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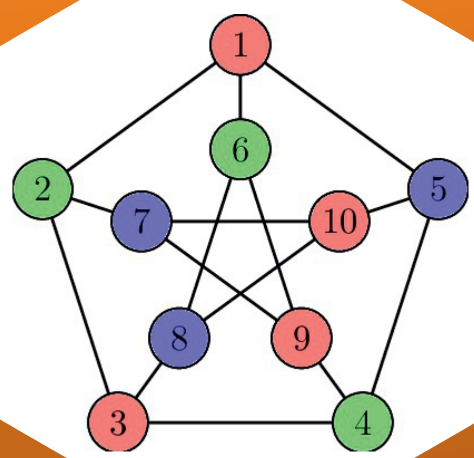
Discrete Mathematics

Text Book : Theory with worked out Examples and Practice Questions



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Discrete Mathematics

(Solutions for Text Book Practice Questions)

1. Mathematical Logic

01. Ans: (d)

Sol: The contrapositive of $(A\rightarrow B)$ is $(\sim B\rightarrow \sim A)$.

and $(A \rightarrow B) \equiv (\sim B \rightarrow \sim A)$.

The statement given in option(d) is contrapositive of p.

$\therefore$ The statement given in option(d) is equivalent to p.

02. Ans: (a)

Sol: S_1: The given argument is

1. $r \rightarrow (q \rightarrow p)$
2. $\underline{\sim p}$

$\therefore (\sim r \vee \sim q)$

3. $(r \wedge q) \rightarrow p$ (1), equivalence
4. $\sim(r \wedge q)$ (3) and (2), modus tollens
5. $(\sim r \vee \sim q)$ (4), demorgan's law

$\therefore S_1$ is valid

S_2: When p has truth value false, q has truth value false and r has truth value true; we have, all the primeses are true but conclusion is false.

$\therefore S_2$ is not valid.

03. Ans: (b)

Sol: Quine's method:

Case1:

When a has truth value true, the given formula becomes

$$c \wedge (\sim b \wedge \sim c)$$

$$\Leftrightarrow (c \wedge \sim c) \wedge \sim b$$

$$\Leftrightarrow F \wedge \sim b$$

$$\Leftrightarrow F$$

Case2:

When a has truth value false, the given formula becomes

$$b \wedge \sim(b \vee c)$$

$$\Leftrightarrow (b \wedge \sim b) \wedge \sim c$$

$$\Leftrightarrow F \wedge \sim c$$

$$\Leftrightarrow F$$

$\therefore$ The given formula is a contradiction.

04. Ans: (a)

Sol: Let $S_1 = (P \rightarrow Q)$

where, $P = ((a \vee b) \rightarrow c)$ and

$Q = (a \wedge b) \rightarrow c$

Here, Q is false only when a is true, b is true and c is false.

For these truth values P is also false.

$\therefore S_1$ is valid

Let $S_2 = (R \rightarrow S)$

where, $R = (a \wedge b) \rightarrow c$ and

$S = (a \vee b) \rightarrow c$

Here, when a is true, b is false and c is false; we have, R is true and S is false.

i.e., $(R \rightarrow S)$ is false.

$\therefore S_2$ is not valid

05. Ans: (c)

Sol: **S_1**: The given formula is equivalent to the following argument.

1. $\sim p \rightarrow (q \rightarrow \sim w)$ premise
2. $(\sim s \rightarrow q)$ premise
3. $\sim t$ premise
4. $\underline{(\sim p \vee t)}$ premise

$\therefore (w \rightarrow s)$ conclusion

We can derive the conclusion from the premises as follows.

5. $\sim p$ (3), (4), Disjunctive syllogism
6. $(q \rightarrow \sim w)$ (1), (5), Modues ponens
7. $(\sim s \rightarrow \sim w)$ (2), (6), Transitivity
8. $(w \rightarrow s)$ (7), Contra positive equivalence

$\therefore$ The given formula is valid

S_1: The given formula can be written as

$\{(q \rightarrow t) \wedge (s \rightarrow r) \wedge (q \vee s)\} \rightarrow (t \vee r)$ which is valid by the rule of constructive dilemma.

