lambda^3)$,

We get the eigen values of $(A^4 - 3A^3)$ are –2, 4, 0

Trace of $(A^4 - 3A^3)$ = Sum of eigen values of $(A^4 - 3A^3) = 2$

28. Ans: 8

Sol: Given $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

The characteristic equation is $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

By Caley-Hamilton's theorem,

$A^3 - A^2 - 4A + 4I = O$

adding 2I on both sides

$A^3 - A^2 - 4A + 6I = 2I$

Let $B = A^3 - A^2 - 4A + 6I$

Now $B = 2I$

$\therefore |B| = |2I| = 8$

29. Ans: 2

Sol: $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Clearly $\lambda = 2$

30. Ans: (d)

Sol: We have, $A^T = -A$

($\because$ A is skew-symmetric)

$\Rightarrow A + A^T = (A - A) = O$

Rank of $(A + A^T) = 0$

$\therefore$ Number of linearly independent eigen vectors = n – rank of $(A + A^T) = n$

31. Ans: (a)

Sol: For upper triangular matrix the eigen values are same as the elements in the principal diagonal.

Let $A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$

Then $(I + A) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow |I + A| = 1$

$\therefore$ I + A is non-singular and hence invertible.

32. Ans: 8

Sol: The characteristic equation of M is

$\lambda^3 - 12\lambda^2 + a\lambda - 32 = 0$ .... (1)

Substituting $\lambda = 2$ in (1), we get a = 36

Now, the characteristic equation is

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$

$\Rightarrow \lambda = 2, 2, 8$

$\therefore$ The largest among the absolute values of the eigen values of M = 8.

33. Ans: (d)

Sol: Now,

$|A| = \begin{vmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{vmatrix}$

$\Rightarrow |A| = 2(3 - 6) - 0 + 1(8 - 0)$

$\therefore |A| = 8 - 6 = 2$

$\therefore$ Option (d) not true and other options are true

34. Ans: (d)

Sol: Applying $C_4 - 3C_1$ the determinant becomes

$$= \begin{bmatrix} -1 & 2 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & -3 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

Expanding the determinant by 2nd row we will get the value 0.

$\therefore$ Option (a) is correct

By deleting 1st row 1st coloumn of A, we get a 3rd order non zero minor.

$\therefore$ The rank of A is 3.

If |A| = 0 then the system AX = 0 has infinitely many non zero solutions

$\therefore$ Option (c) is correct.

If Rank of A is 3, then the system AX = B cannot have unique solution.

35. Ans: (a)

Sol: Given $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & -5 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$\Rightarrow \rho(A) = 3$

= number of linearly independent rows

$\therefore$ The set of vectors is linear set and it forms a basis of R^3

36. k ≠ 0

Sol: If the given vectors form a basis, then they are linearly independent

$$\Rightarrow \begin{vmatrix} k & 1 & 1 \\ 0 & 1 & 1 \\ k & 0 & k \end{vmatrix} \neq 0$$

$\Rightarrow k^2 + k - k \neq 0$

$\therefore k \neq 0$

02. Calculus

01. Ans: (a)

Sol: $\lim\limits_{x\to 5/4}(x - [x]) = \lim\limits_{x\to 5/4} x - \lim\limits_{x\to 5/4}[x]$

$$= \frac{5}{4} - 1 = \frac{1}{4}$$

02. Ans: (d)

Sol: $\lim\limits_{x\to 2} \frac{|x-2|}{x-2}$

Left Limit = $\lim\limits_{x\to 2-} \frac{-(x-2)}{x-2} = -1$

Right Limit = $\lim\limits_{x\to 2+} \frac{x-2}{x-2} = 1$

$\therefore$ Left Limit ≠ Right Limit

$\Rightarrow$ Limit does not exist

03. Ans: (d)

Sol: $\lim\limits_{x\to 4}[x]$

Left Limit = 3, Right Limit = 4

$\Rightarrow$ Left Limit ≠ Right Limit

$\Rightarrow$ Limit does not exist

04. Ans: 2

Sol: $$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\lim_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)}{\left(\frac{1-\cos x}{x^2}\right)}=\frac{1}{\frac{1}{2}}=2$$

05. Ans: 0

Sol:

$$\lim_{x\to\pi/2}[\text{Tan}x-\text{Sec}x]=\lim_{x\to\pi/2}\left[\frac{\text{Sin}x-1}{\text{Cos}x}\right]\left(\frac{0}{0}\text{for m}\right)$$

Using L'Hospital Rule,

$$=\lim_{x\to\pi/2}\left[\frac{\text{Cos}x}{-\text{Sin}x}\right]=\frac{0}{-1}=0$$

06. Ans: (c)

Sol: $\lim_{x\to\infty}[1+x^2]^{e^{-x}}$ ($\alpha^{\circ}$for m)

Let $y=[1+x^2]^{e^{-x}}$

$\Rightarrow \log y = e^{-x}\log(1+x^2)$

$$\Rightarrow \lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\log(1+x^2)}{e^x}\left[\frac{\infty}{\infty}\text{for m}\right]$$

Using L'Hospital Rule

$$\Rightarrow \log\left(\lim_{x\to\infty}(1+x^2)^{e^{-x}}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+x^2}(2x)}{e^x}\left[\frac{\infty}{\infty}\text{for m}\right]$$

$$=\lim_{x\to\infty}\frac{2}{e^x(2x)+e^x(1+x^2)}$$

$$=\frac{2}{\infty}=0$$

$$\Rightarrow \lim_{x\to\infty}(1+x^2)^{e^{-x}}=e^0=1$$

07. Ans: (b)

Sol: $y=\underset{x\to 0}{\text{Lt}}\ x^x$

$$\log y=\underset{x\to 0}{\text{Lt}}\ \log x^x$$

$$=\underset{x\to 0}{\text{Lt}}\ x\log x$$

$$=\underset{x\to 0}{\text{Lt}}\left[\frac{\log x}{\left(\frac{1}{x}\right)}\right]$$

$$=\underset{x\to 0}{\text{Lt}}\left[\frac{\frac{1}{x}}{\left(\frac{-1}{x^2}\right)}\right]\text{ By L'hospital rule}$$

$$=\underset{x\to 0}{\text{Lt}}(-x)=0$$

$$y=e^o=1$$

08. Ans: (a)

Sol:

$$\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}=\lim_{x\to 0}\frac{(a+x)-(a-x)}{x\left[\sqrt{a+x}+\sqrt{a-x}\right]}$$

$$=\lim_{x\to 0}\frac{2x}{x\left[\sqrt{a+x}+\sqrt{a-x}\right]}=\frac{2}{\sqrt{a}+\sqrt{a}}=\frac{1}{\sqrt{a}}$$

09. Ans: (a)

Sol: (a) f(2) = 3,

RL = 2(2) – 1= 3, $LL=\frac{3+7}{3}=3$

$\Rightarrow$ f(x) is continuous at x = 2

(b) f(2) = 2, $\lim_{x\to 2}f(x)=8-2=6\neq f(2)$

$\Rightarrow$ f(x) is discontinuous at x = 2

(c) f(2) = 2 + 2 = 4, LL = 2 + 2 = 4,

RL = 2 – 4 = –2

⇒ LL ≠ RL ⇒ Limit does not exist

⇒ f(x) is discontinuous at x = 2

(d) f(2) is not defined

⇒ f(x) is discontinuous at x = 2

10. Ans: (a)

Sol: Given $(f \circ g)$ (x) = f [g(x)]

In (–∞, 0), g(x) = –x

⇒ f[g(x)] = f(–x)

⇒ f[g(x)] = x^2

∴ f[g(x)] has no points of discontinuities in (–∞, 0).

11. Ans: (a)

Sol: We have $\underset{x\to 1-}{Lt} f(x) = \underset{x\to 1+}{Lt} f(x) = f(1) = 1$

∴ f(x) is continuous at x = 1

Now, $f'(1^-) = \underset{x\to 1^-}{Lt} \dfrac{f(x)-f(1)}{x-1}$

$= \underset{x\to 1}{Lt} \dfrac{x-1}{x-1} = 1$

and $f'(1^+) = \underset{h\to 1^+}{Lt} \dfrac{f(x)-f(1)}{x-1}$

$= \underset{x\to 1}{Lt} \dfrac{(2x-1)-1}{x-1} = 2$

Here, $f'(1^-) = 1 \neq f'(1^+) = 2$

∴ f(x) is not differentiable at x = 1.

12. Ans: (a)

Sol: Since, f is differentiable at x = 2,

$f'(2^-) = f'(2+)$

⇒ $(2x)_{x=2} = m$

∴ m = 4

Since, f is continuous at x = 2

i.e., $(x^2)_{x=2} = (mx + b)_{x=2}$

⇒ 4 = 2m + b

∴ b = – 4

Hence, option (A) is correct.

13. Ans: (c)

Sol: (A) f(x) = |x| is not differentiable at x = 0

(B) f(x) = cot x is neither continuous nor differentiable at x = 0

(C) f(x) = sec x is differentiable in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and hence in the interval [–1, 1]

(D) f(x) = cosec x is neither continuous nor differentiable at x = 0

∴ Option (c) is correct.

14. And: (b)

Sol: Let f ′(x) = sin (x) + 2.sin (2x) + 3.

$\sin(3x) - \dfrac{8}{\pi} = 0$ be the given equation.

Then,

$f(x) = -\cos(x) - \cos(2x) - \cos(3x) - \dfrac{8}{\pi}(x) + k$

Here, if the function f(x) satisfies the all the three conditions of the Rolle's theorem in [a, b], then the equation f ′(x) = 0 has at least one real root in (a, b).

As cos(ax) is continuous & differentiable function and $a_0 + a_1x$ is continuous & differentiable function for all x, the function f(x) is continuous and differentiable for all x.

Here, (i) f(x) is continuous on $\left[0, \dfrac{\pi}{2}\right]$

(ii) f(x) is differentiable on $\left(0, \frac{\pi}{2}\right)$

(iii) $f(0) = -3 + k = f\left(\frac{\pi}{2}\right)$

$\therefore$ By a Rolle's theorem, the given equation has at least one root in $\left(0, \frac{\pi}{2}\right)$.

Hence, option (B) is correct.

15. Ans: (c)

Sol: By Lagrange's theorem, we have

$$f'(C) = \frac{f(8)-f(1)}{8-1}$$

$$\Rightarrow \quad 1-\frac{4}{C^2}=\frac{8.5-5}{7} \quad \left(\because f'(x) = 1-\frac{4}{x^2}\right)$$

$$\Rightarrow \quad C = \pm 2\sqrt{2} \qquad (\because C = -2\sqrt{2} \notin (1,8))$$

$$\therefore C = 2\sqrt{2} \in (1, 8)$$

16. Ans: (b)

Sol: Let f(x) be defined on

$[a, b] = [0, 2] \ni f^1(x) = \frac{1}{1+x^2} \forall x$.

Then by Lagrange's mean value theorem,

$\exists\, c \in (0,2) \ni f^1(c) = \frac{f(2)-f(0)}{2-0}$

$$\Rightarrow \frac{1}{1+c^2} = \frac{f(2)}{2}$$

$\because c \in (0, 2)$

$\Rightarrow 0 < c < 2$

$\Rightarrow 0^2 < c^2 < 2^2$

$\Rightarrow 1 + 0^2 < 1 + c^2 < 1 + 2^2$

$$\Rightarrow \frac{1}{1+0^2} > \frac{1}{1+c^2} > \frac{1}{1+2^2}$$

$$\Rightarrow \frac{1}{1+2^2} < \frac{1}{1+c^2} < \frac{1}{1+0^2}$$

$$\Rightarrow \frac{1}{5} < \frac{f(2)}{2} < \frac{1}{1}$$

$$\therefore \frac{2}{5} < f(2) < 2 \quad \text{(or)} \quad f(2) \in (0.4, 2)$$

17. Ans: 2.5 range 2.49 to 2.51

Sol: By Cauchy's mean value theorem,

$$\frac{f'(c)}{g'(c)} = \frac{f(3)-f(2)}{g(3)-g(2)}$$

$$\Rightarrow -e^{2c} = \frac{e^3-e^2}{e^{-3}-e^{-2}}$$

$$\therefore \quad c = 2.5 \in (2, 3)$$

18. Ans: (a)

Sol: $f(x) = e^{Sinx} \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^{Sinx}. Cosx \Rightarrow f'(0) = 1$

$f''(x) = e^{Sinx}. Cos^2x + e^{Sinx}(-Sinx) \Rightarrow f''(0) = 1 - 0 = 1$

Taylor's Series for f(x) about x = 0 is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ...$$

$$= 1 + x + \frac{x^2}{2!} + ...$$

19. Ans: (a)

Sol: Coefficient of $x^4 = \frac{f^{IV}(0)}{4!}$

Given f(x) = log (Secx)

$$\Rightarrow f'(x) = \frac{1}{Secx} Secx\, Tanx = Tanx$$

$\Rightarrow f''(x) = \text{Sec}^2x$

$\Rightarrow f'''(x) = 2\text{Sec}^2x\,\text{Tan}x$

$\Rightarrow f^{iv}(x) = 2[\text{Sec}^2x\,\text{Sec}^2x + \text{Tan}x.\,2\text{Sec}x.\,\text{Sec}x\,\text{Tan}x]$

$\Rightarrow f^{iv}(0) = 2$

$\therefore$ Coefficient of $x^4 = \dfrac{f^{iv}(0)}{4!} = \dfrac{2}{24} = \dfrac{1}{12}$

20. Ans: (a)

Sol: Given $f(x) = \text{Tan}^{-1}x \Rightarrow f(0) = 0$

$\Rightarrow f'(x) = \dfrac{1}{1+x^2} \Rightarrow f'(0) = 1$

$\Rightarrow f''(x) = \dfrac{-1}{1+x^2}(2x) \Rightarrow f''(0) = 0$

$\Rightarrow f'''(0) = -2$

$\Rightarrow f'''(x) = -2\left[\dfrac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}\right]$

Taylor's Series of f(x) about x = 0 is,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...\infty$$

$$= 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + ...$$

$$= x - \frac{x^3}{3} = ...\infty$$

21. Ans: (c)

Sol: $e^{x+x^2} = 1 + \dfrac{(x+x^2)}{1!} + \dfrac{(x+x^2)^2}{2!} + \dfrac{(x+x^2)^3}{3!} + ...$

$$\left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right)$$

$\therefore\ e^{x+x^2} = 1 + x + \dfrac{3x^2}{2} + \dfrac{7x^3}{6}$

22. Ans: (a)

Sol: Given $u = \text{Sin}\left(\dfrac{x^2+y^2}{x+y}\right) \Rightarrow \text{Sin}u = \dfrac{x^2+y^2}{x+y}$

$\Rightarrow$ f(u) = Sinu is homogeneous with deg, n = 1

By Eulen's theorem

$$x.u_x + y.u_y = n\frac{f(u)}{f'(u)} = 1 - \frac{\text{Sin}u}{\text{Cos}u} = \text{Tan } u$$

23. Ans: (a)

Sol: Given $u = x^{-2}\text{Tan}\left(\dfrac{y}{x}\right) + 3y^3\text{Sin}^{-1}\left(\dfrac{x}{y}\right)$

= f(x, y) + 3 g(x,y)

Where f(x, y) is homogeneous with deg m = –2

and g(x, y) is homogeneous with deg n = 3

$\Rightarrow x^2.\ U_{xx} + 2xy\ U_{xy} + y^2\ U_{yy} = m(m-1) f(x,y) + 3\,n(n-1)\,g(x,y)$

$= -2(-2-1)\,f(x,y) + 3[3(3-1)g(x,y)]$

$= 6\,[f(x,y) + 3\,g(x,y)]$

$= 6u$

24. Ans: (a)

Sol: $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dt}$

$= (3x^2 + z^2 + yz)\,e^t + (3y^2 + xz)\,(-\text{Sin}t) + (2xz + xy)3t^2$

At t = 0,

$$\frac{du}{dt} = (3(1) + 0 + 0)(1) + [3(1) + 0](0) + [0 + 1](0)$$

$= 3$

25. Ans: (c)

Sol: Given $x^y + y^x = -C$

Let $f(x,y) = x^y + y^x$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$$

At (1,1), $\frac{dy}{dx} = -\left[\frac{1+0}{0+1}\right] = -1$

26. Ans: (a)

Sol: $u = x \log(xy)$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \left[x.\frac{1}{xy}(y) + \log(xy)\right](1) + \left[x.\frac{1}{xy}.(x).\frac{dy}{dx}\right]$$

Given

$$\underbrace{x^3 + y^3 + 3xy}_{f(x,y)} = 1 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\left[\frac{3x^2 + 3y}{3y^2 + 3x}\right]$$

$$\therefore \frac{du}{dx} = [1 + \log xy] - \frac{x}{y}\left[\frac{x^2 + y}{y^2 + x}\right]$$

27. Ans: (b)

Sol:
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 - \frac{y^2}{x} & \frac{2y}{x} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

$$= \frac{2y}{x}\left[1 - \frac{y^2}{x^2} - \left(-\frac{y^2}{x^2}\right)\right]$$

$$= \frac{2y}{x}$$

28. Ans: (c)

Sol:
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 3(1 - 2) - 2(-1 - 1) - 1(2 + 1)$

$= -2$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{-1}{2}$$

29. Ans: (b)

Sol: $f(x) = \frac{x^3}{3} - x \Rightarrow f'(x) = \frac{3x^2}{3} - 1 = x^2 - 1 = 0$

$\Rightarrow x \pm 1$ are stationary points

$f''(x) = 2x$

$f''(1) = 2 > 0 \Rightarrow$ minimum at x = 1

$f''(-1) = -2 < 0 \Rightarrow$ maximum at x = –1

30. Ans: 5

Sol: $f(x) = x^3 - 6x^2 + 9x + 1$

$\Rightarrow f'(x) = 3x^2 - 12x + 9 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow x = 1,3$

The extreme value of f(x) in [a,b] may lie either at the stationary points or at the end points of the interval.

$\therefore$ f(0) = 1, f(2) = 3, f(1) = 5, f(3) = 1

$\therefore$ Maximum value = f(1) = 5

31. Ans: (c)

Sol: Given $f(x) = (k^2 - 4)x^2 + 6x^3 + 8x^4$

$\Rightarrow f'(x) = 32x^3 + 18x^2 + 2(k^2-4)x$

and $f''(x) = 96x^2 + 36x + 2(k^2-4)$

f(x) has local maxima at x = 0

$\Rightarrow f''(0) < 0$

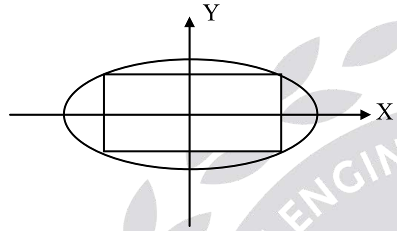
$\Rightarrow 2(k^2 - 4) < 0$

$\Rightarrow k^2 - 4 < 0$ (or) $(k-2)(k+2) < 0$

$\therefore -2 < k < 2$

32. Ans: 1

Sol: Let 2x & 2y be the length & breadth of the rectangle.



Let A = 2x×2y = 4xy be the area of the rectangle.

Then $A^2 = 4x^2y^2 = x^2(1-x^2) = x^2 - x^4$

Let $f(x) = x^2 - x^4$

Then $f'(x) = 2x - 4x^3$ and $f''(x) = 2 - 12x^2$

For maximum, we have

$$f'(x) = 0$$

$$\Rightarrow 2x(1-2x^2) = 0$$

$$\Rightarrow x = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

Here $f''(0) > 0$, $f''\left(\frac{1}{\sqrt{2}}\right) < 0$

$$\therefore \text{Area } A = 4xy = 4x \times \frac{\sqrt{1-x^2}}{2}$$

$$= 2x\sqrt{1-x^2}$$

$$= 2 \times \frac{1}{\sqrt{2}} \times \sqrt{1-\frac{1}{2}} = 1$$

33. Ans: 112

Sol: Let $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Then $p = f_x = 3x^2 + 3y^2 - 30x + 72$, $q = f_y = 6xy - 30y$

and $r = f_{xx} = 6x - 30$, $s = f_{xy} = 6y$, $t = f_{yy} = 6x - 30$

Consider p = 0 and q = 0

$\Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$ and $6xy - 30y = 0$

$\Rightarrow x^2 + y^2 - 10x + 24 = 0$ and $y(x-5) = 0$

$\Rightarrow x^2 + y^2 - 10x + 24 = 0$ and x = 5, y = 0

If y = 0 then $x^2 + y^2 - 10x + 24 = 0$ becomes $x^2 - 10x + 24 = 0$

$\Rightarrow x = 4, 6$

If x = 5 then $x^2 + y^2 - 10x + 24 = 0$ becomes $25 + y^2 - 50 + 24 = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow y = 1, -1$

$\therefore$ The stationary points are (5, 1), (5, –1), (4, 0), (6, 0)

At (x, y) = (5, 1); r = 0, s = 6, t = 0

$\Rightarrow rt - s^2 = 0 - 36 = -36 < 0$

$\therefore$ (5, 1) is a saddle point

At (x, y) = (5, –1); r = 0, s = –6, t = 0

$\Rightarrow rt - s^2 = 0 - 36 = 36 < 0$

$\therefore$ (5, –1) is a saddle point

At (4, 0); r = – 6, s = 0, t = – 6

$\Rightarrow rt - s^2 = 36 - 0 = 36 > 0$ and r < 0

$\therefore$ (4, 0) is a point of maxima

At (6, 0); r = 6, s = 0, t = 6

$\Rightarrow rt - s^2 = 36 - 0 = 36 > 0$ and r > 0

$\therefore$ (6, 0) is a point of minima

Hence, the maximum value of f(x, y) at (4, 0) is

$f(4, 0) = (4)^3 + (0) - 15(4)^2 - (0) + (72)(4)$
$= 112$

34. Ans: 25

Sol: Let $A = \begin{pmatrix} x & y \\ y & 10-x \end{pmatrix}$

Det A = $x(10-x) - y^2$

For maximum value of Det A, y = 0

Now, $A = \begin{pmatrix} x & 0 \\ 0 & 10-x \end{pmatrix}$

$\Rightarrow |A| = x(10-x) = 10x - x^2$

Let $f(x) = 10x - x^2$

$\Rightarrow f'(x) = 10 - 2x$

$\Rightarrow f''(x) = -2$

Consider, $f'(x) = 0$

$\Rightarrow x = 5$

At x = 5, $f''(x) = -2 < 0$

$\therefore$ At x = 5, the function f(x) has a maximum and is equal to 25.