06. Ans: (c)

Sol: (a) When p has truth value false and r has truth value true, then all the premises are true and conclusion is false.

$\therefore$ The argument is not valid.

(b) When p has truth value false, q is true and r is true: then all the premises are true and conclusion is false.

$\therefore$ The argument is not valid.

(c) The given argument is written as

1. $\{p \rightarrow (q \rightarrow r)$
2. $\underline{(p \wedge q)\}}$

$\therefore r$

3. $(p \wedge q) \rightarrow r$ (1), equivalence
4. r (2) and (3), modus ponens

$\therefore$ The argument is valid

(d) When p has truth value ***false*** and q has truth value ***false***, then both the premises are true and conclusion is false.

$\therefore$ The argument is not valid.

07. Ans: (d)

Sol: **S_1**: The contra-positive of $(P \rightarrow Q)$ is $(\sim Q \rightarrow \sim P)$

The contra-positive of $\{(\sim r) \vee (\sim s)\} \rightarrow q$ is

$\sim q \rightarrow \sim\{(\sim r) \vee (\sim s)\}$

$\Leftrightarrow \sim q \rightarrow (r \wedge s)$ Demorgan's Law

$\Leftrightarrow q \vee (r \wedge s)$ $(\because (P \rightarrow Q) \cong (\sim P \vee Q))$

Similarly, we can verify other statements.

08. Ans: (b)

Sol: From the truth table

$(p * q) \Leftrightarrow (p \wedge \sim q)$

Now, $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$

$\Leftrightarrow \sim(p \wedge \sim q)$

$\Leftrightarrow \sim(p * q)$

09. Ans: (d)

Sol: The given formula can be written as

$(\overline{p} . q) + (p. \overline{q}) + (p.q)$

$= (\overline{p} . q) + p. (\overline{q} + q)$

(By distributive law)

$= (\overline{p} . q) + p$ $(\because \overline{q} + q = 1)$

$= (p + \overline{p}) . (p + q)$

(By distributive law)

$= p + q$

10. Ans: (c)

Sol: Quine's Method:

Case 1: When p is true, given formula

$\{T \wedge (F \vee \sim q) \wedge (F \vee q \vee r) \wedge \sim r\}$

$\Leftrightarrow \sim q \wedge (q \vee r) \wedge \sim r$

$\Leftrightarrow \sim(q \vee r) \wedge (q \vee r)$

$\Leftrightarrow F$

Case 2: When p is false, the given formula is also false

$\therefore$ The given formula is not satisfiable.

11. Ans: (a)

Sol: $S_1 = ((P \to Q) \to P) \to Q$

$\Leftrightarrow \sim\{\sim(\sim P \vee Q) \vee P\} \vee Q$

$\Leftrightarrow \{(\sim P \vee Q) \wedge \sim P\} \vee Q$

$\Leftrightarrow (\sim P \vee Q \vee Q) \wedge (\sim P \vee Q)$

$\Leftrightarrow (\sim P \vee Q) \wedge (\sim P \vee Q)$

$\Leftrightarrow (\sim P \vee Q)$

$\Leftrightarrow (P \to Q)$

$S_2 = P \to (Q \to (P \to Q))$

$\Leftrightarrow P \to T$ $[\because Q \to (P \to Q)$ is a tautology$]$

$\Leftrightarrow T$

$\therefore S_1 \Rightarrow S_2$

12. Ans: (c)

Sol: **S_1**: Condition proof:

1. $(a \vee b) \to c$ premise
2. $\underline{c \to (d \wedge e)}$ premise

$(a \to d)$ conclusion

3. a new premise to apply conditional proof
4. $a \vee b$ (3), addition
5. c (1), (4), modus pones
6. $d \wedge e$ (2), (5), modus pones
7. d (6), simplification

$\therefore S_1$ is valid

S_2: Indirect proof:

1. $(p \to q)$ premise
2. $\sim(p \wedge q)$ =premise
3. p new premise to apply indirect proof
4. q (1), (3), modus pones
5. $\sim p$ (2), (4), conjunctive syllogism
6. F (3), (5), contradiction

$\therefore S_2$ is valid

13. Ans: (c)

Sol: Quine's method:

Case 1: When P is true, the given formula has truth value ***true.***

Case 2: When P is false, the given formula has truth value ***true;*** whether Q is false or Q is true.