35. Ans: (c)

Sol: $\int_{-4}^{y} |x| dx + \int_{-4}^{0} -x\, dx + \int_{0}^{y} x\, dx$

$$= -\left.\frac{x^2}{2}\right]_{-4}^{0} + \left.\frac{x^2}{2}\right]_{0}^{7}$$

$$= 0 - \left[-\frac{16}{2}\right] + \left[\frac{49}{2} - 0\right]$$

$= 8 + 24.5 = 32.5$

36. Ans: (d)

Sol:

$$\int_0^{1.5} x[x^2]\, dx = \int_0^1 x[x^2]dx + \int_1^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx$$

$$= 0 + \int_0^{\sqrt{2}} x\, dx + \int_{\sqrt{2}}^{1.5} 2x\, dx = \frac{3}{4}$$

37. Ans: (d)

Sol: $\int_0^{\pi} x \underbrace{\sin^8 x \cos^6 x}_{f(x)} dx$

$$\left[\because \int_0^a x f(x) dx = \frac{a}{2}\int_0^a f(x) dx \text{ if } f(a-x) = f(x)\right]$$

$$= \frac{\pi}{2}\int_0^{\pi} \sin^8 x \cos^6 x\, dx$$

$$= \frac{\pi}{2} \times 2 \times \int_0^{\pi/2} \sin^8 x \cos^6 x\, dx$$

$$= \pi\left[\frac{(7.5.3.1)(5.3.1)}{14.12.10.8.6.4.2}\right]\frac{\pi}{2} = \frac{5\pi^2}{4096}$$

38. Ans: (a)

Sol: Given that, $x \sin(\pi x) = \int_0^{x^2} f(t) dt$

Differentiating both sides, we get

$x \cos(\pi x).\pi + \sin(\pi x) = f(x).2x$

Putting x = 4

$4\pi \cos(4\pi) = f(4).8$

$\therefore f(4) = \frac{\pi}{2}$

39. Ans: (b)

Sol: $$\lim_{x\to 0}\left[\frac{\int_0^{x^2}\text{Sin}\sqrt{x}\,dx}{x^3}\right]\left(\frac{0}{0}\text{for m}\right)$$

Using L'Hospital Rule,

$$= \lim_{x\to 0}\frac{(\text{Sin}x)2x-(\text{Sin}0)(0)}{3x^2}\left(\frac{0}{0}\text{for m}\right)$$

$$= \lim_{x\to 0}\frac{2\text{Cos}x}{3}=\frac{2}{3}$$

40. Ans: 0.785 range 0.78 to 0.79

Sol: $$\int_0^{\frac{\pi}{4}}\frac{\sin 2x}{\cos^4 x+\sin^4 x}dx$$

$$=2\int_0^{\frac{\pi}{4}}\frac{\tan x}{\cos^2 x(1+\tan^4 x)}dx$$

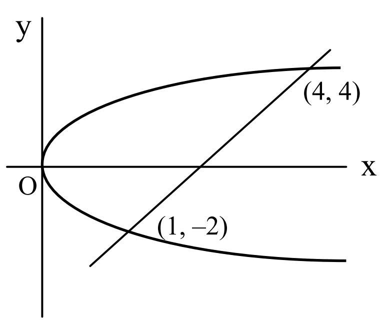
$$=\int_0^1\frac{2t}{1+t^4}dt \quad \text{(by putting tan x = t)}$$

$$=\frac{\pi}{4}=0.785$$

41. Ans: 9

Sol: The required area

$$=\int x\,dy=\int_{-2}^{4}\left(\frac{1}{2}(y+4)-\frac{1}{4}y^2\right)dy=9$$



42. Ans: (d)

Sol: $$\int_{-\infty}^{0}e^{x+e^x}dx=\int_{-\infty}^{0}e^x.e^{e^x}dx \quad \text{Put } e^x=t\Rightarrow e^x\,dx=0$$

$$=\int_0^1 e^t dt=e^t\Big]_0^1=e-1$$

43. Ans: (c)

Sol: $$\int_{-\infty}^{\infty}\frac{dx}{(1+a^2+x^2)^{3/2}}=\int_0^{\infty}\frac{dx}{(K^2+x^2)^{3/2}}$$

where $K^2 = 1+a^2$

Put $x = K\,\text{Tan}\theta \Rightarrow dx = K\,\text{Sec}^2\theta d\theta$

$$2\int_0^{\pi/2}\frac{K\,\text{Sec}^2\theta}{(K^2+K^2\text{Tan}^2\theta)^{3/2}}d\theta=\frac{2}{K^2}.\int_0^{\pi/2}\cos\theta d\theta=\frac{2}{1+a^2}$$

44. Ans: (a)

Sol: $$\int_0^{\infty}\frac{1}{(x^2+4)(x^2+9)}dx=K\pi$$

$$\Rightarrow\int_0^{\infty}\frac{1}{5}\left[\frac{1}{x^2+4}-\frac{1}{x^2+9}\right]dx=K\pi$$

$$\Rightarrow\frac{1}{5}\left[\frac{1}{2}\text{Tan}^{-1}\left(\frac{x}{2}\right)-\frac{1}{3}\text{Tan}^{-1}\left(\frac{x}{3}\right)\right]_0^{\infty}=K\pi$$

$$\Rightarrow\frac{1}{5}\left[\frac{1}{2}.\frac{\pi}{2}-\frac{1}{3}.\frac{\pi}{2}\right]=K\pi$$

$$\Rightarrow\frac{1}{10}\left[\frac{1}{6}\right]=K\Rightarrow K=\frac{1}{60}$$

45. Ans: (c)

Sol: $$\int_0^1 x\log x\,dx=\log x.\frac{x^2}{2}-\int\frac{1}{x}.\frac{x^2}{2}dx$$

$$=\frac{x^2}{2}\log x-\frac{x^2}{4}\Big]_0^1=-\frac{1}{4}$$

46. Ans: (a)

Sol: Given $I = \int_1^3 \frac{\sqrt{1+x^2}}{(x-1)^2} dx$

Let $f(x) = \frac{\sqrt{1+x^2}}{(x-1)^2}$, $f(x) \to \infty$ as $x \to 1$

Let $g(x) = \frac{1}{(x-1)^2}$

$$\underset{x\to 1}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 1}{Lt} \frac{\sqrt{1+x}}{(x-1)^2} \times (x-1)^2 = \sqrt{2}$$

But $\int_1^3 \frac{1}{(x-1)^2}$ is known to be divergent.

$\therefore$ By comparison test, the given integral also divergent.

47. Ans: (a)

Sol: Given $I = \int_1^2 \frac{x^3+1}{\sqrt{2-x}} dx$

Let $f(x) = \frac{x^3+1}{\sqrt{2-x}}$

$f(x) \to \infty$ an $x \to 2$

Let $g(x) = \frac{1}{\sqrt{2-x}}$

$$\underset{x\to 2}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 2}{Lt} \left(\frac{x^3+1}{\sqrt{2-x}} \times \sqrt{2-x} \right) = 9 \text{ finite}$$

But $\int_1^2 g(x)\, dx$ is known to be convergent

$\therefore$ By comparison test, the given integral also convergent.

48. Ans: (d)

Sol: Given $I = \int_1^\infty \frac{e^{-x}}{x^2} dx$

Let $f(x) = \frac{e^{-x}}{x^2}$

Choose $g(x) = \frac{1}{x^2}$

$\int_1^\infty g(x)\, dx = \int_1^\infty \frac{1}{x^2} dx = -1$ is known to be convergent.

$\therefore$ By comparison test, the given integral also convergent.

49. Ans: (c)

Sol: Let $f(a) = \int_0^\infty e^{-x} \frac{\sin ax}{x} dx$

Differentiating partially w.r.t. a

$$f'(a) = \int_0^\infty e^{-x} \frac{\sin ax}{x}\, x\, dx$$

$$= \int_0^\infty e^{-x} \sin ax\, dx = \frac{1}{a^2+1}$$

Integrating both sides

$f(a) = \tan^{-1} a + c$

$f(0) = 0$

$\Rightarrow c = 0$

$\therefore f(a) = \tan^{-1} a$

50. Ans: (a)

Sol: $\int_{x=1}^{a} \int_{y=1}^{b} \frac{1}{xy} dx\, dy$

$$= \int_{x=1}^{a} \frac{1}{x} dx \int_{y=1}^{b} \frac{1}{y} dy$$

$= [\log x]_1^a \; [\log y]_1^b$

$= (\log a - \log 1)(\log b - \log 1)$

$= \log a \log b$

51. Ans: (b)

Sol: $\int_0^1 \int_0^{x^2} x \; dx\, dy$

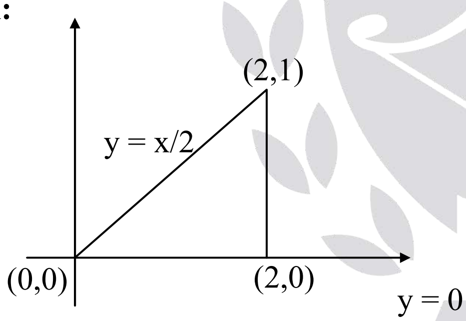
$= \int_0^1 x [y]_0^{x^2} dx$

$= \int_0^1 x^3 dx$

$= \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

52. Ans: (c)

Sol:



$\iint_R xy^2 \, dR$

$\iint_R xy^2 \, dx\, dy$

$= \int_0^2 \int_0^{x/2} x\, y^2 \, dy\, dx$

$= \int_0^1 x\left[\frac{y^3}{3}\right]_0^{x/2} dx = \frac{1}{24}\int_0^2 x^4 \; dx = \frac{32}{24\times 5} = \frac{4}{15}$

53. Ans: (b)

Sol: $\iint_R \frac{x^2y^2}{x^2+y^2} dxdy$

R– $x^2 + y^2 = 1$ & $y \geq 0$

Changing into polar coordinates

$x = r\cos\theta,\ y = r\sin\theta$

$dx.dy = r\ dr\ d\theta$

$\iint_R \frac{x^2y^2}{x^2+y^2} dxdy$

$= \int_0^1 \int_0^{\pi} \frac{r^2 \cos\theta \; r^2 \; \sin^2\theta}{r^2} rd\,rd\,\theta$

$= \int_0^1 \int_0^{\pi} r^3 dr \; d\theta \cos^2 \theta \, \text{siom}^2 \theta \; d\theta$

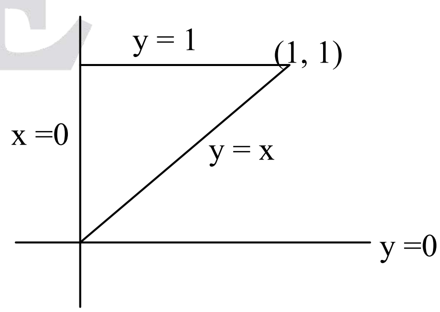
$= \int_0^1 r^3 dr \int_0^{\pi} \cos^2\theta \sin^2\theta \, d\theta$

$= \left[\frac{r^4}{4}\right]_0^1 2\left[\frac{1\times 1}{4\times 2}\times\frac{\pi}{2}\right]$

$= \frac{1}{4}\times\frac{1}{4}\times\frac{\pi}{2} = \frac{\pi}{32}$

54. Ans: (d)

Sol: $\int_0^1 \int_x^1 e^{y^2} dy\, dx$



By changing the order of integration we have

$$\int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{y} e^{y^2}\,dy\,dx$$

$$= \int_0^1 e^{y^2}[x]_0^y\,dy$$

$$= \int_0^1 e^{y^2}\,y\,dy$$

$$= \frac{1}{2}\int_0^1 e^{y^2}(2y\,dy)$$

$$= \frac{1}{2}\left[e^{y^2}\right]_0^1$$

$$= \frac{1}{2}\left[e^1 - e^0\right]$$

$$= \frac{1}{2}[e-1]$$

55. Ans: (a)

Sol: $\int_0^1\int_0^y\int_0^{1+x+y} y\,dz\,dx\,dy$

$$= \int_0^1\int_0^y y(1+x+y)\,dx\,dy$$

$$= \int_0^1\int_0^y (h + y^2 + xy)\,dx\,dy$$

$$= \int_0^1 (y+y^2)[x]_0^y + y\left[\frac{x^2}{2}\right]_0^y dy$$

$$= \int_0^1 \left[(y+y^2)y + \frac{y^2}{2}\right]dy$$

$$= \int_0^1 (2y^2 + 3y^3)\,dy$$

$$= \int_0^1 \left[y^2 + \frac{3}{2}y^3\right]dy$$

$$= \left[\frac{y^3}{3} + \frac{3y^4}{8}\right]_0^1$$

$$= \left[\frac{1}{3} + \frac{3}{8}\right]_0^1 = \frac{17}{24}$$

56. Ans: (c)

Sol: $V = \iint_R Z\,dy\,dx$

$$= \int_0^1\int_{-x}^{x} z\,dy\,dx$$

$$= \int_0^1\int_{-x}^{x} (3 + x^2 - 2y)\,dy\,dx$$

$$= \int_0^1 \left[(3+x^2)y - y^2\right]_{-x}^{x} dx$$

$$= \int_0^1 \left[(3+x^2)(2x) - 0\right]dx$$

$$= \int_0^1 (6x + 2x^3)\,dx$$

$$= \left[\frac{6x^2}{2} + \frac{2x^4}{4}\right]_0^1$$

$$V = 3 + \frac{1}{2} = \frac{7}{2}$$

57. Ans: (d)

Sol: Length $= \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^3 \sqrt{1+x}\,dx$$

$$= \frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^3 = \frac{14}{3}$$

58. Ans: 25.12

Sol: Volume = $\int_0^4 \pi y^2 \; dx$

$= \int_0^4 \pi x \; dx$

= 8π cubic units

59. Ans: 1.88

Sol: Volume = $\int_0^1 \pi \; x^2 \; dy$

$= \pi \int_0^1 y^{\frac{2}{3}} \; dy \approx 1.88$

60. Ans: (a)

Sol: $x^2 + y^2 + z^2 = 9$

let $\phi = x^2 + y^2 + z^2 = 9$

$\nabla\phi = 2xi + 2yj + 2zk$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi + xyj + 2zk}{2\sqrt{x^2 + y^2 + z^2}}$$

Required unit-normal = $\frac{xi + yj + zk}{3}$

61. Ans: (b)

Sol: $\phi = x^2yz + 4xz^2$

$\vec{a} = 2i - j - 2k$

$\nabla\phi = i[2xyz + 4z^2] + j[x^2z] + k[x^2y + 82x]$

$[\nabla\phi]_{(1-2-1)} = i[4 + 4] + j[-1] + k[-10]$

$= 8i - j + 10\,k$

Required directional derivative

$$= (8i - j + 10k)\frac{(2i - j - 2k)}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$= \frac{16 + 1 + 20}{\sqrt{9}} = \frac{37}{3}$$

62. Ans: (c)

Sol: $\phi(xy) = e^{xy} \sin(x + y)$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{dz}$$

$\nabla\phi$ = i[e^{xy} (y) sin (1+y) + e^{xy}(b) (x + y) + j[e^{xy}(x) sin (x + y) + e^{xy}(x) (x+y)]+k[0]]

$(\nabla\theta)_{\left(0,\, \frac{\pi}{2}\right)}$

$$= i\left[e^0\frac{\pi}{2}\sin\frac{\pi}{2} + e^0 cs\frac{\pi}{2} + j\left[e^0 0 \sin\frac{\pi}{2} + e^o\cos\pi\right]\right]$$

$$\nabla\phi = i\left[\frac{\pi}{2}\right] + j[0] + k[0]$$

Required direction = $\nabla\phi = \frac{\pi}{2}i$

63. Ans: (b)

Sol:

$$\nabla \times \bar{v} = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 + 6y + 0z & 4x + 2(y + z) & 2bx - 3y + 2z \end{bmatrix} = \bar{0}$$

$= i[2c - 2] - j[4 - a] + k[2b - 6] = \bar{0}$

$\Rightarrow c = 1, a = 4, b = 3$

$\Rightarrow a = 4; b = 3; c = 1$

64. Ans: (b)

Sol: $\bar{V} = e^x i + 2yj - k$

div $\bar{V} = e^x + 2 - 0 = e^x + 2 \neq 0$

and $\nabla \times \bar{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = \bar{0}$

65. Ans: (a)

Sol: $\nabla[f(r)] = f'(r)\frac{\vec{r}}{r}$

$\nabla(\sin r) = (\cos r)\frac{\vec{r}}{r}$

66. Ans: (c)

Sol: div $\left[e^r.\vec{r}\right]$

$\nabla.(\phi\bar{A}) = (\nabla\phi).\bar{A} + \phi(\nabla.\bar{A})$ (Identity)

$\nabla.(e^r\ \bar{r}) = (\nabla e^r).\bar{r} + e^r(\nabla.\ \bar{r})$

$= e^r\frac{\vec{r}}{r}.\vec{r} + e^r(3)$

$= e^r(3) + e^r\frac{r^2}{r}$

$\nabla(e^2.\vec{r}) = e^r(3+r)$

67. Ans: (d)

Sol: $\text{curl}(r^4\bar{r}) = ?$

curl $[\phi\,\bar{F}]$

$= \phi\ \text{curl}\,\bar{F} + (\text{grad}\,\phi)\times\bar{F}$ (Identity)

$= \text{curl}\ (r^4\bar{r})$

$= r^4(\text{curl}\,\bar{r}) + \text{grad}(r^4)\times\bar{r}$

$= r^4\cdot 0 + 4r^3\frac{\vec{r}}{r}\times\bar{r}$

$= \bar{0} + \bar{0} = \bar{0}$

68. Ans: (b)

Sol: $\text{L.I} = \int_C \bar{f}.d\bar{r} = \int_A^B (f_1\ dx + f_2\ dy + f_3\,dz)$

$= \int_{(0,2,1)}^{(4,1,-1)} [(2z)dx + (2y)dy + (2x)dx]$

$= \int_{(0,2,1)}^{(4,1,-1)} [(2z\ dx + 2x\,dz) + (2y)dy]$

$= \int_{(0,2,1)}^{(4,1,-1)} [2(z\ dx + x\,dx) + (2y)dy]$

$= \int_{(0,2,1)}^{(4,1,-1)} [2\ d(xz) + 2y\ dy]$

$= \left(2(xz) + 2\frac{y^2}{2}\right)_{(0,2,1)}^{(4,1,-1)}$

$= [2\ (4)\ (-1) + (1)^2] - [(2)\ (0)\ (1) + (2)^2]$

$= -11$

69. Ans: 202

Sol: Given $\bar{F} = (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$

$$\text{Curl}\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$= \bar{i}[0-0] - \bar{j}[3z^2 - 3z^2] + \bar{k}[2x - 2x] = \bar{0}$

$\Rightarrow \bar{F}$ is irrotational

$\Rightarrow$ Work done by $\bar{F}$ is independent of path of curve

$\Rightarrow \bar{F} = \nabla\phi$

where $\phi(x, y, z)$ is scalar potential

$\Rightarrow (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\Rightarrow d\phi = (2xy + z^3)\,dx + x^2\,dy + 3xz^2\,dz$

$\Rightarrow \int d\phi = \int (2xy + z^3)dx + x^2\,dy + 3xz^2\ dz$

$\Rightarrow \int d\phi = \int d(x^2y + xz^3)$

$\Rightarrow \phi(x, y, z) = x^2y + xz^3$

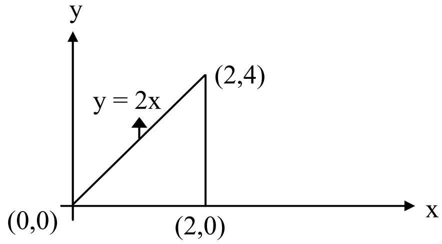
$\therefore$ Workdone $= \int_C \bar{F}.d\bar{r} = \phi(3,1,4) - \phi(1,-2,1)$

$= [9(1)+3(64)] - [1(-2) + 1(1)]$

$= 202$

70. Ans: (d)

Sol:



By Green's Theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

where $M = x + y$, $N = x^2$ and

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 1$$

The given integral $= \int_{x=0}^{2} \int_{y=0}^{2x} (2x - 1)\,dy\,dx$

$$= \int_0^2 [2xy - y]_0^{2x}\,dx$$

$$= \int_0^2 [4x^2 - 2x]\,dx$$

$$= \frac{20}{3}$$

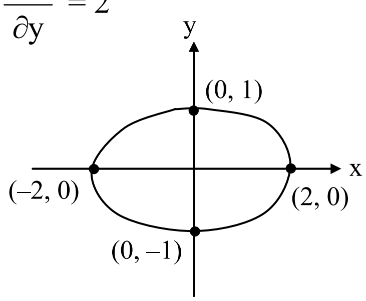
71. Ans: (c)

Sol: By Green's Theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here, $M = 2x - y$ and $N = x + 3y$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$$



The given integral $= \iint_R 2\,dx\,dy$

$= 2$ Area of the given ellipse

$= 2\,(\pi.\,2.\,1) = 4\pi$

72. Ans: 0

Sol: Given $\overline{A} = \nabla\phi$

$\Rightarrow$ Curl $\overline{A} = \overline{0}$

$\Rightarrow$ $\overline{A}$ is Irrotational

$\therefore$ Line integral of Irrotational vector function along a closed curve is zero

i.e. $\int_C \overline{A}.d\overline{r} = 0$, where $C : \frac{x^2}{4} + \frac{y^2}{9} = 1$ is a closed curve.

73. Ans: (b)

Sol: Using Gauss-Divergence Theorem,

$$\int_S \overline{F}.\overline{N}\;ds = \int_V \text{div}\;\overline{F}\;dv$$

$$= \int_V 3\;dv = 3\;V$$

$$= 3 \times \frac{4}{3}\pi r^3 = 4\pi(4)^3 = 256\pi$$

74. Ans: 264

Sol: Using Gauss-Divergence Theorem,

$$\iint_S xy\,dy\,dz + yz\,dz dx + zx\,dx\,dy = \iiint_V \text{div}\;\overline{F}\;dv$$

$$= \iiint_V (y + z + x)\,dv$$

$$= \int_{x=0}^{4} \int_{y=0}^{3} \int_{z=0}^{4} (x + y + z)\,dzdydx$$

$$= \int_{x=0}^{4} \int_{y=0}^{3} [4x + 4y + 8]\,dy\,dz$$

$$= \int_{x=0}^{4} [12x + 18 + 24]\,dx = 264$$

75. Ans: 0

Sol: By Stokes' theorem, we have

$$\int_C \bar{f}.d\bar{r} = \iint_S \left(\nabla\times\bar{f}\right).\bar{n}\ ds$$

Here, $\nabla \times \bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+yz) & (y^2+xz) & (z^2+xy) \end{vmatrix} = \bar{0}$$

$\Rightarrow \bar{f}$ is an irrotational

$\therefore \int \bar{f}.d\,\bar{r} = 0$

76. Ans: (d)

Sol: Curl $\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy-y & -yz^2 & -y^2z \end{vmatrix}$

$= \bar{i}[-2yz+2yz]-\bar{j}[0]+\bar{k}[0+1]$

$\Rightarrow$ Curl $\bar{F} = \bar{k}$

Using Stokes' theorem,

$$\int_C \bar{F}.d\bar{r} = \int_S \text{curl } \bar{F}.\bar{N}\ ds = \int_S \bar{k}.\bar{N}\ ds$$

Let R be the protection of s on xy plane

$$\Rightarrow \int_S \bar{k}.\bar{N}\ ds = \iint_R \bar{k}.\bar{N}\frac{dxdy}{|\bar{N}.\bar{k}|} = \iint_R 1\,dx\,dy$$

= Area of Region

$= \pi r^2 = \pi(1)^2 = \pi$

77. Ans: (d)

Sol: The function f(x) = x^2 cos(x) is even function

$\therefore$ The fourier series of f(x) contain only cosine terms.

The coefficient of sin 2x = 0

78. Ans: (d)

Sol: Given $f(x) = \begin{cases} -1, -2 \le x \le 0 \\ 2\ \ , 0 < x \le 0 \end{cases}$

The fourier series of f(x) in [c, c+ 2l] is given by

f(x) =

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

The term independent of 'x' in above fourier series is given by "$\frac{a_0}{2}$".

Now, $\frac{a_0}{2} = \frac{1}{2}\left[\frac{1}{\ell}\int_c^{c+2\ell} f(x)\,dx\right]$

$\Rightarrow \frac{a_0}{2} = \frac{1}{2}\left[\frac{1}{2}\int_{-2}^{2} f(x)\,dx\right]$

$\Rightarrow \frac{a_0}{2} = \frac{1}{4}\left[\int_{-2}^{0}(-1)\ dx + \int_{-2}^{0}(2)\ dx\right]$

$\Rightarrow \frac{a_0}{2} = \frac{1}{4}\left[(-x)_{-2}^{0} + (2x)_0^2\right]$

$\therefore$ The constant term is $\frac{a_0}{2} = \frac{1}{2}$

79. Ans: (b)

Sol: The given function is odd in (–π, π)

$\therefore$ Fourier series of f(x) contains only sine terms.

80. Ans: (b)

Sol: $f(x) = \sum_{n=1}^{\infty} \frac{k}{\pi}\left[\frac{2-2(-1)^n}{n}\right]\sin(nx)$

At $x = \frac{\pi}{2}$

$$k = \frac{k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right]$$

$$\therefore\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\infty = \frac{\pi}{4}$$

81. Ans: (c)

Sol: $f(x) = \pi x - x^2$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\pi x - x^2\right)\sin nx\ dx$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\left[\left(\pi x - x^2\right)\sin x\right]dx$$

$$\frac{2}{\pi}\left[\left(\pi x - x^2\right)(-\cos x) - (\pi - 2x)(-\sin x) + (-2)\cos x\ \right]_0^{\pi}$$

$$= \frac{8}{\pi}$$

82. Ans: (b)

Sol: $f(x) = (x - 1)^2$

The Half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x-1)^2\ \cos\ (n\pi x)\ dx$$

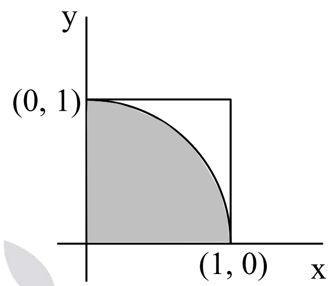
$$\frac{2}{\pi}\left[(x-1)^2.\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1).\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}$$

03_Probability & Statistics

01. Ans: (a)

Sol:



Let X and Y are two numbers in the interval (0, 1)

We have choose X and Y such that

$$X^2 + Y^2 < 1.$$

$$\text{Required probability} = \frac{\text{Area of the shaded region}}{\text{Area of the square}}$$

$$= \frac{\pi/4}{1} = \frac{\pi}{4}$$

02. Ans: (a)

Sol: A non-decreasing sequence can be described by a partition $n = n_0 + n_1 + n_2$

where n_i is number of times the digit i appear in the sequence.

There are (n + 1) choices for n_0, and given n_0 there are $n - n_0 + 1$ choices for n_1.

So, the total number of possibilities is

$$\sum_{n_0=0}^{n}\left(n - n_0 + 1\right) = (n+1).(n+1) - \sum_{n_0=0}^{n} n_0$$

$$= (n + 1)\ .\ (n + 1) - \frac{n^2 + n}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$\text{Required probability} = \frac{n^2 + 3n + 2}{2\left(3^n\right)}$$

03. Ans: (d)

Sol: Number of ways, we can choose R = C(n, 3)

We have to count number of ways we can choose R, so that median (R) = median (S).

Each such set R contains median S, one of the $\left(\frac{n-1}{2}\right)$ elements of S less than median (S), and one of the $\left(\frac{n-1}{2}\right)$ elements of S greater than median (S).

So, there are $\left(\frac{n-1}{2}\right)^2$ choices for R.

Required probability = $\frac{\left(\frac{n-1}{2}\right)^2}{C(n,3)}$

$= \frac{3(n-1)}{2n(n-2)}$

04. Ans: (a)

Sol: For each i $\in$ {1, 2, ..., n},

let A_i heads be the event that the coin comes up heads for the first time and continues to come up heads there after.

Then, the desired event is the disjoint union of A_i.

Since, each A_i occurs with probability 2^{-n}.

The required probability = n. 2^{-n}

05. Ans: (b)

Sol: Probability of the event that we never get the consecutive heads or tails

= P(HT HT HT ....) + P(TH TH TH ....)

$= \left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n.\left(\frac{2}{3}\right)^n$

$= 2\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n$

The required probability = $1 - 2\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n$

$= \frac{3^{2n}-2^{n+1}}{3^{2n}}$

06. Ans: (c)

Sol: Number of ways of selecting three integers $= {}^{20}C_3$

We know that, product of three integers is even, if atleast one of the number is even.

Number of ways of selecting 3 odd integers $= {}^{10}C_3$

$\therefore$ Required probability $= 1 - \frac{{}^{10}C_3}{{}^{20}C_3}$

$= 1 - \frac{2}{19} = \frac{17}{19}$

07. Ans: (c)

Sol: Given that P(A|B) = 1

$\Rightarrow \frac{P(A\cap B)}{P(B)} = 1$

$\Rightarrow P(A \cap B) = P(B)$ .... (1)

$P(B^C \mid A^C) = \frac{P(B^C\cap A^C)}{P(A^C)} = \frac{1-P(A\cup B)}{1-P(A)}$

$= \frac{1-\{P(A)+P(B)-P(A\cap B)\}}{1-P(A)}$

$= \frac{1-P(A)}{1-P(A)}$ [from (1)]

= 1

08. Ans: (a)

Sol: Let A = Getting electric contract and

B = Getting plumbing contract

$P(A) = \frac{2}{5}$; $P(\overline{B}) = \frac{4}{7}$; $P(B) = \frac{3}{7}$

$P(A \cup B) = \frac{2}{3}$;

$P(A \cap B) = \frac{2}{5} + \frac{3}{7} - \frac{2}{3} = \frac{17}{105}$

09. Ans: (d)

Sol: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A) = \frac{33}{100}$

$P(B) = \frac{14}{100}$

$P(A \cap B) = \frac{4}{100}$

$(A \cap B)$ is not empty set.

Therefore, A and B are not mutually exclusive.

$P(A \cap B) \neq P(A) . P(B)$

Therefore, A and B are not independent.

10. Ans: 0.2

Sol: To find the number of favourable cases consider the following partition of the given set {1, 2, ..., 100}

$S_1 = \{1, 6, 11, ..., 96\}$

$S_2 = \{2, 7, 12, ..., 97\}$

$S_3 = \{3, 8, 13, ..., 98\}$

$S_4 = \{4, 9, 14, ..., 99\}$

$S_5 = \{5, 10, 15, ..., 100\}$

Each of the above sets has 20 elements. If one of the two numbers selected from S_1 then the other must be chosen from S_4.

If one of the two numbers selected from S_2 then the other must be chosen from S_3.

Number of favourable cases = C(20, 1).C(20, 1) + C(20, 1).C(20, 1) + C(20, 2)

= 400 + 400 + 190 = 990

$\therefore$ Required probability = $\frac{990}{C(100,2)}$

$= \frac{990}{50 \times 99} = 0.2$

11. Ans: 0.66 Range 0.65 to 0.67

Sol: Let N = the number of families

Total No. of children = $\left(\frac{N}{2} \times 1\right) + \left(\frac{N}{2} \times 2\right)$

$= \frac{3N}{2}$

$\therefore$ The Required Probability $= \frac{\left(\frac{N}{2} \times 2\right)}{\frac{3N}{2}}$

$= \frac{2}{3} = 0.66$

12. Ans: 0.125

Sol: Total number of outcomes = 6^3

Number of outcomes in which sum of the numbers is 10 = Number of non-negative integer solutions to the equation a+b+ c =10 where $1 \le a, b, c \le 6$

= Co-efficient of x^{10} in the function

$(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$

$(x+x^2+x^3+x^4+x^5+x^6)^3 = x^3(1+x+x^2+x^3+x^4+x^5)^3$

$= x^3(1-x^6)^3(1-x)^{-3}$

$= x^3(1-3x^6+3x^{12}-x^{18})\sum_0^\infty \frac{(n+1)(n+2)}{2}.x^n$

$= (x^3-3x^9+3^{18}-x^{21})\sum_0^\infty \frac{(n+1)(n+2)}{2}.x^n$

Co-efficient of $x^{10} = 36 - 3\times 3 = 27$

$\therefore$ Required probability $= \frac{27}{216} = 0.125$

13. Ans: (a)

Sol: If A and B be disjoint events then $A\cap B=\{\ \}$

Probability of $A\cap B = 0$ .... (1)

If A and B are independent then

$P(A\cap B) = P(A).P(B)$ .... (2)

From (1) and (2)

$P(A).P(B) = 0$

$\Rightarrow Pr(A) = 0$ or $Pr(B) = 0$

14. Ans: 2.916 range 2.9 to 2.92

Sol: $E(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$

$E(X^2) = \frac{1}{6}(1^2+2^2+3^3+4^4+5^2+6^2) = \frac{91}{6}$

$\therefore$ Variance $= E(X^2) - \{E(X)\}^2$

$= \frac{91}{6} - (3.5)^2 = 2.916$

15. Ans: (c)

Sol: Total number of counters = 1 + 2 + ... + n

$= \frac{n(n+1)}{2}$

Probability of choosing counter k and winning $k^2 = \frac{2k}{n(n+1)}$

Expectation $= \sum_{k=1}^{n}\left\{k^2.\ \frac{2k}{n(n+1)}\right\}$

$= \frac{2}{n(n+1)}.\frac{n^2(n+1)^2}{4}$

$= \frac{n(n+1)}{2}$

16. Ans: (b)

Sol: The probability that she gives birth between 8 am and 4 pm in a day $= \frac{1}{3}$

By Total theorem of probability,

The required probability $= \left(\frac{1}{3}\times\frac{3}{4}\right)+\left(\frac{2}{3}\times\frac{1}{4}\right)$

$= \frac{5}{12}$

17. Ans: (b)

Sol: Let A = getting red marble both times

B = getting both marbles of same colour

$P(A\cap B) = \frac{3}{10}.\frac{2}{10}$

$P(B) = \frac{7}{10}.\frac{6}{10}+\frac{3}{10}.\frac{2}{10}$

Required probability $= \frac{P(A\cap B)}{P(B)} = \frac{6}{48} = \frac{1}{8}$

18. Ans: (d)

Sol: Let E_1 = The item selected is produced machine C and E_2 = Item selected is defective

$P(E_1 \wedge E_2) = \frac{20}{100}.\frac{5}{100}$

$$P(E_2) = \frac{50}{100}\cdot\left(\frac{3}{100}\right) + \frac{30}{100}\cdot\left(\frac{4}{100}\right) + \frac{20}{100}\cdot\left(\frac{5}{100}\right)$$

Required probability

$$= P(E_1/E_2) = \frac{P(E_1 \wedge E_2)}{P(E_2)} = \frac{100}{370} = \frac{10}{37}$$

19. Ans: 0.75 (No range)

Sol: Total probability = $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_0^2 cx\ dx = 1$$

$$\Rightarrow c = \frac{1}{2}$$

$$P(X > 1) = \int_1^{\infty} f(x)dx = \int_1^2 \frac{1}{2}x(dx) = \frac{3}{4} = 0.75$$

20. Ans: 1.944 range 1.94 to 1.95

Sol: The probability distribution for Z is

Z	0	1	2	3	4	5
P(Z)	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

$$E(Z) = \sum Z.P(Z)$$

$$= \frac{1}{36}(0.(6) + 1.(10) + 2(8) + 3(6) + 4(4) + 5(2))$$

$$= \frac{70}{36} = \frac{35}{18} = 1.944$$

21. Ans: (c)

Sol: $E(a^x) = \sum_{k=0}^{n} a^k . P(X = k)$

$$= \sum_{k=0}^{n} a^k\, C(n,k) \left(\frac{1}{2}\right)^k . \left(\frac{1}{2}\right)^{n-k}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} a^k\, C(n,k)\, a^k . (1)^{n-k}$$

$$= \left(\frac{a+1}{2}\right)^n$$

22. Ans: (d)

Sol: Given that mean = E(X) = 1

and Variance = V(X) = 5

$E((2 + X)^2) = E[X^2 + 4X + 4]$

$= E(X^2) + 4\,E(X) + 4$

Given V(X) = 5

$\Rightarrow E(X^2) - (E(X))^2 = 5$

$\Rightarrow E(X^2) = 5 + 1 = 6$

$E((2 + X)^2) = 6 + 4(1) + 4 = 14$

23. Ans: (a)

Sol: Total Probability = $\sum_{x=1}^{\infty} P(X = x) = 1$

$$\Rightarrow \sum_{x=1}^{\infty} K(1-\beta)^{x-1} = 1$$

$$\Rightarrow K(\ 1 + (1-\beta) + (1-\beta)^2 + \ldots\ldots\ \infty) = 1$$

$$\Rightarrow \frac{K}{1-(1-\beta)} = 1$$

$$\Rightarrow K = \beta$$

24. Ans : 209

Sol: $E(X) = \sum x\,P(x) = (-3)\times\frac{1}{6} + 6\times\frac{1}{2} + 9\times\frac{1}{3}$

$$= \frac{11}{2}$$

$$E(X^2) = \sum x^2\,P(x) = 9\times\frac{1}{6} + 36\times\frac{1}{2} + 81\times\frac{1}{3}$$

$$= \frac{93}{2}$$

$\therefore$ E(2X + 1)^2 = E(4X^2 + 4X + 1)

$= 4E(X^2) + 4E(X) + 1$

$= 4\times\frac{93}{2}+4\times\frac{11}{2}+1 = 209$

25. Ans: 0.1

Sol: $E(W) = \int_0^{10} 0.003\,V^2\, f(V)\, dV$

$= \int_0^{10} 0.003\,V^2 \frac{1}{10} dV$

= 0.1 *lb/ft*^{2}

Where f(V) = probability density function of V

26. Ans: (c)

Sol: Let X = Number of rupees you win on each throw.

The probability distribution of X is

X	0	1	2	3	4	5
P(X)	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

$E(X) = \sum X.P(X) = \frac{35}{18}$

27. Ans: 0.23 Range 0.22 to 0.24

Sol: Let X = number of ones in the sequence

n = 5

p = probability for digit 1 = 0.6

q = 0.4

Required probability = P(X = 2)

= C(5, 2). (0.6)^2. (0.4)^3

= 0.23

28. Ans: 0.25 Range 0.24 to 0.26

Sol: Given that, mean = 2(variance)

$\Rightarrow$ np = 2(npq) .... (1)

further, np + npq = 3 .... (2)

Solving, n = 4, $p = q = \frac{1}{2}$

$P(X = 3) = C(4, 3).\left(\frac{1}{2}\right)^3.\left(\frac{1}{2}\right) = \frac{1}{4} = 0.25$

29. Ans: (d)

Sol: Let X = Number of times we get negative values.

By using Binomial Distribution,

$P(X = k) = C(n, k)\, p^k q^{n-k}$

Where $p = \frac{1}{2}$, $q = \frac{1}{2}$, n = 5

Required probability = P(X ≤ 1)

= P(X = 0) + P(X = 1)

$= {}^5C_0 \times \left(\frac{1}{2}\right)^5 + {}^5C_1 \times \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)$

$= \frac{1+5}{32} = \frac{6}{32}$

30. Ans: (d)

Sol: We can choose four out of six winning in C(6, 4) different ways and if the probability of winning a game is p, then the probability of winning four out of six games

= C(6, 4) $p^4(1-p)^2$

= 15($p^4 - 2p^5 + p^6$)

31. Ans: 0.5706

Sol: The odds that the program will run is 2 : 1. Therefore, Pr(a program will run) = $\frac{2}{3}$. Let B denote the event that four or more programs will run and A_j denote that exactly j program will run. Then,

$Pr(B) = Pr(A_4 \cup A_5 \cup A_6)$

$= Pr(A_4) + Pr(A_5) + Pr(A_6)$

$$= C(6,4)\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + C(6,5)\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) + C(6,6)\left(\frac{2}{3}\right)^5$$

$= 0.5706$

32. Ans: 4

Sol: If n missiles are fired then probability of not hitting the target $= [1 - (0.3)]^n = (0.7)^n$

$\Rightarrow$ Probability of hitting the target atleast once $= 1 - (0.7)^n$

We have to fired the smallest +ve integer n so that, $\{1 - (0.7)^n\} > \frac{75}{100}$

$\Rightarrow \{1 - (0.7)^n\} > 0.75$

The smallest +ve integer satisfying this inequality is n = 4

33. Ans: 0.224 range 0.2 to 0.3

Sol: Average calls per minute $= \frac{180}{60} = 3$

Here, we can use poisson distribution with $\lambda = 3$.

Required Probability $= P(X = 2) = \frac{e^{-3}.3^2}{2\,!}$

$$= \frac{e^{-3}.9}{2} = 4.5\,e^{-3} = 0.224$$

34. Ans: 0.168

Sol: λ = average number of cars pass that point in a 12 min period $= \frac{15}{60/12} = 3$

Using the Poisson distribution,

$Pr(k) = e^{-\lambda}\frac{\lambda^k}{k\,!}$

$\therefore$ Required probability $Pr(4) = e^{-3}\frac{3^4}{4\,!} = 0.168$

35. Ans: 0.865 range 0.86 to 0.87

Sol: Let X = number of cashew nuts per biscuit. We can use Poisson distribution with mean

$$= \lambda = \frac{2000}{1000} = 2$$

$$P(X = k) = \frac{e^{-\lambda}.\lambda^k}{\angle k} \quad (k = 0,1,2...)$$

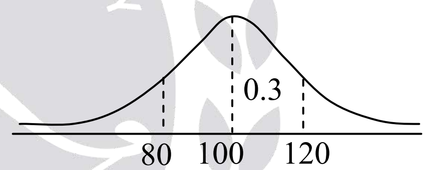
Probability that the biscuit contains no cashew nut = P (X = 0)

$= e^{-\lambda} = e^{-2} = 0.135$

Required probability = 1 – 0.135 = 0.865

36. Ans: 0.2

Sol: The area under normal curve is 1 and the curve is symmetric about mean.