$\therefore$ The given formula is a tautology.

14. Ans: (d)

Sol: (a) Let $A = (p \to q) \to r$ and

$B = p \to (q \to r)$

Here, B is false only when p is true, q is true and r is false.

For this set of truth values, A is also false.

$\therefore A \to B$ is a tautology.

(b) Let $A = p \to (r \vee q)$ and

$B = (p \to r) \vee (p \to q)$

Here, B is false only when p is true, q is false and r is false.

For this set of truth values, A is also false.

$\therefore$ $A \to B$ is a tautology.

(c) Let $A = p \to (r \wedge q)$ and

$B = (p \to r) \vee (p \to q)$

Here, B is false only when p is true, q is false and r is false.

For this set of truth values, A is also false.

$\therefore$ $A \to B$ is a tautology.

(d) Let $A = p \to (q \to r)$ and

$B = (p \to q) \to r$

When p is false, q is true and r is false; then A is true and B is false.

$\therefore$ $A \to B$ is not a tautology.

15. Ans: (c)

Sol: S1: $\sim(p \vee q) \to (p \to q)$

Let us represent this as $A \to B$, where $A = \sim(p \vee q)$ and $B = (p \to q)$

Here, B is false only when p is true and q is false.

For this set of truth values, A is also false.

$\therefore$ S1 is a tautology

S2: $\sim(q \to \sim p) \to (p \to \sim q)$

Let us represent this as $A \to B$. where $A = \sim(q \to \sim p)$ and $B = (p \to \sim q)$

Here, B is false only when p is true and q is true.

For this set of truth values, A is true.

$\therefore$ S2 is not a tautology

S3: $\sim(p \to q) \to (p \vee q)$

Let us represent this as $A \to B$.

Here, B is false only when p is false and q is false.

In this case, A is also false.

$\therefore$ S3 is a tautology

S4: $(p \wedge \sim q) \to (p \leftrightarrow q)$

Let us represent this as $A \to B$.

Here, A is true only when p is true and q is false.

In this case, B is false.

$\therefore$ S4 is not a tautology

16. Ans: (b)

Sol: The truth table of a propositional function in n variables contain 2^n rows. In each row the function can be true or false.

By product rule, number of non equivalent propositional functions (different truth tables) possible $= 2^{(2^n)}$.

If we put n = 3, we get 256.

17. Ans: (b)

Sol: In the given formula, if we replace $(c \to \sim d)$ with $(\sim c \vee \sim d)$, then the given formula is a substitution instance of destructive dilemma.

$\therefore$ The given formula is valid.

18. Ans: (c)

Sol: If $(p \to q)$ is false then p is true and q is false.

(a) $((\sim p) \wedge q) \leftrightarrow (p \vee q)$

replacing p with true and q with false, we have $(F \wedge F) \leftrightarrow (T \vee F)$

$\Leftrightarrow F \leftrightarrow T$

$\Leftrightarrow F$

(b) $(p \leftrightarrow q)$

replacing p with true and q with false, we have

$$(T \leftrightarrow F) \Leftrightarrow F$$

(c) $(p \vee q) \vee r$

replacing p with true and q with false, we have

$$(T \vee F) \vee r \Leftrightarrow T$$

(d) $(p \wedge q) \vee r$

replacing p with true and q with false, we have

$$(T \wedge F) \vee r \Leftrightarrow r$$

19. Ans: (b)

Sol: S_1: When a is false and b is false, then the given formula has truth value false.

$\therefore$ S_1 is not valid

S_2: $(a \leftrightarrow b)$ is equivalent to $((a \wedge b) \vee (\neg a \wedge \neg b))$

$\therefore$ S_2 is valid

20. Ans: (a)

Sol: S_1: Let us denote the given formula by $P \Rightarrow Q$

Here, Q is false only when a is true, b is true and c is false.

For these truth values, P also has truth value false.

$\therefore$ S_1 is valid.