$\therefore P(100 < X < 120) = P(80 < X < 120)$

$= 0.3$

Now, $P(X < 80) = 0.5 - P(80 < X < 120)$

$= 0.5 - 0.3 = 0.2$

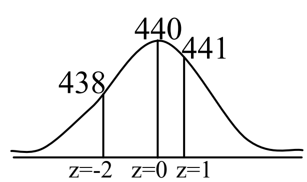
37. Ans: (a)

Sol: The standard normal variable Z is given by

$$Z = \frac{x-\mu}{\sigma}$$

When x = 438

$$Z = \frac{438-440}{1} = -2$$



When x = 441

$$Z = \frac{441-440}{1} = 1$$

The percentage of rods whose lengths lie between 438 mm and 441 mm

= P(438 < x < 441)

= P(–2 < Z < 1)

= P(–2 < Z< 0) + P(0 < Z< 1)

$= \frac{0.9545}{2} + \frac{0.6826}{2} = 0.81855$

≈ 81.85 %

38. Ans: (d)

Sol: The parameters of normal distribution are $\mu = 68$ and $\sigma = 3$

Let X = weight of student in kgs

Standard normal variable = $Z = \frac{X-\mu}{\sigma}$

(a) When X = 72, we have Z = 1.33

Required probability = P(X > 72)

= Area under the normal curve to the right of Z = 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh greater than 72 kgs = 300 × 0.0918

= 28

(b) When X = 64, we have Z = –1.33

Required probability = P(X ≤ 64)

= Area under the normal curve to the left of Z = – 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

(By symmetry of normal curve)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh less than 68 kgs = 300 × 0.0918

= 28

(c) When X = 65, we have Z = –1

When X = 71, we have Z = +1

Required probability = P(65< X < 71)

= Area under the normal curve to the left of Z = – 1 and Z = +1

= 0.6826

(By Property of normal curve)

Expected number of students who weighs between 65 and 71 kgs

= 300 × 0.6826

≈ 205

39. Ans: 0.8051

Sol: The probability of population has Alzheimer's disease is

p = 0.04, q = 0.96, n = 3500

$\mu = np = (3500)(0.04) = 140$

$\sigma^2 = npq = (3500)(0.04)(0.96)$

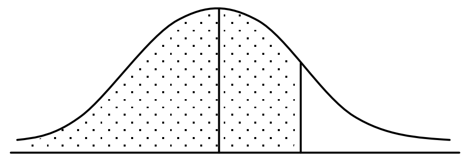
$\sigma^2 = 134.4$, $\sigma \cong 11.59$

Let X = number of people having Alzheimer's disease

$$P(X < 156) = P\left(\frac{X-\mu}{\sigma} < \frac{150-\mu}{\sigma}\right)$$

$$= P\left(Z < \frac{150-140}{11.59}\right)$$

= P(Z < 0.86)



= 0.5 + Area between z = 0 & z = 0.86
= 0.5 + 0.3051
= 0.8051

40. Ans: 0.7 range 0.65 to 0.75

Sol: The probability density function of

$$X = f(x) = \begin{cases} \frac{1}{10} & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P\left\{\left(X + \frac{10}{X}\right) \ge 7\right\} = \{P(X^2 + 10 \ge 7X)$$

$= P(X^2 - 7X + 10 \ge 0)$

$= P\{(X - 5)(X - 2) \ge 0\}$

$= P(X \le 2 \text{ or } X \ge 5)$

$= 1 - P(2 \le X \le 5)$

$= 1 - \int_2^5 f(x)\,dx$

$= 1 - \int_2^5 \frac{1}{10}\,dx$

$= 1 - \frac{3}{10} = 0.7$

41. Ans: (d)

Sol: (a) $E(x) = \int_0^1 x\, f(x)dx$

$f(x) = \frac{1}{b-a} = 1$

$E(x) = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$

(b) $E(x^2) = \int_0^1 x^2\, f(x)dx$

$f(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1$

$E(x^2) = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$

(c) $E(x^2) = \int_0^1 x^3\, f(x)dx$

$= \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}$

(d) Variance = $E(x^2) - (E(x))^2$

$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

42. Ans: (d)

Sol: If point chosen is (0,0) then length of position vector (minimum value of P can be O) will be 0 and the maximum value of P be $\sqrt{5}$ when point chosen is (1,2)

Minimum value of P = 0 at (0,0) point

Maximum value of P = $\sqrt{5}$ at (1,2) point

Probability Density function of P =

$f(P) = \frac{1}{\sqrt{5}-0} = \frac{1}{\sqrt{5}}$ as P is random variable

$$E(P^2) = \int_0^{\sqrt{5}} P^2\, f(P)\,dP$$

$$E(P^2) = \int_0^{\sqrt{5}} P^2 \frac{1}{\sqrt{5}}\,dP$$

$$= \frac{1}{\sqrt{5}}\left(\frac{P^3}{3}\right)_0^{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} \times \frac{1}{3}(\sqrt{5})^3 = \frac{1}{3} \times \sqrt{5} \times \sqrt{5}$$

$$E(P^2) = \frac{5}{3}$$

43. Ans: (i) a (ii) b

Sol: Given that passenger derives at a bus stop at 10 AM:

While stop arrive time is uniformly distributed between 10AM to 10:30AM

$$f(x) = \frac{1}{b-a} = \frac{1}{30-0} = \frac{1}{30}$$

(i) As we know passenger arrives bus stop at 10:00AM. But as given he want to wait more than 10 minutes means 10:10AM to 10:30AM

$$P(X \geq 10\,\text{min}) = \int_{10}^{30} f(x)\,dx$$

$$= \int_{10}^{30} \frac{1}{30} dx$$

$$= \frac{1}{30}(x)_{10}^{30}$$

$$= \frac{20}{30}$$

$$= \frac{2}{3}$$

(ii) As per given condition passenger will has to wait 10 : 15 AM to 10 : 25 AM.

$$P(15 \leq x \leq 25) = \int_{15}^{25} f(x)\,dx$$

$$= \int_{15}^{25} \frac{1}{30}\,dx$$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

44. Ans: (a)

Sol: We can use Exponential Distribution with mean $\mu = 5$

Let X is waiting time in minutes.

Probability Density function of X is

$$f(x) = 0.2\, e^{-(0.2)x} \quad \text{if } x \geq 0$$
$$= 0 \quad \text{if } x < 0$$

The required probability = P(0 < X < 1)

$$= \int_0^1 0.2\, e^{-(0.2)x} dx = 0.1813$$

45. Ans: (b)

Sol: In case of exponential distribution

$$f(x)\begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = \frac{1}{\theta}$$

$$\text{Variance} = \frac{1}{\theta^2}$$

$$f(x) = 2\, e^{-2x}$$

$$\text{mean} = \frac{1}{\theta} = \frac{1}{2}$$

$$\text{variance} = \frac{1}{\theta^2} = \frac{1}{4}$$

46. Ans: 0.3678

Sol: The probability density function is

$$P(x) = \theta\, e^{-\theta x} \quad x \geq 0$$
$$= 0 \quad x < 0$$

Probability that x exceeds th expected value

$$P\left(X \geq \frac{1}{\theta}\right) = \int_{x=\frac{1}{\theta}}^{\infty} \theta\, e^{-\theta x}\,dx$$

$$= \theta\left[\frac{e^{-\theta x}}{-\theta}\right]_{x=\frac{1}{\theta}}^{\infty}$$

$$= \frac{\theta}{-\theta}\left(0 - e^{-1}\right)$$

$= e^{-1}$

$= 0.3678$

47. Ans: 0.0025

Sol: $f(x) = 2\,e^{-2x}$

The probability that shower more than three minutes

$$= \int_3^{\infty} f(x)\,dx$$

$$= \int_3^{\infty} 2\,e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_3^{\infty}$$

$= -1\ (0 - e^{-6})$

$= e^{-6} = 0.0025$

48. Ans: (a)

Sol: $\sum_{r=1}^{\infty} P(X = r) = 1$

$\Rightarrow k(1 + (1-\beta) + (1-\beta)^2 + \ldots\ldots\ \infty) = 1$

$$\Rightarrow k\left\{\frac{1}{1-(1-\beta)}\right\} = 1$$

$\Rightarrow k = \beta$

$\therefore P(X = r) = \beta(1-\beta)^{r-1}$

This function is maximum when r = 1.

$\therefore$ mode = 1

49. Ans: Mean = 34, Median = 35, Modes = 35, 36 & SD = 4.14

Sol: Mean $= \frac{\sum x_i}{n} = 34$

Median is the middle most value of the data by keeping the data points in increasing order or decreasing order.

Mode = 36

S.D = 4.14

50. Ans:

Sol: μ = Mean $= \sum_{k=1}^{5}\{x_k . P(X = k)\}$

$= 1(0.1) + 2(0.2) + 3(0.4) + 4(0.2) + 5(0.1)$

$= 3$

$P(X \le 2) = 0.1 + 0.2 = 0.3$

$P(X \le 3) = 0.1 + 0.2 + 0.4 = 0.7$

$\therefore$ Median $= \frac{2+3}{2} = 2.5$

Mode = The value of X at which P(X) is maximum = 3

Variance $= \sum_{k=1}^{5} x_k^2 . P(X = k) - \mu^2$

$= 10.2 - 9 = 1.2$

Standard deviation $= \sqrt{1.2} = 1.095$

51. If the probability density function of a random variable X is given by

$$f(x) = \begin{cases} kx(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

then find k, mean, median, mode and standard deviation.

51. Ans: k = 6, Mean = $\frac{1}{2}$, Median = $\frac{1}{2}$, Mode = $\frac{1}{2}$ and S.D = $\frac{1}{2\sqrt{5}}$

Sol: We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 k(x-x^2)\,dx = 1$$

$$\Rightarrow k\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1\right] = 1$$

$$\Rightarrow k\left(\frac{1}{2}-\frac{1}{3}\right) = 1$$

$$\Rightarrow k\left(\frac{3-2}{6}\right) = 1 \Rightarrow k = 6$$

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 6(x^2-x^3)\,dx$$

$$= 6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{1}{2}$$

Median is that value 'a' for which

$$P(X \le a) = \frac{1}{2} \quad \int_0^a 6(x-x^2)\,dx = \frac{1}{2}$$

$$\Rightarrow 6\left(\frac{a^2}{2}-\frac{a^3}{3}\right) = \frac{1}{2}$$

$$\Rightarrow 3a^2 - 2a^3 = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{2}$$

Mode a that value at which f(x) is max/min

$\therefore$ $f(x) = 6x - 6x^2$

$f^1(x) = 6 - 12x$

For max or min $f^1(x) = 0 \Rightarrow 6 - 12x = 0$

$$\Rightarrow x = \frac{1}{2} \qquad f^{11}(x) = -12 \qquad f^{11}\left(\frac{1}{2}\right) = -12 < 0$$

$\therefore$ maximum at x = 1/2

$\therefore$ mode is 1/2

$$S.D = \sqrt{E(x^2)-(E(x))^2} = \frac{1}{2\sqrt{5}}$$

52. Ans: (i) a (ii) c (iii) d

Sol: The regression line of x and y is

$$2x - y - 20 = 0$$

$$2x = y + 20$$

$$x = \frac{1}{2}y + 10$$

The regression coefficient of x and y is

$$b_{xy} = \frac{1}{2}$$

The regression line of y on x is

$$2y - x + 4 = 0$$

$$2y = x - 4$$

$$y = \frac{1}{2}x - 2$$

The regression coefficient of y on x is

$$b_{yx} = \frac{1}{2}$$

(i) The correlation coefficient is

$$r = \sqrt{b_{yx}\ b_{xy}} = \sqrt{\frac{1}{4}}$$

$$r = \frac{1}{2}$$

(ii) Given $\sigma_y = \frac{1}{4}$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$\frac{1}{2}=\frac{1}{2}\frac{\frac{1}{4}}{\sigma_x}$$

$$\therefore\ \sigma_x=\frac{1}{4}$$

(iii) Both regression lines passing through $(\bar{x},\bar{y})$, we have

$$2\bar{x}-\bar{y}-20=0$$
$$2\bar{y}-\bar{x}+4=0$$

By solving these two equations, we get

$\bar{x}=12$ and $\bar{y}=4$

53. Ans: 0.18

Sol: Given: b_{yx} = 1.6 and b_{xy} = 0.4

$$r=\sqrt{b_{yx}\,b_{xy}}$$
$$r=\sqrt{1.6\times0.4}$$
$$r = 0.8$$

Now, $b_{yx}=r\dfrac{\sigma_y}{\sigma_x}$

$$1.6=0.8\frac{\sigma_y}{\sigma_x}$$
$$\frac{\sigma_y}{\sigma_x}=\frac{1.6}{1.8}=\frac{2}{1}$$

$\Rightarrow \sigma_x = 1$ and $\sigma_y = 2$

The angle between two regression lines is

$$\tan\theta=\left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x\ \sigma_y}{\sigma_x^{\ 2}\ \sigma_y^{\ 2}}\right)$$

$$=\left\{\frac{1-(0.8)^2}{0.8}\right\}\left\{\frac{(1)\,(2)}{(1)^2+(2)^2}\right\}=0.18$$

54.

Sol:

X	65	66	67	68	69	70	71
Y	67	68	66	69	72	72	69

We effect change of origin in respect of both x and y the new origins are chosen at or near the average of extreme values thus we take $\frac{65+71}{2}=68$ as the new origin for x and $\frac{66+72}{2}=69$ as the new origins for y. viz; we put

u = x – 68 and v = y– 69

X	Y	u = x-68	v = y-69	u^2	v^2	uv
65	67	–3	–2	9	4	6
66	68	–2	–1	4	1	2
67	66	–1	–3	1	9	3
68	69	0	0	0	0	0
69	72	1	3	1	9	3
70	72	2	3	4	9	6
71	69	3	0	9	0	0
	Total	0	0	28	31	20

$$r=\frac{\sum uv}{\sqrt{\sum u^2}.\sqrt{\sum v^2}}$$

$$=\frac{20}{\sqrt{28}.\sqrt{31}}\approx 0.67$$

55. Ans: (b)

Sol: Null Hypothesis H_0: The sample has been drawn from a population with mean μ = 280 days

Alternate Hypothesis H_1: The sample is not drawn from a population with mean μ = 280 i.e. $\mu\neq280$

Two-tailed test should be used.

Now the test statistic $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

$\mu = 280$, $\bar{x}$ = mean of the sample = 265

$\sigma = 30$, n = size of the sample = 400

$$Z = \frac{265 - 280}{\frac{30}{\sqrt{400}}} = -10$$

$\Rightarrow |Z| = 10$

$Z_\alpha = 1.96$

Since $|Z| = 10 > 1.96$, we reject null hypothesis

The sample is not drawn from population.

56. Ans: (c)

Sol: $H_0 : P = \frac{1}{5}$, i.e., 20% of the product manufactured is of top quality.

$H_1 : P \neq \frac{1}{5}$.

p = proportion of top quality products in the sample

$= \frac{50}{400} = \frac{1}{8}$

From the alternative hypothesis H_1, we note that two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$.

$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{\frac{1}{8} - \frac{1}{5}}{\sqrt{\frac{1}{5} \times \frac{4}{5} \times \frac{1}{400}}}$$

Since the size of the sample is equal to 400.

i.e., $z = \frac{3}{40} \times 50 = -3.75$

Now $|z| = 3.75 > 1.96$.

The difference between p and P is significant at 5% level.

Also H_0 is rejected. Hence H_0 is wrong or the production of the particular day chosen is not a representative sample.

95% confidence limits for P are given by

$$\frac{|p - P|}{\sqrt{\frac{pq}{n}}} \leq 1.96$$

Note:

We have taken $\sqrt{\frac{pq}{n}}$ in the denominator, because P is assumed to be unknown, for which we are trying to find the confidence limits and P is nearly equal to p.

i.e. $\left(p - \sqrt{\frac{pq}{n}} \times 1.96\right) \leq P \leq \left(p + \sqrt{\frac{pq}{n}} \times 1.96\right)$

i.e. $\left(0.125 - \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96\right) \leq P$

$\leq \left(0.125 + \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96\right)$

i.e. $0.093 \leq P \leq 0.157$

Therefore, 95% confidence limits for the percentage of top quality product are 9.3 and 15.7.

57. Ans: (d)

Sol: H_0: p = P, i.e. the hospital is not efficient.

H_1: $p < P$

One-tailed (left-tailed) test is to be used.

Let LOS be 1%.

Therefore, $z_\alpha = -2.33$.

$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}, \text{ where } p = \frac{63}{640} = 0.0984$$

P = 0.1726, Q = 0.8274

$$z = \frac{0.0984 - 0.1726}{\sqrt{\frac{0.1726 \times 0.8274}{640}}} = -4.96$$

$\therefore\ |z| > |z_\alpha|$

Therefore, difference between p and P is significant. i.e., H_0 is rejected and H_1 is accepted.

That is, the hospital is efficient in bringing down the fatality rate of typhoid patients.

04_Differential Equations

01. Ans: (a)

Sol: Given y dx + (1 + x^2) (1 + log y) dy = 0 .... (1)

Dividing by y(1+x^2)

$$\Rightarrow \frac{1}{1+x^2}dx + \left(\frac{1}{y} + \frac{\log y}{y}\right) dy = 0$$

$$\Rightarrow \int \frac{1}{1+x^2}\ dx + \int \left[\frac{1}{y} + \frac{1}{y}(\log y)\right] dy = c$$

$\therefore\ \tan^{-1}(x) + \log y + \frac{(\log y)^2}{2} = c$ is a general solution of equation (1)

02. Ans: (d)

Sol: Given $\frac{dy}{dx} = (x + y - 1)^2$ ………… (1)

Put x + y –1 = t .... (2)

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1 \text{ .... (3)}$$

Using (2) & (3), (1) becomes

$$\frac{dt}{dx} - 1 = t^2$$

$$\Rightarrow \frac{dt}{dx} = 1 + t^2 \Rightarrow \frac{dt}{1+t^2} = dx$$

Integrating both sides

$$\int \frac{dt}{1+t^2} = \int dx$$

$\Rightarrow \tan^{-1} t = x + c$

$\therefore$ The general solution of (1) is

$\tan^{-1}(x + y - 1) = x + c$

(or) x + y–1 = tan(x + c)

(or) y = 1 – x + tan(x+c)

03. Ans: (d)

Sol: Given $\frac{dy}{dx} + y = 1$ .... (1)

with y = 0 at x = 0 .... (2)

$$\Rightarrow \frac{dy}{dx} = 1 - y$$

$$\Rightarrow \frac{dy}{1-y} = dx$$

$$\Rightarrow \int \frac{dy}{y-1} = -\int dx$$

$\Rightarrow$ log(y – 1) = –x + c

$\Rightarrow y - 1 = e^{-x+c} = k\,e^{-x}$ where $k = e^c$

$\Rightarrow x = 1 + ke^{-x}$ ……………(3)

Using (2), (3) becomes

0 = 1 + k (or) k = –1

$\therefore$ The solution of (1) with (2) is $y = 1 - e^{-x}$.

04. Ans: (a)

Sol: Given $\frac{dy}{dx}+7x^2y=0$ ---------- (1)

with $y(0)=\frac{3}{7}$ ---------- (2)

Now, (1) is written as

$\Rightarrow \int\frac{1}{y}dy+\int 7x^2dx=C$

$\Rightarrow \log y+\frac{7x^3}{3}=C$

$\Rightarrow y=e^{-\frac{7x^3}{3}+C}$ -------- (3)

Using (2), (3) becomes

$\frac{3}{7}=e^0.e^C$ (or) $e^C=\frac{3}{7}$ -------- (4)

$\therefore$ The solution of (1) with (3) & (4) is given by

$$y=y(x)=e^{\frac{-7x^3}{3}+C}=e^{\frac{-7x^3}{3}}.e^C=\frac{3}{7}.e^{\frac{-7x^3}{3}}$$

Hence, $y(1)=y=\frac{3}{7}.e^{-\frac{7}{3}}$

05. Ans: (a)

Sol: $\frac{dy}{dx}=\frac{3x^2-2xy}{x^2}$

$x^2\,dy=(3x^2-2\,xy)\,dx$

$(x^2\,dy+2xy\,dx)=3x^2\,dx$

$\int d(x^2y)=\int 3x^2dx$

$x^2y=3\left(\frac{x^3}{3}\right)+c$

$(x^2y-x^3)=c$

$x^2(\,y-x)=c$

06. Ans: (d)

Sol: $\frac{dy}{dx}=\frac{y}{x}+\sec\left(\frac{y}{x}\right)$ -------- (1)

Let $\left(\frac{y}{x}\right)=z \Rightarrow y=zx$

$\frac{dy}{dx}=\left(z+x\frac{dz}{dx}\right)$

$\therefore$ (1) becomes

$\left(z+x\frac{dx}{dx}\right)=z+\sec z$

$x\frac{dz}{dx}=\sec z$

$\int\cos z\,dz=\int\frac{dx}{x}$

$\sin z=\log x+c$

$\sin\left(\frac{y}{x}\right)=\log x+c$

07. Ans: (b)

Sol: $(3x^2y^2+x^2)dx+(2x^3y+y^2)dy=0$

$(3x^2y^2\,dx+2x^3y\,dy)+x^2\,dx+y^2\,dy=0$

$\int d(x^3y^2)+\int x^2dx+\int y^2dy=c$

$x^3y^2+\left(\frac{x^3}{3}\right)+\left(\frac{y^3}{3}\right)=c$

08. Ans: (a)

Sol: $(x^2y^2+y)dx+(2x^3y-x)dy=0$

$(x^2\,y^2\,dx+2x^3ydy)+(ydx-x\,dy)=0$

$(y^2\,dx+2xy\,dy)+\left(\frac{ydx-xdy}{x^2}\right)=0$

$\int d(xy^2)-\int d\left(\frac{y}{x}\right)=c$

$xy^2-\left(\frac{y}{x}\right)=c$

09. Ans: (b)

Sol: $(y - xy^2)dx + (x + x^2y)\,dy = 0$

$(ydx + xdy) + xy\,(xdy - ydx) = 0$

$$\frac{(ydx + xdy)}{xy} + (xdy - ydx) = 0$$

$$\frac{d(xy)}{(xy)^2} + \left(\frac{xdy - ydx}{xy}\right) = 0$$

$$\int \frac{d(xy)}{(xy)^2} + \int d\log\left(\frac{y}{x}\right) = c$$

$$-\frac{1}{xy} + \log\left(\frac{y}{x}\right) = c$$

10. Ans: (a)

Sol: $2xy^3\,dx + (3x^2y^2 + x^2y^3 + 1)dy = 0$

$(3x^2y^2 + x^2y^3 + 1)\,dy = -\,2\,xy^3\,dx$

$$\frac{dx}{dy} = -\frac{3x}{2y} - \frac{x}{2} - \frac{1}{2xy^3}$$

$$\frac{dx}{dy} + \left(\frac{3}{2y} + \frac{1}{2}\right)x = \frac{-1}{2xy^3}$$

$$2x\frac{dx}{dy} + \left(\frac{3}{y} + 1\right)x^2 = \frac{-1}{y^3} \quad ----(1)$$

Let $x^2 = z \Rightarrow 2x\frac{dx}{dy} = \frac{dz}{dy}$

(1) becomes

$$\frac{dz}{dy} + \left(\frac{3}{y} + 1\right)z = \left(-\frac{1}{y^3}\right)$$

$$\text{I.F} = e^{\int\left(\frac{2}{y}+1\right)dy} = e^{3\log y + y} = y^3e^y$$

$$\therefore\ z\left(y^3e^y\right) = \int\left(\frac{-1}{y^3}\right)y^3e^y dy$$

$$= -\,e^y + c$$

$x^2\,y^3\,e^y + e^y = c$

11. Ans: (c)

Sol: Given that

$r\sin\theta\,d\theta + (r^3 - 2r^2\cos\theta + \cos\theta)dr = 0$

Let $M = r\sin\theta$ and $N = r^3 - 2r^2\cos\theta + \cos\theta$

$$\frac{\partial M}{\partial r} = \sin\theta$$

$$\frac{\partial N}{\partial \theta} = (2r^2 - 1)\sin\theta$$

$$\frac{\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}}{M} = -2\left(r - \frac{1}{r}\right)$$

$$\text{I.F} = e^{\int 2\left(r - \frac{1}{r}\right)dr}$$

$$= \frac{e^{r^2}}{r^2}$$

Multiplying the given equation by I.F

$$\frac{e^{r^2}}{r^2}\sin\theta\,d\theta + \frac{e^{r^2}}{r^2}\,(r^3 - 2r^2\cos\theta + \cos\theta)dr = 0$$

The above equation is exact.

$$\int \frac{e^{r^2}}{r^2}\sin\theta\,d\theta + \int e^{r^2}\,r\,dr = c$$

$$\Rightarrow \frac{-e^{r^2}\cos\theta}{r} + \frac{e^{r^2}}{2} = c$$

12. Ans: (a)

Sol: Given equation

$(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$

$$\frac{dy}{dx} = \frac{-(x^2y - 2xy^2)}{(3x^2y - x^3)} = \frac{2y^2 - xy}{3xy - x^2}$$

The above equation is homogenous

Put y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{2v^2 - v}{3v-1} - v = \frac{-v^2}{3v-1}$$

$$\frac{3v-1}{v^2}dv + \frac{dx}{x} = 0$$

Integrating $3\log v + \frac{1}{v} + \log x = c$

$$\left(\frac{x}{y}\right) + \log\left(\frac{y^3}{x^2}\right) = c$$

13. Ans: (a)

Sol: $x^4y^1 + 4x^3y = x^8$

$$x^4\frac{dy}{dx} + 4x^3y = x^8$$

$$\frac{dy}{dx} + \frac{4y}{x} = x^4$$

$$\text{I.F : } e^{\int P dx} = e^{4\int \frac{1}{x}dx}$$

$$= e^{4\log x} = x^4$$

$$\therefore y(x^4) = \int x^4.x^4 dx$$

$$= \left(\frac{x^4}{9}\right) + c$$

$$y = \left(\frac{x^5}{9} + \frac{c}{x^4}\right)$$

14. Ans: (a)

Sol: $y^1 + y = \sin x$

$$\frac{dy}{dx} + y = \sin x$$

$$\text{I.F : } e^{\int 1.dx} = e^x$$

$$ye^x = \int e^x \sin x \; dx$$

$$= \frac{e^x}{2}(\sin x - \cos x) + c$$

$$y = \frac{1}{2}(\sin x - \cos x) + ce^{-x}$$

$$y(\pi) = 1 \Rightarrow 1 = +\frac{1}{2} + ce^{-\pi}$$

$$\frac{1}{2}e^{\pi} = c$$

$$\therefore y = \frac{1}{2}\left(\sin x - \cos x + e^{\pi - x}\right)$$

15. Ans: (b)

Sol: $y^1 + \frac{3y}{2} = \frac{3xy^{\frac{1}{3}}}{2}$

$$y^{-1/3}\frac{dy}{dx} + \frac{3}{2}y^{2/3} = \left(\frac{3x}{2}\right) \text{ ------- (1)}$$

Let $y^{2/3} = z$

$$\Rightarrow \frac{2}{3}y^{-1/3}\frac{dy}{dx} = \frac{dz}{dx} \text{ ----------(2)}$$

From (1) & (2)

$$\frac{dz}{dx} + z = x$$

$$\text{I.F: } e^{\int 1.dx} = e^x$$

$$\therefore ze^x = \int xe^x dx$$

$$y^{2/3}e^x = e^x(x-1) + c$$

$$y^{2/3} = (x-1) + c\,e^{-x}$$

16. Ans: (b)

Sol: Given $\tan y\frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x \text{ ...... (1)}$$

Put sec y = v .... (2)

$\Rightarrow$ sec y tan y $\frac{dy}{dx} = \frac{dv}{dx}$ .... (3)

Using (2) and (3), (1) becomes

$$\frac{dv}{dx} + (\tan x)v = \cos^2 x$$

I.F = $e^{\int \tan x\, dx}$ = sec x

The solution is

v. sec x = $\int \cos^2 x$ .sec x dx + c

$\therefore$ sec y = cos x(sin x + c) is a G.S

17. Ans: (d)

Sol: $y'' + 4y' + 5y = 0$

$(D^2 + 4D + 5)\,y = 9$

AE has roots $(-2 \pm i)$

$\therefore \quad y = e^{-2x}(c_1 \cos 2x + c_2 \sin x)$

18. Ans: (c)

Sol: Given equation $y'' - 4y' - 6y = 0$

The auxiliary equation is

$m^2 - 4m - 6 = 0$

$$m = \frac{4 \pm \sqrt{16+24}}{2} = 2 \pm \sqrt{10}$$

The solution is

$$y = c_1\, e^{(2+\sqrt{10})x} + c_2 e^{(2-\sqrt{10})x}$$

By algebraic manipulation

$$= e^{2x}[c_1 \cosh(\sqrt{10})x + c_2 \sinh(\sqrt{10})x]$$

19. Ans: (a)

Sol: $y''' - 6y'' + 11y' - 6y = 0$

$(D^3 - 6D^2 + 11D - 6)\,y = 0$

$(D-1)(D^2 - 5D + 6)\,y = 0$

$(D-1)(D-2)(D-3)\,y = 0$

AE has roots 1, 2, 3

$\therefore \quad y = (c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$

20. Ans: (b)

Sol: Given equation of $(D^2 + 1)^2 y = 0$

The auxiliary equation is

$(m^2 + 1)^2 = 0$

$m = \pm i, \pm i$

$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

21. Ans: 4.54

Sol: Given $2\frac{d^2y}{dt^2} + 8y = 0$ .... (1)

with $y(0) = 0$ .... (2)

and $y'(0) = 10$ .... (3)

Now, (1) can be written as f(D)y = 0,

where $f(D) = D^2 + 4$ and $d = \frac{d}{dt}$

The auxiliary equation of (1) is given by f(m) = 0

$\Rightarrow m^2 + 4 = 0$

$\Rightarrow m = 0 \pm 2i$

$\therefore$ The general solution of (1) is given by

$y = c_1 \cos(2t) + c_2 \sin(2t)$ .... (4)

$\Rightarrow y' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$ .... (5)

Using (2), (4) becomes

$0 = c_1 + 0$ .... (6)

Using (3), (5) becomes

$10 = 0 + 2c_2$

$\Rightarrow c_2 = 5$ .... (7)

$\therefore$ The solution of (1) with (2) & (3) is given by $y = y(t) = 5\sin(2t)$

Hence, $y(1) = 5\sin(2) = 5(0.9092) = 4.54$

22. Ans: (b)

Sol: The given equation is

$(D^2 + 1)\, y = 0$

$\therefore$ A.E has roots $\pm\, i$

$\therefore$ $y = (c_1 \cos t + c_2 \sin t)$

$1 = c_1$ ……(1) ($\because$ y = 1 at t = 0)

$$\frac{dy}{dt} = (-c_1 \sin t + c_2 \cos t)$$

$0 = c_2$……(2) $\left(\because \dfrac{dy}{dt} = 0 \text{ at } t = 0\right)$

$\therefore$ from (1) & (2)

$y = \cos t$

23. Ans: (b)

Sol: $c_1 e^x + e^{\frac{-x}{2}}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\right)x + c_3 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$

$\Rightarrow$ AE has roots $1, \left(\frac{-1}{2} \pm \frac{\sqrt{3}}{2} i\right)$

$\Rightarrow (D - 1)(D^2 + D + 1)\, y = 0$

$(D^3 - 1)\, y = 0$

24. Ans: (b)

Sol: $y = (c_1 e^x + c_2\, e^x \cos x + c_3\, e^x \sin x)$ is the general solution from the given independent solutions

$\therefore$ $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$

$\therefore$ A.E. has roots $1, (1 \pm i)$

$\therefore$ $(D - 1)(D^2 - 2D + 2)\, y = 0$

$(D^3 - 3D^2 + 4D - 2)\, y = 0$

25. Ans: (c)

Sol: $y'' - 4y' + 13y = e^{2x}$

$(D^2 - 4D + 13)\, y = e^{2x}$

A.E has roots $(2 \pm 3i)$

$\therefore$ $y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

$$y_p = \frac{e^{2x}}{(D^2 - 4D + 13)} = \frac{e^{2x}}{(4 - 8 + 13)} = \frac{e^{2x}}{9}$$

$\therefore$ $y = (y_c + y_p)$

$$= e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{e^{2x}}{9}$$

26. Ans: 5

Sol: $y^{iv} + 4y''' + 8y'' + 8y' + 4y = 20$

$(D^4 + 4D^3 + 8D^2 + 8D + 4)\, y = 20\, e^{0.x}$

$$y_p = \frac{20.e^{0.x}}{(D^4 + 4D^3 + 8D^2 + 8D + 4)}$$

$$= \frac{20.e^{0.x}}{4} = 5$$

27. Ans: (a)

Sol: $y^v - y' = 12e^x$

$(D^5 - D)\, y = 12\, e^x$

$$\therefore\quad y_p = \frac{12e^x}{D(D^4 - 1)}$$

$$= \frac{12e^x}{D(D-1)(D+1)(D^2+1)}$$

$$= \frac{12xe^x}{2.2}$$

$y_p = 3x\, e^x$

28. Ans: (c)

Sol: $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh(x)$$

$(D^2 + 4D + 5)\,y = -(e^x + e^{-x})$

$$y_p = \frac{-(e^x + e^{-x})}{(D^2+4D+5)}$$

$$= \frac{-e^x}{(1+4+5)} - \frac{e^{-x}}{(1-4+5)}$$

$$= \frac{-e^x}{10} - \frac{e^{-x}}{2}$$

29. Ans: (d)

Sol: The given equation is $\frac{d^2y}{dx^2} = e^x$

$\Rightarrow \frac{dy}{dx} = e^x + c_1$

$\Rightarrow y = e^x + c_1x + c_2$ .... (i)

$\because y(0) = 1 \Rightarrow c_2 = 0$

$\because y^1(0) = 2 \Rightarrow c_1 = 1$

Substituting the values of c_1 & c_2 in (i), we get $y = e^x + x$

30. Ans: (a)

Sol: $\frac{d^2y}{dx^2} + y = \cos(x)$

$(D^2 + 1)\,y = \cos x$

A.E has roots $\pm i$

$\therefore\ y_c = (c_1 \cos x + c_2 \sin x)$

$\therefore\ y = (y_c + y_p) = (c_1 \cos x + c_2 \sin x + \frac{x}{2}\sin x)$

$y(0) = 1 \Rightarrow 1 = c_1$

$y(\pi/2) = 0 \Rightarrow 0 = c_2 + \frac{\pi}{4} \Rightarrow c_2 = -\frac{\pi}{4}c$

$$\therefore\ y = \left(\cos x - \frac{\pi}{4}\sin x + \frac{x}{2}\sin x\right)$$

31. Ans: (*)

Sol: $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin(2x)$

$(D^3 + 4D)\,y = \sin 2x$

$$\therefore y_p = \frac{\sin 2x}{(D^3+4D)}$$

$$= \frac{1}{D}\frac{\sin 2x}{(D^2+4)}$$

$$y_p = \frac{1}{D}\left(\frac{-x}{4}\cos 2x\right)$$

$$= -\frac{1}{4}\left(\frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right)$$

$$= -\frac{1}{8}(2x\sin 2x + \cos 2x)$$

32. Ans: (a)

Sol: $y''' + y = \sin(3x)$

$(D^3 + 1)\,y = \sin 3x$

$$y_p = \frac{\sin 3x}{(D^3+1)}$$

$$= \frac{\sin 3x}{(-9D+1)} \quad \text{(Replacing } D^2 \text{ by -9)}$$

$$= (1+9D)\frac{\sin 3x}{(1-81D^2)}$$

$$= \frac{(1+9D)\sin 3x}{1-81(-9)}$$

$$= \frac{1}{730}(\sin 3x + 27\cos 3x)$$

33. Ans: (c)

Sol: $y''' + 8y = x^4 + 2x + 1$

$(D^2 + 8)\,y = (x^4 + 2x + 1)$

$$y_p = \frac{(x^4+2x+1)}{(D^3+8)}$$

$$= \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+2x+1)$$

$$= \frac{1}{8}\left(1-\frac{D^3}{8}\right)(x^4+2x+1)$$

$$= \frac{1}{8}(x^4+2x+1-3x)$$

$$= \frac{1}{8}(x^4-x+1)$$

34. Ans: (b)

Sol: $y'' - 4y' - 2y = x^2$

$(D^2 - 4D - 2)\, y = x^2$

$$y_p = \frac{x^2}{(D^2-4D-2)} = -\frac{1}{2}\left(1+\left(2D-\frac{D^2}{2}\right)\right)^{-1}(x^2)$$

$$= -\frac{1}{2}\left[1-\left(2D-\frac{D^2}{2}\right)+4D^2\right](x^2)$$

$$= -\frac{1}{2}\left[1-2D+\frac{9}{2}D^2\right](x^2)$$

$$= \frac{-1}{2}\left[x^2-4x+9\right]$$

35. Ans: (d)

Sol: $(y'' + 2y' + y) = x^2e^{-x}$

$(D^2 + 2D + 1)y = x^2\, e^{-x}$

$(D + 1)^2\, y = x^2\, e^{-x}$

$$y_p = \frac{e^{-x}x^2}{(D+1)^2}$$

$$= e^{-x}\left(\frac{x^2}{D^2}\right)$$

$$= e^{-x}\left(\frac{1}{3}.\frac{x^4}{4}\right)$$

$$= \frac{x^4.e^{-x}}{12}$$

36. Ans: (a)

Sol: $y'' - 2y' + 5y = e^x \cos(3x)$

$(D^2 - 2D + 5)y = e^x \cos 3x$

$$\therefore \quad y_p = \frac{e^x \cos 3x}{(D^2-2D+5)}$$

$$= e^x \frac{\cos 3x}{[(D+1)^2-2(D+1)+5]}$$

$$= e^x \frac{\cos 3x}{(D^2+4)}$$

$$= \frac{e^x}{-5}\cos 3x$$

37. Ans: (b)

Sol: $y'' + 4y = x \sin(x)$

$(D^2+4)y = x \sin x$

$$y_p = \frac{x \sin x}{(D^2+4)}$$

$$= x\left(\frac{\sin x}{D^2+4}\right)-\left(\frac{2D}{(D^2+4)^2}\right)\sin x$$

$$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

38. Ans: (a)

Sol: $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$$\left(D^2 - 6D + 9\right)y = \frac{e^{3x}}{x^2}$$

$$(D-3)^2 y = \frac{e^{3x}}{x^2}$$

$y_c = (c_1 + c_2 x)\, e^{3x}$

$= (c_1 e^{3x} + c_2 xe^{3x} = c_1 y_1 + c_2 y_2$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$$

Let $y_p = (A\ y_1 + B\ y_2)$ from method of variations of parameters.

$$\text{Where } A = -\int \frac{e^{3x}}{x^2} - \frac{xe^{3x}}{e^{6x}} dx$$

$= -\log x$

$$\text{and } B = \int \frac{e^{3x}}{x^2} + \frac{e^{3x}}{e^{6x}} dx = \frac{-1}{x}$$

$$y_p = (-\log x)e^{3x} + \left(\frac{-1}{x}\right)xe^{3x}$$

$= -e^{3x}(\log x + 1)$

39. Ans: (a)

Sol: $x^2y'' - 2xy' + 2y = 4$

$$x(D-1)y - 2Dy + 2y = 4e^{0.z}$$

$(D^2 - 3D + 2)\ y = 4\ e^{0.z}$

$(D - 1)\ (D - 2)\ y = 4\ .\ e^{0.z}$

$y_c = (c_1 e^z + c_2 e^{2z})$

$$y_p = \frac{4e^{0.z}}{(D-1)(D-2)} = 2$$

$$y = \left(y_c + y_p\right) = \left(c_1x + c_2x^2 + 2\right)$$

40. Ans: (c)

Sol: $x^2y'' - 4xy' + 6y = 0$

Put $x = e^z$ and $D = \frac{d}{dz}$

$D(D - 1)\ y - 4\ D\ y + 6y = 0$

$(D^2 - 5\ D + 6)\ y = 0$

$(D - 2)\ (D - 3)\ y = 0$

$\therefore\quad y = (c_1 e^{2z} + c_2 e^{3z})$

$= (c_1 x^2 + c_2 x^3)$

41. Ans: (b)

Sol: $z = ax^n + by^n$ ---------- (1)

Differentiating equation (1) partially with respect to x

$p = n\ a\ x^{n-1} \Rightarrow ax^{n-1} = \frac{p}{n}$ ----------- (2)

where $p = \frac{\partial z}{\partial x}$

Differentiating equation (1) partially with respect to y

$q = n\ b\ y^{n-1} \Rightarrow by^{n-1} = \frac{q}{n}$ ----------- (3)

where $q = \frac{\partial z}{\partial y}$

From (1)

$z = a\ x^{n-1}.\ x + b\ y^{n-1}\ .y$

$$z = \frac{px}{n} + \frac{qy}{n}$$

$px + qy = nz$

42. Ans: (b)

Sol: $f(x^2 + y^2 + z^2\ ,\ x + y + z) = 0$

Or $x + y + z = f(x^2 + y^2 + z^2)$ ---------- (1)

Differentiating (1) partially with respect to ‘x’

$1 + p = f'(x^2 + y^2 + z^2)(2x + 2zp)$ --------(2)

Differentiating (1) partially w . r. t 'y'

$1 + q = f'(x^2 + y^2 + z^2)(2y + 2zq)$

$$f'(x^2 + y^2 + z^2) = \frac{1+q}{2y+2zq} \to (3)$$

Sub (3) in (2)

$$1 + p = \left(\frac{1+q}{2y+2zq}\right)(2x + 2zp)$$

$(1 + p)\,2\,(y + zq) = (1 + q)\,2\,(x + pz)$

$y + py + zq + pqz = x + xq + pz + pqz$

$p(y - z) + q\,(z - x) = x - y$

43. Ans: (b)

Sol: $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$

$B^2 - 4AC = 36 - 4(1)(9) = 36 - 36 = 0$

Equation (1) is parabolic

44. Ans: (c)

Sol: $3u_{xx} + 6u_{xy} - 16u_{yy} = 0 \quad \to \quad (1)$

$B^2 - 4AC = 36 - 4(3)(-16)$

$= 36 + 192 = 228 > 0$

Equation (1) is hyperbolic

45. Ans: (a)

Sol: $6u_{xx} + 7u_{yy} - 3u_{xy} = 4u_x + u_y$

$B^2 - 4AC = 9 - 4(6)(7) = 9 - 168 = -159 < 0$

Equation (1) is elliptic

46. Ans: (c)

Sol: $x^5 u_{xx} - xu_{yy} + 2u_y = 0 \; x > 0$

$B^2 - 4AC = 0 - 4(x^5)(-x) = 4x^6 > 0$

Equation (1) is hyperbolic

47. Ans: (c)

Sol: Sol: $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ --------- (1)

$u(x, 0) = 6e^{-3x}$ -------- (2)

$u = XT$ ------- (3) where X is a function of 'x' only and T is a function of 't' only

Sub (3) in (1)

$X'T = 8\,XT' + XT$

$X'T = X\,(2T' + T)$

$$\frac{X'}{X} = \frac{2T' + T}{T} = K$$

$$\frac{X'}{X} = k \;\&\; \frac{8T' + T}{T} = K$$

$$\frac{X'}{X} = k \Rightarrow \frac{dX}{dx} = kX$$

$$\frac{dX}{X} = k\,dx$$

On integrating

$\log X = kx + \log C_1$

$X = c_1 e^{kx} \to (4)$

$$\frac{2T' + T}{T} = k \Rightarrow 2T' + T = kT$$

$$\frac{dT}{dt} = \frac{(k-1)T}{2}$$

$$\frac{1}{T}dT = \left(\frac{k-1}{2}\right)dt$$

On integrating

$$\text{Log } T = \left(\frac{k-1}{2}\right)t + \log C_2$$

$$T = c_2\, e^{\left(\frac{k-1}{2}\right)t} \to (5)$$

Sub (4) & (5) in (3)

$$u = c_1 e^{kx}\; c_2\, e^{\left(\frac{k-1}{2}\right)t}$$

Given $u(x, 0) = 6e^{-3x}$

$6e^{-3x} = u(x, 0) = C_1C_2\, e^{kx}$

$c_1c_2 = 6 \;\; \& \; k = -3$

$u = 6e^{-3x} \,.\, e^{\left(\frac{-3-1}{2}\right)t}$

$u = 6e^{-3x}\, e^{-2t}$

48. Ans: (b)

Sol: The given equation is $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ .... (1)

Let u = X(x).T(t) be the solution of (1)

Then $\frac{\partial^2 u}{\partial x^2} = X''T$ and $\frac{\partial u}{\partial t} = XT'$

Substituting in equation (i)

$X''T = \alpha\, XT'$

$\frac{X''}{X} = \frac{\alpha T'}{T} = k$

$\frac{X''}{X} = \frac{k}{\alpha}$ and $\frac{T'}{T} = k$

$\Rightarrow T = c_1\, e^{kt}$ and $X = c_2\, e^{x\sqrt{\frac{k}{\alpha}}} + c_3\, e^{-x\sqrt{\frac{k}{\alpha}}}$

The solution of equation (1) is

$$u = c_1e^{kt}\left[c_2e^{\left(\sqrt{\frac{k}{\alpha}}\right)x} + c_3e^{-\left(\sqrt{\frac{k}{\alpha}}\right)x}\right]$$

49. Ans: (a)

Sol: Given $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ .... (i)

Let u = X(x).Y(y) be the solution of (i)

Then $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Substituting in equation (i)

$X^1Y = 4XY'$

$\frac{X'}{X} = \frac{4Y'}{Y} = k$

$\frac{X'}{X} = k$ and $\frac{4Y'}{Y} = k$

$\Rightarrow X = c_1\, e^{kx}$ and $Y = c_2\, e^{\frac{ky}{4}}$

Now, the solution is,

$u = c_1c_2\, e^{kx}\, e^{\frac{ky}{4}}$

$u = c_3\, e^{kx}\, e^{\frac{ky}{4}}$ .... (ii)

Given $u(0, y) = 8e^{-3y}$

$\Rightarrow 8e^{-3y} = u(0,y) = c_3\, e^{\frac{k}{4}y}$

$\Rightarrow c_3 = 8,\; k = -12$

$\therefore\; u = 8\, e^{-12x-3y}$

50. Ans: (a)

Sol: The given equation is

$p - q = \log(x + y)$

$\therefore$ The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

Consider $\frac{dx}{1} = \frac{dy}{-1}$

$\therefore\; x + y = C$

Consider $\frac{dx}{1} = \frac{dz}{\log(x+y)}$

$\Rightarrow dx = \frac{1}{\log C}\, dz$

$\Rightarrow x = \frac{z}{\log C} + C_1$

$\Rightarrow x - \frac{z}{\log x + y} = C_1$

$\therefore$ The solution is

$$\phi\left[x+y, x-\frac{z}{\log(x+y)}\right]=0$$

51. Ans: (b)

Sol: The auxiliary equations are

$$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x} \text{ .... (i)}$$

Using the multipliers 1, 1, 1 each of the fractions in (i) $=\frac{dx+dy+dz}{0}$

$\Rightarrow dx+dy+dz=0$

$\Rightarrow x+y+z=C_1$ .... (ii)

Using the multipliers x, y, z each of the fractions in (i)

$$=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\Rightarrow x\,dx+y\,dy+z\,dz=0$

$\Rightarrow x^2+y^2+z^2=C_2$ .... (iii)

$\therefore$ The solution is

$f(x+y+z, x^2+y^2+z^2)=0$

52. Ans: (c)

Sol: The given equation is

$q=3p^2$ (Type-I)

Let the solution be

$z=ax+by+c$ .... (1)

$\Rightarrow p=a$ and $q=b$

Substituting in the equation Type-I, we have

$b=3a^2$ ....(2)

Eliminating b from (1) & (2)

The solution is $z=ax+3a^2y+c$

53. Ans: (d)

Sol: Given $p^2z^2+q^2=1$ ………..(1)

where $p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}$

Let $q=ap$ ……… (2)

$p^2z^2+a^2p^2=1$

$\Rightarrow p^2(z^2+a^2)=1$

$\Rightarrow p=\pm\frac{1}{\sqrt{a^2+z^2}}$

$\Rightarrow q=ap=\pm\frac{a}{\sqrt{a^2+z^2}}$

Consider $dz=p\,dx+q\,dy$

$\Rightarrow dz=\frac{\pm1}{\sqrt{a^2+z^2}}dx+\frac{\pm a}{\sqrt{a^2+z^2}}dy$

$\Rightarrow \int\sqrt{a^2+z^2}\,dz=\int\pm\,dx\pm\int a\,dy+c$

$\therefore \frac{z}{2}\sqrt{a^2+z^2}+\frac{a^2}{2}\sinh^{-1}(z/a)=\pm(x+ay)+c$

is a required solution.

54. Ans: (b)

Sol: The given equation is

$p^2+q^2=x+y$ (Type-III)

$\Rightarrow p^2-x=y-q^2=a$ (say)

$\Rightarrow p=\sqrt{a+x}$ and $q=\sqrt{y-a}$

Consider $dz=p\,dx+q\,dy$

$\Rightarrow dz=\sqrt{a+x}\,dx+\sqrt{y-a}\,dy$

Intégrating,

$$z=\left(\frac{2}{3}\right)(a+x)^{3/2}+\left(\frac{2}{3}\right)(y-a)^{3/2}+b$$

55. Ans: (c)

Sol: The given equation can be written as

$$z = px + qy + \frac{1}{p-q} \quad \text{(Type-IV)}$$

The solution is

$$z = ax + by + \frac{1}{(a-b)} \qquad \text{for } p = a \ \& \ q = b$$

56. Ans: (d)

Sol: Given $\frac{\partial^2 u}{\partial x^2} = 25\frac{\partial^2 u}{\partial t^2}$

(or) $\frac{\partial^2 u}{\partial t^2} = \frac{1}{25}\frac{\partial^2 u}{\partial x^2}$ .... (1)

with u(0) = 3x .... (2)

and $\frac{\partial u(0)}{\partial t} = 3$ .... (3)

If the given one dimensional wave equation is of the form $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial u^2}{\partial x^2}$, $-\infty < x < \infty$, t > 0 and c > 0, satisfying the conditions u(x, 0) = f(x) and $\frac{\partial u(x,0)}{\partial t} = g(x)$, where f(x) & g(x) are given functions representing the initial displacement and initial velocity, respectively then its general solution is given by

$$u(x, t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds$$

Comparing the given problem with above general problem, we have

$c = \frac{1}{5}$, f(x) = 3x, g(x) = 3

Now,

$$u(1, 1) = \frac{1}{2}\left[f\left(1-\frac{1}{5}\right)+f\left(1+\frac{1}{5}\right)\right] + \frac{1}{2\left(\frac{1}{5}\right)}\int_{1-\frac{1}{5}}^{1+\frac{1}{5}} 3\ ds$$

$$\Rightarrow u(1, 1) = \frac{1}{2}\left[3\left(\frac{4}{5}\right)+3\left(\frac{6}{5}\right)\right] + \frac{5}{2}(3)\ (s)_{\frac{4}{5}}^{\frac{6}{5}}$$

$$\Rightarrow u(1, 1) = \frac{1}{2}\left[\frac{3}{5}\times(4+6)\right] + \frac{15}{2}\left[\frac{6}{5}-\frac{4}{5}\right]$$

$$\Rightarrow u(1, 1) = 3 + \frac{15}{2}\left(\frac{2}{5}\right)$$

$\therefore$ u(1, 1) = 6

57. Ans: (a)

Sol: Given that $\frac{\partial u}{\partial t} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}$ .... (1)

$\left(\because \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}\right)$

with B.C's : u(0, t) = 0 $(\because$ u(0, t) = 0)

u(1, t) = 0 $(\because u(l, t) = 0)$

and I.C's : u(x, 0) = sin(πx) .... (2)

$(\because$ u(x, 0) = f(x))

Now, the solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n.\sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n^2\pi^2c^2}{\ell^2}\right)t}$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x).e^{-n^2 t} \text{ .... (3)}$$

where $a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

Put t = 0 in (3), we get

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$\Rightarrow \sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$\Rightarrow \sin(\pi x) = a_1 \sin(\pi x) + a_2 \sin(2\pi x) + ...$

Comparing coefficients of sin on both sides of above, we get

$a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 0,$ ....(4)

$\therefore$ The solution of (1) with (2) from (3) and (4) is

$$u(x, t) = \sin(\pi x).\ e^{-\left[\frac{\pi^2}{1}\cdot\left(\frac{1}{\pi^2}\right)\right].t} = e^{-t}\sin(\pi x)$$

58. Ans: (c)

Sol: Given $u_t = \left(\sqrt{2}\right)^2 u_{xx}$ ……………(1)

$(\because u_t = c^2\ u_{xx})$

with B.C's: $u(0, t) = 0$

$(\because u(0,t) = 0)$

$u(\pi, t) = 0 \qquad (\because u(\ell, t) = 0)$

and I.C: $u(x, 0) = \sin(x)$ …........(2)

$(\because u(x, 0) = f(x))$

The solution of (1) is given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n.\sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n\pi c}{\ell}\right)^2 t}$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} a_n.\sin(nx)\ e^{-2n^2 t} \text{ .... (3)}$$

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} a_n.\sin(nx) \quad (\text{for } t = 0)$$

$\Rightarrow \sin(x) = a_1\sin x + a_2\sin(2x) +$ ....

$\Rightarrow a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0,$ .... (4)

$\therefore$ The solution of (1) with (2) from (3) and (4) is

$u(x, t) = a_1 \sin(x).\ e^{-2t}$

Hence, $u(\pi/2, \log 5) = 1.\ \sin(\frac{\pi}{2}).\ e^{-2\log 5}$

$= 5^{-2} = 0.04$

59. Ans: (b)

Sol: Given $u_{tt} = 2^2 u_{xx}$ .... (1)

$(\because u_{tt} = c^2 u_{xx})$

with B.C's : $u(0,t) = 0 \qquad (\because u(0,t) = 0)$

$u(\pi,t) = 0 \qquad (\because u(\ell,t) = 0)$

and I.C's : $u(x, 0) = 0 \qquad (\because u(x,0) = 0)$

$$\frac{\partial}{\partial t}u(x,0) = 2\sin(x) \quad \left(\because \frac{\partial u}{\partial t}(x,0) = g(x)\right)$$

The solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{\ell}\right).\sin\left(\frac{n\pi ct}{\ell}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx).\sin(n2t) \text{ .... (2)}$$

$$\Rightarrow \frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx).\cos(2nt).2n$$

$$\Rightarrow \frac{\partial}{\partial t}u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)(2n) \quad \text{for } t = 0$$

$\Rightarrow \sin(x) = b_1\sin(x).\ 2 + b_2.\ 2(2).\ \sin(2x)$ …

$\Rightarrow b_1 = \frac{1}{2}, b_2 = 0, b_3 = 0,$ .... (3)

$\therefore$ The solution of (1) with given conditions from (2) and (3) is given by

$u(x, t) = b_1.\sin(x)\ .\ \sin(2t) + 0 + 0$…….

$= \frac{1}{2}\sin(x).\sin(2t)$

Hence, $u\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \frac{1}{2}\sin\left(\frac{\pi}{3}\right).\sin\left(\frac{2\pi}{6}\right)$

$= \frac{1}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} = \frac{3}{8}$

60. Ans: (a)

Sol: Given $u_{tt} = u_{xx}$ ……..(1) $(\because u_{tt} = c^2 u_{xx})$

with B.C's: $u(0, t) = 0$ $(\because u(0,t) = 0)$

$u(\pi, t) = 0$ $(\because u(\ell, t) = 0)$

and I.C's: $u(x,0) = 2\sin(x)$ ....(2)

$(\because u(x,0) = f(x))$

$$\frac{\partial}{\partial t}u(x, 0) = 0$$

$$\left(\because \frac{\partial u}{\partial t}(x,0) = 0\right)$$

Now, the solution of the wave equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right).\cos\left(\frac{n\pi ct}{\ell}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx).\cos(nt) \;……(3)$$

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{for } t = 0$$

$$\Rightarrow 2\sin(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$\Rightarrow 2\sin(x) = a_1 .\sin x + a_2 \sin(2x) + ………$

$\Rightarrow a_1 = 2, a_2 = 0. a_3 = 0$ ……….(4)

$\therefore$ The solution of (1) with (2) from (3) and (4) is given by

$u(x, t) = a_1.\sin(x) \cos(t) = 2.\sin(x) \cos(t)$

61. Ans: (a)

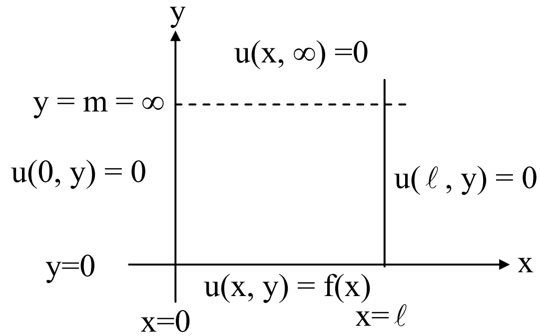
Sol: Given $u_{xx} + u_{yy} = 0 \rightarrow (1)$

$u(0, y) = 0 \rightarrow (2) \quad \forall\, y > 0$

$u(l, y) = 0 \rightarrow (3) \quad \forall\, y > 0$

$u(x, 0) = f(x) = u_0 \rightarrow (4) \quad 0 < x\, l$

$u(x, \infty) = 0 \rightarrow (5) \quad 0 < x < l$



The G.S of (1) satisfying above all boundary conditions is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n\pi y}{\ell}\right)} \rightarrow (6)$$

where $b_n = \frac{2}{\ell}\int_0^{\ell} f(x).\sin\left(\frac{n\pi x}{\ell}\right)dx$

Now, $b_n = \frac{2}{\ell}\int_0^{\ell} u_0.\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$\Rightarrow b_n = \frac{2u_0}{\ell}\left[\frac{-\cos\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}\right]_0^{\ell}$$

$$\Rightarrow b_n = \frac{2u_0}{n\pi}\left[1 - \cos(n\pi)\right]$$

$$\Rightarrow b_n = \frac{2u_0}{n\pi}\left[1 - (-1)^n\right] \rightarrow (7)$$

Using (7) (i.e. the value of b_n in (6), the required solution is), the equation (6) becomes

$$u(x,y) = \sum_{n=1}^{\infty}\frac{2u_0}{n\pi}\left[1-(-1)^n\right].\sin\left(\frac{n\pi x}{\ell}.\right)e^{\left(\frac{-2\pi y}{\ell}\right)}$$

(or)

$$u(x,y) = \sum_{n=1}^{\infty}\frac{2u_0}{(2n-1)\pi}(2).\sin\left[\frac{(2n-1)\pi x}{\ell}\right].e^{-\left[\frac{(2n-1)\pi y}{\ell}\right]}$$

62. Ans: (a)

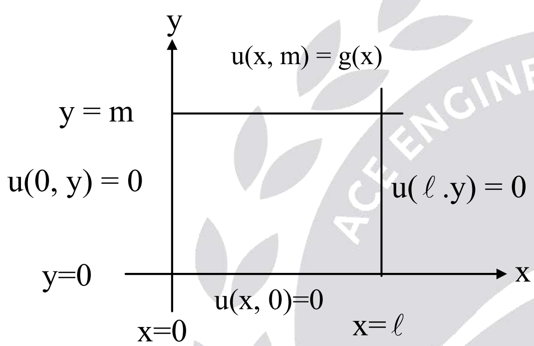
Sol: Given $u_{xx} + u_{yy} = 0$ …..(1)

with B.C's

$$\left.\begin{aligned} u(0,y) &= 0 \\ u(\ell,y) &= 0 \end{aligned}\right\} \quad 0 \le y \le m$$

$$\left.\begin{aligned} u(x,0) &= 0 \\ u(x,a) &= \sin\frac{n\pi x}{\ell} \end{aligned}\right\} \quad 0 \le x \le \ell$$

The solution of (1) is given by



$$u(x, y) = \sum_{n=1}^{\infty} b_n . \sin\left(\frac{n\pi x}{\ell}\right) . \sinh\left(\frac{n\pi y}{\ell}\right)$$

where

$$b_n = \frac{2}{\ell \sinh\left(\frac{n\pi m}{\ell}\right)} \int_0^{\ell} g(x) . \sin\left(\frac{n\pi x}{\ell}\right) dx$$

Now,

$$b_n = \frac{2}{\ell . \sinh\left(\frac{n\pi m}{\ell}\right)} \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) . \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell . \sinh\frac{n\pi m}{\ell}} \int_0^{\ell} \frac{1 + \cos\frac{2n\pi x}{\ell}}{2} dx$$

$$= \frac{2}{\ell . \sinh\left(\frac{n\pi m}{\ell}\right)} \left[\frac{x}{2} + \frac{1}{2} \frac{\sin\left(\frac{2n\pi x}{\ell}\right)}{\frac{2n\pi}{\ell}}\right]_0^{\ell}$$

$$= \frac{1}{\ell . \sin h\left(\frac{n\pi m}{\ell}\right)} \left[(\ell + 0) - (0 + 0)\right]$$

$$b_n = \frac{1}{\sinh\left(\frac{n\pi m}{\ell}\right)} = \frac{1}{\sinh\frac{n\pi a}{\ell}} \quad \text{for } m = a$$

∴ The solution of (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{1}{\sinh\left(\frac{n\pi a}{\ell}\right)} . \sin\left(\frac{n\pi x}{\ell}\right) . \sinh\frac{n\pi y}{\ell}$$

63. Ans: (a)

Sol: L { e^{-t} sint }

$$L\{\sin t\} = \frac{1}{s^2 + 1}$$

$$L\{e^{-t} \sin t\} = \frac{1}{(s+1)^2 + 1}$$

64. Ans: (b)

Sol: L {t cos t}

$$L\{\cos t\} = \frac{s}{s^2 + 1}$$

$$L\{t \cos t\} = (-1)' \frac{d}{ds}\left[\frac{s}{s^2 + 1}\right]$$

$$= -\left[\frac{s^2 + 1 - s(2s)}{(s^2 + 1)^2}\right] = \frac{s^2 - 1}{(s^2 + 1)^2}$$

65. Ans: (c)

Sol: $L\left\{\frac{\sin at}{t}\right\}$

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2 + a^2} ds$$

$$= \left(a.\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right)_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{a} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

66. Ans: (d)

Sol: $L\left\{\int_0^t e^{-t}\sin t\, dt\right\}1$

$$L\{\sin t\} = \frac{1}{s^2 + 1}$$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2 + 1}$$

$$L\left\{\int_0^t e^{-t}\sin t dt\right\} = \frac{1}{s}\left(\frac{1}{(s+1)^2 + 1}\right)$$

67. Ans: (b)

Sol: $L\ \{t\, e^{-t}\sin t\}$

$$L\ \{\sin t\} = \frac{1}{s^2 + 1}$$

$$L\ \{t \sin t\} = (-1)\ \frac{d}{ds}\left[\frac{1}{s^2 + 1}\right] = (-1)\frac{(-1)2s}{(s^2+1)^2}$$

$$= \frac{2s}{(s^2+1)^2}$$

$$L\{e^{-t}\, t \sin t\} = \frac{2(s+1)}{[(s+1)^2 + 1]^2}$$

68. Ans: (c)

Sol: Sol: $L(\sin t) = \dfrac{1}{s^2 + 1}$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1} ds$$

$$= [\tan^{-1} s]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\{f^1(t)\} = s.\ L\{f(t)\} - f(0)$$

$$\Rightarrow L\{f^1(t)\} = s.L\left\{\frac{\sin t}{t}\right\} - f(0)$$

$$L\{f^1(t)\} = s\cot^{-1} s - f(0) = s \cot^{-1} s - 1$$

69. Ans: (a)

Sol: $f(t) = \begin{cases} t, & 0 < t \le 1 \\ 0, & 1 < t < 2 \end{cases}$

$\because$ f(t) is periodic function with period 2

$$L\{f(t)\} = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st}\ f(t)\ dt$$

$$= \frac{1}{1 - e^{-2s}} \int_0^1 t.e^{-st}\ dt$$

$$= \frac{1}{1 - e^{-2s}}\left[t.\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_0^1$$

$$= \frac{1}{1 - e^{-2s}}\left[\left(\frac{e^{-s}}{-s}\right) - \left(\frac{e^{-s}}{s^2}\right) + \frac{1}{s^2}\right]$$

70. Ans: (d)

Sol: By definition of laplace transform

$$L\{u(t-a)\} = \int_0^\infty e^{-st}.u(t-a)\,dt$$

$$= \int_a^\infty e^{-st}.1\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-as}}{s}$$

71. Ans: (b)

Sol: $L\{e^t\} = \frac{1}{s-1}$

$e^t\,u(t-4) = [e^{t-4}.\,u(t-4)]e^4$

By second shifting property

$L[e^t.u(t-4)] = e^4.\,L[e^{t-4}.\,u(t-4)]$

$$= e^4.\left(\frac{e^{-4s}}{s-1}\right) = \frac{e^{4-4s}}{s-1} = \frac{e^{-4(s-1)}}{s-1}$$

72. Ans: 0.08

Sol: $L(\cos t) = \frac{s}{s^2+1}$

$$L(t\cos t) = \int_0^\infty e^{-st}(t\cos t)\,dt$$

$$\Rightarrow (-1).\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \int_0^\infty e^{-st}(t\cos t)\,dt$$

$$\Rightarrow \frac{s^2-1}{(s^2+1)^2} = \int_0^\infty e^{-st}(t\cos t)\,dt$$

Put s = 3

$$\Rightarrow \frac{9-1}{(3^2+1)^2} = \int_0^\infty e^{-st}\,t\cos t\,dt$$

$$\therefore \int_0^\infty e^{-st}\,t.\cos t\,dt = \frac{2}{25} = 0.08$$

73. Ans: (a)

Sol: we have

$$L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin 3t}{3}$$

By first shifting property

$$L^{-1}\left(\frac{1}{(s-2)^2+9}\right) = e^{2t}\,\frac{\sin 3t}{3}$$

74. Ans: (a)

Sol: $L^{-1}\left(\frac{s+3}{s^2+2s+1}\right) = L^{-1}\left(\frac{s+1+2}{(s+1)^2}\right)$

$$= L^{-1}\left(\frac{1}{s+1}\right) + 2L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$= e^{-t} + 2\,e^{-t}.t$ (By first shifting property)

75. Ans: (b)

Sol: $L^{-1}\left(\frac{1}{s^2+2s}\right) = L^{-1}\left(\frac{1}{s(s+2)}\right)$

$$= \frac{1}{2}L^{-1}\left(\frac{1}{s} - \frac{1}{(s+2)}\right)$$

$$= \frac{1}{2}\left(1-e^{-2t}\right)$$

$$= \frac{1}{2}\left(\frac{e^t - e^{-t}}{e^t}\right) = e^{-t}.\sinh(t)$$

76. Ans: (c)

Sol: $L^{-1}\left(\frac{1}{(s+1)}\right) = e^{-t}$

$$L^{-1}\left(\frac{1}{s(s+1)}\right) = \int_0^t e^{-t}dt = 1 - e^{-t}$$

$$L^{-1}\left(\frac{1}{s^2(s+1)}\right) = \int_0^t \left(1-e^{-t}\right)dt = t + e^{-t} - 1$$

77. Ans: (a)

Sol: $L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$

$$L^{-1}\left(\frac{e^{-4s}}{s+3}\right) = e^{-3(t-4)}u(t-4)$$

By 2[nd] shifting property

$$\begin{cases} e^{-3(t-4)} & \text{when } t \geq 4 \\ 0 & \text{other wise} \end{cases}$$

78. Ans: (b)

Sol: Let $L^{-1}\left(\log\left(\frac{s-a}{s-b}\right)\right) = f(t)$

$$\Rightarrow L[f(t)] = \log\left(\frac{s-a}{s-b}\right)$$

$$= \log(s-a) - \log(s-b)$$

$$\Rightarrow L[t.f(t)] = (-1)\frac{d}{ds}\left(\log(s-a) - \log(s-b)\right)$$

$$= \frac{1}{s-b} - \frac{1}{s-a}$$

$$t.\,f(t) = L^{-1}\left(\frac{1}{s-b} - \frac{1}{s-a}\right)$$

$$= e^{bt} - e^{at}$$

$$\therefore\ f(t) = \frac{e^{bt} - e^{at}}{t}$$

79. Ans: (c)

Sol: $L^{-1}\left(\frac{1}{(s-1)(s-2)}\right) = L^{-1}\left(\frac{1}{s-2} - \frac{1}{s-1}\right)$

$$= e^{2t} - e^{t}$$

80. Ans: (a)

Sol: $L^{-1}\left(\frac{s+3}{(s+1)(s-2)}\right)$

$$= L^{-1}\left(\frac{s+1+2}{(s+1)(s-2)}\right)$$

$$= L^{-1}\left(\frac{1}{s-2} + \frac{2}{(s+1)(s-2)}\right)$$

$$= L^{-1}\left(\frac{1}{s-2} - \frac{2}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s-2}\right)$$

(Partial fractions)

$$= L^{-1}\left(\frac{5}{3}\left(\frac{1}{s-2}\right) - \frac{2}{3}\left(\frac{1}{s+1}\right)\right)$$

$$= \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}$$

81. Ans: (d)

Sol: Given $y^1(t) + 5\,y(t) = u(t)$……………..(1)

with $y(0) = 1$ ……………………(2)

Applying L. T on both sides of (1), we get

$L\{y^1(t) + 5\,y(t)\} = L\{u(t)\}$

$$\Rightarrow L\{y^1(t)\} + 5\,L\{y(t)\} = \frac{1}{s}$$

$$\Rightarrow [s.\bar{y}(s) - y(0)] + 5.\bar{y}(s) = \frac{1}{s}$$

$$\Rightarrow (s+5)\,\bar{y}(s) - 1 = \frac{1}{s}$$

$$\Rightarrow (s+5)\bar{y}(s) = 1 + \frac{1}{s} = \frac{s+1}{5}$$

$$\Rightarrow \bar{y}(s) = \frac{s+1}{s.(s+5)}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{5}.\frac{1}{s} + \frac{4}{5}\,\frac{1}{s+5}$$

Applying inverse Laplace transform on both sides of above, we get

$$L^{-1}\{\bar{y}(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{5}L^{-1}\left\{\frac{1}{s+5}\right\}$$

$\therefore\ y(t) = \frac{1}{5} + \frac{4}{5}e^{-5t}$ is a solution of (1)

05_Complex Variables

01. Ans: (b)

Sol: $z = x + iy$

Let $x = r\cos\theta$ and $y = r\sin\theta$

Now $z = r.\ e^{i\theta}$

$ze^{i\alpha} = r.\ e^{i\theta}\ .\ e^{i\alpha}$

$= r\ .\ e^{i(\theta+\alpha)}$

$\therefore$ $ze^{i\alpha}$ is the point of rotation of (x, y) through an angle α

02. Ans: (b)

Sol: $\left|\frac{z-3}{z+3}\right| < 2 \Rightarrow |z-3| < 2\,|z+3|$

$\Rightarrow |z-3|^2 < 4\,|z+3|^2$

$\Rightarrow |(x-3) + iy\,|^2 < 4\,|(x+3) + iy\,|^2$

$\Rightarrow (x-3)^2 + y^2 < 4\,((x+3)^2 + y^2)$

$x^2 - 6x + 9 + y^2 < 4\,(x^2 + 9 + 6x + y^2)$

$0 < 3x^2 + 3y^2 + 30\,x + 27$

$0 < 3\,(x^2 + 10\,x + y^2 + 9)$

$0 < (x+5)^2 + y^2 - 16$

$\therefore$ $(x+5)^2 + y^2 - 16 > 0 \Rightarrow$ out side of $(x+5)^2 + y^2 = 16$

03. Ans: (c)

Sol: $f(z) = x^2 + iy^3$

$u = x^2$; $v = y^3$

$u_x = 2x$ & $v_x = 3y^2$

$u_x = v_y$ only at (0,0)

Hence f(z) is differentiable only at (0,0).

$\Rightarrow$ f(z) is not analytic any where.

04. Ans: (d)

Sol: If $f(z) = u + iv$ is analytic then its derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(we get this in the proof of C-R equations).

$\therefore$ Option (c) is correct.

Now $\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x}$

$\therefore$ Options (a) is correct

Again, $\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ (using C-R equations)

$= -i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}\right) = -i\frac{\partial w}{\partial y}$

$\therefore$ Option (b) is correct

05. Ans: (a)

Sol: $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$= (3x^2 - 3y^2) + i(6xy)$

06. Ans: (a)

Sol: Let $u + iv = f(z) = z\,\text{Im}(z) = (x + iy)\,y$

Then $u + iv = f(z) = xy + iy^2$

$\Rightarrow u = xy$ and $v = y^2$

$\Rightarrow u_x = y$ | $v_x = 0$

$u_y = x$ | $v_y = 2y$

Here, $u_x = v_y$ and $v_x = -u_y$ only at one point origin. i.e., C.R equations $u_x = v_y$ and $v_x = -u_y$ are satisfied only at origin. Further u, v, v_x, v_y, u_x, u_y are also continuous at origin.

$\therefore$ f(z) = z Im(z) is differentiable only at origin (0,0).

07. Ans: (d)

Sol: sin(z), cos(z) and polynomial $az^2 + bz+c$ are analytic everywhere.

$\therefore$ sin(z), cos(z) and az^2+bz+c are an entire functions.

Here, $\frac{1}{z-1}$ is analytic at every point except at z = 1 because the function $\frac{1}{z-1}$ is not defined at z = 1.

$\Rightarrow \frac{1}{z-1}$ is not analytic at z = 1

$\therefore \frac{1}{z-1}$ is not an entire function

08. Ans: (a)

Sol: u(x, y) = Ax + By

$u_x = A$; $u_y = B$

$\therefore$ $f'(z) = u_x + i\, v_x$

$= u_x - i\, u_y$ $(\therefore u_y = -v_x)$

$= A - i B$

$f(z) = (A - iB)z + iC$

09. Ans: (d)

Sol: Given that $v = x^3 - 3xy^2$

$\Rightarrow v_x = 3x^2 - 3y^2$ and $v_y = -6xy$

Consider $du = u_x\, dx + u_y\, dy$

$\Rightarrow du = (v_y)\, dx + (-v_x)\, dy$

$(\because u_x = v_y \;\&\; v_x = -u_y)$

$\Rightarrow du = (-6xy)\, dx + (-3x^2 + 3y^2)\, dy$

which is an exact differential form

$\Rightarrow \int du = \int (-6xy)\, dx + \int (3y^2)\, dy + k$

$\therefore u(x, y) = -3x^2y + y^3 + k$

10. Ans: (c)

Sol: u = log r

$$\frac{\partial u}{\partial r} = \frac{1}{r};\ \frac{\partial u}{\partial \theta} = 0$$

$$dv = \frac{\partial v}{\partial \theta} d\theta + \frac{\partial v}{\partial r} dr$$

$$= \left(r\frac{\partial u}{\partial r}\right) d\theta + \left(\frac{-1}{r}\frac{\partial u}{\partial \theta}\right) dr$$

(using C-R equations)

$$u = \int r \times \frac{1}{r} d\theta + \int -\frac{1}{r} \times 0 dr = \theta + c$$

11. Ans: (c)

Sol: Given that $Re\{f^{I}(z)\} = 2x + 2$, f(0) = 2 and f(1) = 1 + 2i

Let $f^{I}(z) = u + iv$, then u = 2x + 2

Consider $f^{II}(z) = u_x + i\, v_x = u_x - i\, u_y$

$= 2 - i\, 0$

$\Rightarrow f^{I}(z) = 2z + c$

$\Rightarrow f(z) = z^2 + cz + k$

$\because f(0) = 2$

$\Rightarrow k = 2$

$\because f(i) = 1 + 2i$

$\Rightarrow (i)^2 + c(i) + k = 1 + 2i$

$\Rightarrow c = 2$

$\therefore f(z) = z^2 + 2z + 2$

$\Rightarrow f^1(z) = 2z + 2$

$\Rightarrow f^1(z) = 2(x + i y) + 2 = 2(x + 1) + i(2y)$

$\therefore$ Imaginary part of $f^1(z) = 2y$

12. Ans: (c)

Sol: $u = e^y \sin x$

$u_x = e^y \cos x$; $u_y = e^y \sin x$

$f'(z) = u_x + i v_x = u_x + i(-u_y)$

$f'(z) = e^y \cos x - i e^y \sin x$

replace $x = z$ & $y = 0$

$f'(z) = \cos z - i \sin z$

$\Rightarrow f(z) = \sin z + i \cos z + i c$

$= i e^{-iz} + ic$

13. Ans: (c)

Sol: $u = xy$

$u_x = y$; $u_y = x$

$f'(z) = u_x + i v_x = u_x + i(-u_y)$

$f'(z) = y - ix$

replace x by z & y with 0

$f'(z) = 0 - iz = -iz$

$\therefore f(z) = -i\dfrac{z^2}{2} + ic$

14. Ans: (a)

Sol: Given that $v = e^x[y \cos y + x \sin y]$

$\Rightarrow v_x = e^x [0 + \sin y] + e^x[y \cos y + x \sin y]$

and $v_y = e^x[-y \sin y + \cos y + x \cos y]$

Consider $f^1(z) = u_x - iu_y$

$\Rightarrow f^1(z) = v_y + i v_x$ ($\because u_x = v_y$ & $v_x = -u_y$)

$\Rightarrow f^1(z) = e^x[-y \sin y + \cos y + x \cos y] + i e^x[\sin y + y\cos y + x\sin y]$

$\Rightarrow \int f^1(z) = ze^z - e^z + e^z + c$

$\therefore f(z) = z e^z + c$, $c = c_1 + ic_2$ is a required analytic function.

15. Ans: (d)

Sol: $u = 2x(1 - y)$

$u_x = 2(1 - y) \Rightarrow u_{xx} = 0$

$u_y = 2x(-1) \Rightarrow u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$

u is Harmonic

$f(z) = u + iv = 2x(1 - y) + i(2y + x^2 - y^2)$

$x = z$ & $y = 0$

$f(z) = 2z + i(0 + z^2 - 0) = 2z + iz^2$

is differentiable every where

$\Rightarrow f(z)$ is Analytic

16. Ans: (a)

Sol: $f(z) = 12z^2 - 4iz$ is entire function

$\therefore \int_c f(z)dz$ is independent of the path

$$\int_c f(z)dz = \int_{1+i}^{2+3i} (12z^2 - 4iz)dz = \left(12\frac{z^3}{3} - 4i\frac{z^2}{2}\right)_{1+i}^{2+3i}$$

$= (4z^3 - 2iz^2)_{1+i}^{2+3i}$

$= 4(2 + 3i)^3 - 2i(2 + 3i)^2 - 4(1 + i)^3 + 2i(1+i)^2$

$= 4(8 - 27i + 36i - 54) - 2i(4 - 9 + 12i) - 4(1 - i + 3i - 3) + 2i(1 - 1 + 2i)$

$= 4(9i - 46) - 2i(-5 + 12i) - 4(-2 + 2i) - 4$

$= 36i - 184 + 10i + 24 + 8 - 8i - 4$

$= 38i - 156$

17. Ans: (b)

Sol: $f(z) = \dfrac{z^2-1}{z^3-z^2+9z-9} = \dfrac{(z-1)(z+1)}{(z-1)(z^2+9)}$

$\therefore$ z = ± 3i are poles which are outside of |z| = 2

$\therefore$ f(z) is Analytic inside and on |z| = 2

According to integral theorem

$$\oint_c \frac{z^2-1}{z^3-z^2+9z-9}dz = 0$$

Options are wrong so answer is '0'.

18. Ans: (d)

Sol: $\oint_c \dfrac{z-1}{z^2+1}dz$

$$= \oint_c \frac{z-1}{(z+i)(z-i)}dz$$

only z = -i is inside of |z + i| = 1

$$= \oint_c \frac{z-1/(z-i)}{(z+i)}dz$$

According to Cauchy's integral formula

= 2 π i ϕ (-i) where $\phi(z) = \dfrac{(z-1)}{(z-i)}$

$$\phi(-i) = \frac{-i-1}{-i-i} = \frac{i+1}{2i}$$

$$= 2\pi i\left(\frac{(i+1)}{2i}\right)$$

= π (i + 1)

19. Ans: (d)

Sol: $f(z) = \dfrac{e^{2z}}{(z+1)^4}$

z = -1 is pole inside of |z| = 2

According to formula

$$\oint_c \frac{\phi(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}\phi^n(a)$$

$$\therefore \oint_c \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!}\phi'''(-1)$$

where $\phi(z) = e^{2z}$

$\phi'''(z) = 8e^{2z}$

$$= \frac{\pi i}{3}(8e^{-2})$$

$$= \frac{8}{3}\pi i e^{-2}$$

20. Ans: (c)

Sol: $\oint_c \dfrac{e^z}{(z^2+\pi^2)^2}dz = \oint_c \dfrac{e^z}{(z+\pi i)^2(z-\pi i)^2}dz$

$\therefore$ z = ± π i are singular points lie inside 'c'

Residue at z = πi is

$$\lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi+i}{4\pi^3}$$

Residue at z = -πi is

$$\lim_{z\to-\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z+\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi-i}{4\pi^3}$$

Then $\oint_c \dfrac{e^z}{(z^2+\pi^2)^2}dz = 2\pi i(\text{sum of residues})$.

$$= 2\pi i\left(\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}$$

21. Ans: (a)

Sol: $f(z) = \dfrac{e^{3z}}{(z-\pi i)}$

z = πi is singular pt.

|z – 2| + |z + 2| = | + πi -2| + |πi + 2|

$$= \sqrt{\pi^2+4} + \sqrt{\pi^2+4}$$

$$= 2\sqrt{\pi^2+4} = 7.44 > 6$$

$\therefore$ z = πi is outside of the path

Hence $\oint_c \frac{e^{3z}}{z-\pi i} dz = 0$

22. Ans: (b)

Sol: $\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{(z^2+1)} dz = \frac{1}{2\pi i}\oint_c \frac{e^{zt}}{(z+i)(z-i)} dz$

$\therefore$ z = ± i are singular points inside of |z| = 2

$$= \frac{1}{2\pi i}\oint_c \left(\frac{e^{zt}}{2i(z-i)} - \frac{e^{zt}}{2i(z+i)}\right) dz$$

$$= \frac{1}{-4\pi}\oint_c \left(\frac{e^{zt}}{(z-i)} - \frac{e^{zt}}{(z+i)}\right) dz$$

According to formula

$$= \frac{-1}{4\pi} 2\pi i\left(e^{it} - e^{-it}\right) = \frac{-1}{4}(-4)\frac{e^{it}-e^{-it}}{2i} = +\sin t$$

23. Ans: (c)

Sol: $\oint_c \frac{\sinh z}{z^4} dz$

z = 0 is singular point inside of |z| = 2

According to formula

$$= \frac{2\pi i}{3!}\phi'''(0) \quad \text{where } \phi(z) = \sin hz$$

$$\phi'''(z) = \cosh z$$

$$= \frac{\pi i}{3}\cosh(0) = \frac{\pi i}{3}$$

24. Ans: (b)

Sol: Let $f(z) = e^z + \sin z$ and $z_0 = \pi$

Then Taylor's series expansion of f(z) about a point $z = z_0$ (or) in power of $(z - z_0)$ is given by $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

where $a_n = \frac{f^{(n)}(z_0)}{n!}$

Here, the coefficient of $(z - z_0)^n$ in the Taylor's series expansion of f(z) about $z = z_0$ is given by $a_n = \frac{f^{(n)}(z_0)}{n!}$.

$$\therefore a_2 = \frac{f''(z_0)}{2!} = \frac{f''(\pi)}{2!}$$

$$= \frac{(e^z - \sin z)_{z=\pi}}{2} = \frac{e^{\pi}}{2}$$

25. Ans: 1

Sol: Let $f(z) = \log\left(\frac{z}{1-z}\right)$ and $|z| > 1$

(or) $\left|\frac{1}{z}\right| < 1$

Then $f(z) = \log\left[\frac{z}{z\left(1-\frac{1}{z}\right)}\right] = \log\left(\frac{1}{1-\frac{1}{z}}\right)$

$$\Rightarrow f(z) = \log\left(1-\frac{1}{z}\right)^{-1} = -\log\left(1-\frac{1}{z}\right), \quad \left|\frac{1}{z}\right| < 1$$

$$\Rightarrow f(z) = -\left[-\left\{\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\ldots\right\}\right],$$

$$\left|\frac{1}{z}\right| < 1$$

$\Rightarrow \log(1-z) = \frac{1}{z} + \frac{1}{2}.\frac{1}{z^2} + \frac{1}{3}.\frac{1}{z^3} + ...$,

$\left|\frac{1}{z}\right| < 1$

$\therefore$ The coefficient of $\frac{1}{z}$ is 1

26. Ans: (c)

Sol: $f(z) = \frac{e^{2z}}{(z-1)^3}$

put z – 1 = w ⇒ z = 1 + w

$$f(z) = \frac{e^{2(1+w)}}{w^3} = \frac{e^2.e^{2w}}{w^3}$$

$$= \frac{e^2}{w^3}\left[1 + 2w + \frac{(2w)^2}{2!} + \frac{(2w)^3}{3!} + \frac{(2w)^4}{4!} + ...\right]$$

$$= e^2\left[\frac{1}{w^3} + \frac{2}{w^2} + \frac{2^2}{2w} + \frac{2^3}{6} + \frac{2^4}{4!}w + ...\right]$$

$\therefore$ by observation of the series

The residue of f(x) at z = 1 is $2e^2$.

27. Ans: (a)

Sol: $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$

put z + 2 = w

$f(w) = (w-5)\sin\left(\frac{1}{w}\right)$

$$= (w-5)\left[\frac{1}{w} - \frac{1}{3!w^3} + \frac{1}{5!w^5}........\right]$$

Residue of

$f(z)\big|_{z=-2}$ = coefficient of $\frac{1}{w}$ i.e., $\frac{1}{z+2}$

= - 5

28. Ans: (a)

Sol: $f(z) = \frac{z}{(z+1)(z+2)}$

Put z + 2 = w ⇒ z = w – 2

$$f(w-2) = \frac{w-2}{(w-1)w} = \frac{-1}{w-1} + \frac{2}{w}$$

$$= \frac{1}{1-w} + \frac{2}{w}$$

$$= (1-w)^{-1} + \frac{2}{w} = 1 + w + w^2 + w^3 + ... + \frac{2}{w}$$

$$\therefore f(z) = \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + ...$$

29. Ans: (b)

Sol: $f(z) = \frac{z - \sin z}{z^3} = \frac{1}{z^2} - \frac{\sin z}{z^3}$

$$= \frac{1}{z^2} - \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...\right]$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - ....$$

30. Ans: 1

Sol: Given $f(z) = \frac{z-1}{(z+1)(z-3)}$

⇒ the singular points are z = –1 & z = –3

⇒ z = –1 & z = –3 are first order poles.

If the algebraic function f(z) has a first order pole at a singular point $z = z_0$ then the residue of f(z) is given by

$$\text{Res}(f(z): z = z_0) = \underset{z\to z_0}{\text{Lt}}\left[(z - z_0).f(z)\right]$$

$R_1 = \text{Res}(f(z) : z = -1)$

$$= \underset{z\to -1}{\text{Lt}}\left[[z-(-1)].\frac{z-1}{(z+1)(z-3)}\right]$$

$$\therefore R_1 = \underset{z\to -1}{\text{Lt}}\left[\frac{z-1}{z-3}\right] = 1/2$$

$R_2 = \text{Res}(f(z): z = 3)$

$$= \underset{z\to 3}{\text{Lt}}\left[(z-3).\frac{z-1}{(z-1)(z-3)}\right]$$

$$= \underset{z\to 3}{\text{Lt}}\left(\frac{z-1}{z+1}\right) = \frac{1}{2}$$

Hence, the sum of the residues of f(z) at its singular points is $R_1 + R_2 = \frac{1}{2}+\frac{1}{2} = 1.$

31. Ans: 0

Sol: The singular points of $f(z) = \frac{\sin z}{z.\cos(z)}$ are given by z.cos(z) = 0

$\Rightarrow z = 0$ and $z = (2n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow z = 0$ and $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$

$\Rightarrow z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ are the given singular points of f(z).

Here, $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ are simple poles of

$$f(z) = \frac{\sin z}{z\cos(z)} = \frac{\phi(z)}{\psi(z)},$$

where $\psi^1(z) = \cos(z) - z \sin z$

$R_1 = \text{Res}(f(z): z = \frac{\pi}{2})$

$$= \frac{\phi\left(\frac{\pi}{2}\right)}{\psi'\left(\frac{\pi}{2}\right)} = \frac{1}{0-\frac{\pi}{2}} = \frac{-2}{\pi}$$

$R_2 = \text{Res}(f(z): z = -\frac{\pi}{2})$

$$= \frac{\phi\left(-\frac{\pi}{2}\right)}{\psi'\left(-\frac{\pi}{2}\right)} = \frac{-1}{0-\frac{\pi}{2}} = \frac{2}{\pi}$$

Hence, the sum of the residues of the function f(z) at given singular points $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ is $R_1 + R_2 = \left(\frac{-2}{\pi}\right) + \left(\frac{2}{\pi}\right) = 0$

32. Ans: 1

Sol: The given singular point z = 0 is a simple pole (or) 1st order pole of

$$f(z) = \frac{1+e^z}{z\cos(z)+\sin(z)}$$

Now $R_1 = \text{Res}\ (f(z) : z = 0) = \underset{z\to 0}{\text{Lt}}(z-0)\, f(z)$

$$\Rightarrow R_1 = \underset{z\to 0}{\text{Lt}}(z-0).\frac{1+e^z}{z\cos(z)+\sin(z)}$$

$$\left(\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow R_1 = \underset{z\to 0}{\text{Lt}}\frac{z(0+e^z)+(1+e^z)}{-z\sin(z)+\cos(z)+\cos(z)}$$

$$\therefore R_1 = \frac{0+1+1}{0+1+1} = 1$$

33. Ans: 2

Sol: $f(z) = \dfrac{2+3\operatorname{cosec} z}{z} = \dfrac{2}{z} + \dfrac{3\operatorname{cosec} z}{z}$

$= \dfrac{2}{z} + \dfrac{3}{z\sin z}$

$\operatorname{Res}\big|_{z=0} = \operatorname{Res}\big|_{z=0}$ of $\dfrac{2}{z} + \operatorname{Res}\big|_{z=0}$ of $\dfrac{3}{z\sin z}$

= coefficient of $\dfrac{1}{z}$ in both terms

($\therefore$ In second term $\dfrac{1}{z}$ term not exist)

= 2 + 0

= 2

34. Ans: (d)

Sol: $f(z) = e^z \tan z = e^z \dfrac{\sin z}{\cos z}$ is not analytic at z

$= \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, ...$

Of these points only $z = \pm\dfrac{\pi}{2}$ lie inside C

Residue at $z = \pi/2$ is

$$\lim_{z\to\pi/2}\left\{\left(z-\frac{\pi}{2}\right)\frac{e^z\sin z}{\cos z}\right\} = e^{\pi/2}$$

Residue at $z = -\pi/2$ is

$$\lim_{z\to-\pi/2}\left\{\left(z+\frac{\pi}{2}\right)\frac{e^z\sin z}{\cos z}\right\} = e^{-\pi/2}$$

Then $\oint_c e^z \tan z\, dz = 2\pi i(\text{sum of residues})$.

$$= 2\pi i\left(e^{\pi/2} + e^{-\pi/2}\right)$$

35. Ans: (a)

Sol: $\oint_c \dfrac{e^{\frac{1}{z}}}{z} dz$ where c is $|z| = 2$

z = 0 is essential singularity

$$f(z) = \frac{e^{1/z}}{z} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + -----\right)$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{2!z^3} + ------$$

$\operatorname{Res}|_{z=0}$ = coefficient of $\dfrac{1}{z} = 1$

According to residue theorem

$2\pi i\,(\operatorname{Res}|_{z=0}) = 2\pi i\,(1) = 2\pi i$

06_Numerical Method

01. Ans: (c)

Sol: $f(x) = x^3 + x^2 + x + 7 = 0$

$f(-3) = -8$ and $f(-2) = 5$

A root lies in $(-3, -2)$

Let $x_1 = \dfrac{-2-3}{2} = -2.5$ is first approximation to the root

$\therefore f(x_1) = f(-2.5) < 0$

Now, Root lies in $[-2.5, -2]$

Let $x_2 = \dfrac{-2.5-2}{2} = -2.25$ is second approximation root.

02. Ans: 0.67

Sol: $f(x) = x^3 + x - 1 = 0$

Let $x_0 = 0.5$, $x_1 = 1$

$f(x_0) = f(0.5) = -0.375$

$f(x_1) = f(1) = 1$

$\therefore\ x_2 = \dfrac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$

is first approximation root

$= \dfrac{1(0.5) - (-0.375)(1)}{1 - (-0.375)}$

$= \dfrac{0.5 + 0.375}{1.375} = \dfrac{0.875}{1.375}$

$= 0.6363$

$f(x_2) = f(0.6363)$

$= (0.6363)^3 + (0.6363) - 1$

$= 0.2576 + 0.6363 - 1$

$= -0.1061 < 0$

Root lies in (0.6363, 1)

$x_3 = \dfrac{f(x_1)x_2 - f(x_2).x_1}{f(x_1) - f(x_2)}$

$= \dfrac{1(0.6363) - (-0.1061)1}{1 + 0.1061}$

$= 0.6711$

03. Ans: (b)

Sol: $f(x) = xe^x - x = 0$

$f(0) = -2 < 0,\ f(1) = 2.7183 - 2 > 0$

Let $x_0 = 0,\ x_1 = 1$

$x_2 = \dfrac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$

$= \dfrac{0.7183(0) - (-2).1}{0.7183 - (-2)}$

$= \dfrac{2}{2.7183}$

$= 0.7357$

$f(x_2) = f(0.7357)$

$= 0.7357\ .e^{0.7357} - 2\ = -0.4644$

Take $x_0 = 0.7357$ & $x_1 = 1$

$\therefore\ x_3 = \dfrac{f(x_2).x_1 - f(x_1).x_2}{f(x_2) - f(x_1)}$

$= \dfrac{0.9929}{1.1827} = 0.8395$

04. Ans: –3.26

Sol: $f(x) = x^5 - 10x + 100$

$x_o = -2 \qquad f'(x) = 5x^4 - 10$

$x_1 = x_o - \dfrac{f(x_o)}{f'(x_o)}$

$= (-2) - \dfrac{88}{70} = -3.26$

05. Ans: 1.57

Sol: $f(x) = x^3 - 5x^2 + 6x - 1$

$f'(x) = 3x^2 - 10x + 6$

$x_o = 1.5$

$x_1 = x_o - \dfrac{f(x_o)}{f'(x_o)}$

$= 1.57$

06. Ans: (a)

Sol: $x = \sqrt[5]{N}$

$x^5 = N$

$f(x) = x^5 - N$

$f'(x) = 5x^4$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$= x_n - \dfrac{(x_n^5 - N)}{5x_n^4} = \dfrac{4x_n^5 + N}{5x_n^4}$

07. Ans: (c)

Sol: Putting n = 0 in the iteration formula of the above example

$$x_1 = \frac{4x_o^5 + N}{5x_o^4}$$

$$= \frac{4(2^5)+30}{5(2^4)} = \frac{158}{80} = 1.975$$

08. Ans: (a)

Sol: **Given** $x_{n+1} = \frac{2x_n^3+1}{3x_n^2+1}$

Suppose the formula converges to the root after n iterations

$x_{n+1} = x_n = x$

$$x = \frac{2x^3+1}{3x^2+1}$$

$\Rightarrow x^3 + x - 1 = 0$

09. Ans: 1.7845

Sol: $\int_0^2 f(x)dx = \frac{h}{2}[(y_o + y_4) + 2(y_1 + y_2 + y_3)]$

$$= \frac{0.5}{2}[(0+1.26)+2(0.794+1+1.145)]$$

$= 1.7845$

10. Ans: (a)

Sol:

x	–1	0	1
f(x) = 5x³ – 3x² + 2x + 1	–9	1	5

$$\int_{-1}^{1} f(x)\, dx = \frac{h}{3}[(y_0 + y_2) + 2(0) + 4(y_1)]$$

$$= \frac{1}{3}[(-4)+4(1)] = 0$$

11. Ans: (a)

Sol: Error = Exact value of the integral – The value of the integral by the simpson's rule

$= 0 - 0 = 0$

12. 10.04

Sol: **Given** $\int_0^{1.2} f(x)dx$

By Simpsons rule $\int_0^{1.2} f(x)dx$

$$= \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{0.2}{3}[(9.6+12.2)+2(7.4+7.6)+4(9.1+6.8+8.8)]$$

$= 10.04$

13. Ans: (c)

Sol:

x	0	1	2	3	4	5	6
f(x) = $\frac{1}{1+x^2}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

$$\int_0^6 \frac{dx}{1+x^2} =$$

$$\frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{37}\right)+2\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}\right)\right]$$

$= 1.4107$

14. Ans: (a)

Sol: The volume of cylinder = $\pi\int_0^1 y^2\ dy$

$$= \pi\frac{h}{2}\left[\left(y_0^2 + y_4^2\right) + 2y_2^2 + 4\left(y_1^2 + y_3^2\right)\right]$$

$$= \pi\frac{0.25}{3}\left[(1+1) + 2(9) + 4(4+1)\right]$$

$$= \pi\frac{0.25}{3}[40]$$

$$= \frac{10\pi}{3}$$

15. Ans: (a)

Sol: Error = $\text{Max}\left|\frac{b-a}{12}\times h^2\times f''(x)\right|$

$$= \frac{1}{12}\times\frac{1}{100}\times 6(2.718)$$

$= 0.0136$

Here,

$f(x) = e^{x^2}$

Max $|f^{11}(x)|_{[0,1]} = 6e$

$\therefore\ h = \frac{b-a}{n}$

$= \frac{1}{10}$

16. Ans: (c)

Sol: $\left|\frac{b-a}{180}\times h^4\times \max f^{iv}(x)\right| \le 10^{-5}$

Let $h = \frac{b-a}{n} = \frac{1}{n}$

$f(x) = \frac{1}{x}$

Max $|f^{iv}(x)|_{\text{at } x=1} = 24$

$$\left(\frac{1}{180}\times\frac{1}{n^4}\times 24\right) \le 10^{-5}$$

$\Rightarrow n \ge 10.738$

$\therefore\ n \ge 10.738$

17. Ans: x = 0.9, y = 1 & z = 1

Sol: Let

$10x + y + z = 12$

$2x + 10y + z = 13$

$2x + 2y + 10z = 14$ and

$x_0 = 0, y_0 = 0, z_0 = 0$

Then first iteration will be

$x_1 = \frac{1}{10}(12 - y_0 - z_0)$

$= 1.2$

$y_1 = \frac{1}{10}(13 - 2x_1 + 10y_0)$

$= \frac{1}{10}(13 - 2(1.2) - 0) = 1.06$

$z_1 = \frac{1}{10}(14 - 2x_1 - 2y_1)$

$= \frac{1}{10}(14 - 2(1.2) - 2.(1.06)) = 0.95$

Second iteration will be

$x_2 = \frac{1}{10}(12 - y_1 - z_1)$

$= 0.90$

$y_2 = \frac{1}{10}(13 - 2x_2 + 10y_1) = 1.00$

$z_2 = \frac{1}{10}(14 - 2x_2 - 2y_2) = 1.00$

The required solution after second iteration is x = 0.9, y = 1 & z = 1

18. Ans: 0.992

Sol: $y^1 = f(x, y) = 4 - 2xy$

$x_0 = 0$, $y_0 = 0.2$, $h = 0.2$

By Taylor's theorem,

$y(x) = y(x_0 + h)$

$= y(x_0) + h\, y^1(x_0) + \frac{h^2}{2\,!} y^{11}(x_0)$

$= 0.2 + 0.24) + \frac{(0.2)^2}{2!}(-0.4)$

$= 0.992$

19. Ans: 1

Sol: $f(x, y) = 4 - 2xy$

$x_0 = 0$, $y_0 = 0.2$, $f_1 = 0.2$

By Euler's formula

$y_1 = y_0 + h\, f(x_0, y_0) = 0.2 + 0.2(4 - 0) = 1$

20. Ans: 1.1

Sol: By Euler's formula,

$y_1 = y_0 + h\, f(x_0, y_0)$

$y_1 = 1 + (0.1)\,(1 - 0) = 1.1$

21. Ans: 0.968

Sol: $\frac{dy}{dx} = f(x, y) = 4 - 2xy$

$x_o = 0$, $y_o = 0.2$, $h = 0.2$

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$k_1 = hf(x_0, y_0) = 0.2\,(4 - 0) = 0.8$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.2\,(4 - 2(0.2)0.8 = 0.736$

$y_1 = 0.2 + \frac{1}{2}(0.8 + 0.736)$

$= 0.968$

22. Ans: 0.02

Sol: $f(x, y) = x + y$

$x_0 = 0$, $y_0 = 0$, $h = 0.2$

$k_1 = h(f_0, y_0)$

$= 0.2(0+0) = 0$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.2(0.2 + (0.0))$

$= 0.04$

$y_1 = 0 + \frac{1}{2}(0 + 0.04) = 0.02$

23. Ans: 0.96

Sol: Let $\frac{dy}{dx} = f(x, y) = 4 - 2xy$

$x_o = 0$, $y_o = 0.2$, $4 = 0.2$

$k_1 = h.f\,(x_o, y_o) = 0.2\,(4 - x_o y_o) = 0.8$

$k_2 = h\, f\left(x_o + \frac{h}{2}, y_o + \frac{k_1}{2}\right)$

$= (0.2)\,(4 - 2(0.1)\,(0.6))$

$= (0.2)\,(3.88) = 0.776$

$k_3 = h\, f(x_o + h, \ y_o + \frac{k_2}{2})$

$= (0.2)\,(4 - 2(0.2)\,(0.976)) = 0.7219$

$k_4 = h.\, f\,(x_o + h, \ y_o + k_3)$

$= (0.2)\,(4 - 2(0.2)\,(0.9219)) = 0.7262$

$y(0.2) = y_1 = y_o + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.2 + \frac{1}{6}(0.8 + 2(0.776 + 0.7219) + 0.7262)$

$= 0.97$

25. Ans: 1.1165

Sol: $f(x, y) = x + y^2$,

$x_0 = 0$, $y_0 = 1$, $f_1 = 0.1$

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1\left[\left(x_0 + \frac{h}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right]$

$= 0.1168$

$k_3 = hf\left(x_0 + h, y_0 + \frac{k_2}{2}\right)$

$= 0.1[0.05 + 1.1185]$

$= 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1347$

$y_1 = y_0 = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + 0.1164$

$y_1 = 1.1164$

26. Ans: 2.6 – 1.3x, 2.3

Sol: The various summations are given as follows:

	x_i	y_i	x_i^2	x_iy_i
	–2	6	4	–12
	–1	3	1	–3
	0	2	0	0
	1	2	1	2
Σ	–2	13	06	–13

Thus, $\Sigma y_i = na + b\,\Sigma x_i$

$\Sigma x_i y_i = a\,\Sigma x_i + b\,\Sigma x_i^2$

These are called normal equations. Solving for a and b, we get

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \bar{y} - b\bar{x}$$

$$b = \frac{4\times(-13) - (-2)\times 13}{4\times 6 - 6}$$

$= -1.3$

$$a = \frac{13}{4} - 1.3\times\frac{(-2)}{4} = 2.6$$

Therefore, the linear equation is

$y = 2.6 - 1.3x$

The least squares error $= \sum_{i=1}^{4}\{y_i - (a + bx_i)\}^2$

$= (6 - 5.2)^2 + (3 - 3.9)^2 + (2 - 2.6)^2 + (2 - 1.3)^2$

$= 2.3$

27. Ans: i. $8x^2 - 19x + 12$ ii. 6 iii. 13

Sol: $f(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$

$f(x) = 8x^2 - 19x + 12$

$f(2) = 6$

$f^1(2) = 13$

$f(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$

$= 1 + (x - 1)\,13 + (x - 1)(x - 3)\,8$

$= 8x^2 - 19x + 12$

$p(2) = 6$

$p^1(2) = 13$

28. Ans: $8x^2 - 19x + 12$, 6, 13

Sol:

x	P(x)	Δp	$\Delta^2 p$
1	1		
		$\frac{27-1}{3-1} = 13$	
3	27		$\frac{37-13}{4-1} = 8$
		$\frac{64-27}{4-3} = 37$	
4	64		

By Newton's divided difference formula

$P(x) = P(x_0) + (x - x_0)\, f[x_0, x_1]$

$\quad + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$

$= 1 + (x - 1)13 + (x - 1)(x - 3).8$

$= 8x^2 - 19x + 12$

$P^1(x) = 16x - 19$

$P(2) = 6$

$P^1(2) = 13$

29. Ans: $x^2 + 2x + 3$, 4.25, 3

Sol: Since the given observations are at equal interval of width unity.

Construct the following difference table.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

Therefore f(x)

$f(x) = f(0) + C(x,1)\,\Delta f(0) + C(x, 2)\, f(0)$

$= 3 + (x \times 3) + \left(\frac{x(x-1)}{2!} \times 2\right)$

$f(x) = x^2 + 2x + 3$

$f^1(x) = 2x + 2$

$f(0.5) = 4.25$

$f^1(0.5) = 3$