S_2: When a is true, b is false and c is false; the given formula has truth value false.

$\therefore$ S_2 is not valid.

21. Ans: (c)

Sol: S_1: $p \to (q \wedge r)$

$\Leftrightarrow \sim p \vee (q \wedge r)$

$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee r)$

$\Leftrightarrow (p \to q) \wedge (p \to r)$

S_2: $[(p \vee q) \to r]$

$\Leftrightarrow \sim(p \vee q) \vee r$

$\Leftrightarrow (\sim p \wedge \sim q) \vee r$

$\Leftrightarrow (\sim p \vee r) \wedge (\sim q \vee r)$

$\Leftrightarrow [(p \to r) \wedge (q \to r)]$

22. Ans: (a)

Sol: $[p \to (q \vee r)]$

$\Leftrightarrow \sim p \vee q \vee r$

$\Leftrightarrow \sim(p \wedge \sim q) \vee r$

$\Leftrightarrow [(p \wedge \sim q) \to r]$

$\therefore$ The given formula is valid.

23. Ans: (c)

Sol: S_1: $p \to (q \wedge r)$

$\Leftrightarrow \sim p \vee (q \wedge r)$

The dual is $\sim p \wedge (q \vee r)$

S_2: $p \leftrightarrow q$

$\Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

The dual is $(\sim p \vee \sim q) \wedge (q \vee p)$

24. Ans: (a)

Sol: $((a \wedge b) \to c)$

$\Leftrightarrow \sim(a \wedge b) \vee c$

$\Leftrightarrow \sim a \vee \sim b \vee c$

$\Leftrightarrow (\sim a \vee c) \vee (\sim b \vee c)$

$\Leftrightarrow ((a \to c) \vee (b \to c))$

25. Ans: (c)

Sol: Argument1: Let p: there was a ball game

q: travelling was difficult.

r: they arrived on time

The given argument in symbolic form is

1. $p \to q$	premise
2. $r \to \sim q$	premise
3. r	premise
$\sim p$	conclusion

Proof:

4. $\sim q$	(2), (3), modus pones
5. $\sim p$	(1), (4), modus tollens

$\therefore$ The argument is valid

Argument2:

Let p: jack misses many classes through illness

q: he fails high school.

r: he is uneducated.

s: jack reads a lot of books

t: jack is smart

The given argument in symbolic form is

1. $p \to q$	premise
2. $q \to r$	premise
3. $s \to \sim r$	premise
4. $p \wedge s$	premise
$\therefore$ t	conclusion

Proof:

5. $p \to r$	(1), (2), transitivity
6. p	(4), simplification
7. s	(4), simplification
8. r	(5), (6), modus ponens
9. $\sim r$	(3), (7), modus ponens

(8) and (9) contradict each other.

$\therefore$ The premises are inconsistent.

Hence, the argument is valid.

26. Ans: (b)

Sol: $(p \wedge (\sim r \vee q \vee \sim q)) \vee ((r \vee t \vee \sim r) \wedge \sim q)$

$\Leftrightarrow (p \wedge T) \vee (T \wedge \sim q)$

$\Leftrightarrow p \vee \sim q$

27. Ans: (a)

Sol: $(p \vee (p \wedge q) \vee (p \wedge q \wedge \sim r)) \wedge ((p \wedge r \wedge t) \vee t)$

In boolean algebra notation

$(p + (p.q) + (p.q.\sim r)).((p.r.t) + t)$

$= p.(1 + q + \bar{r}).t\,(p.r + 1)$

$= p.t$

$= p \wedge t$

28. Ans: (a)

Sol: The given formula is equivalent to the following argument

1. p	premise
2. $(p \to q)$	premise
3. $(s \vee r)$	premise
4. $(r \to \sim q)$	premise
$\therefore$ $(s \vee t)$	conclusion

Proof:

5. q	(1), (2), modus pones
6. $\sim r$	(4), (5), modus tollens
7. s	(3), (6),disjunctive syllogism
8. $s \vee t$	(7), addition

The argument is valid.

$\therefore$ The given formula is valid.

29. Ans: (a)

Sol: The given formula is equivalent to the following argument

1. $((\sim p \vee q) \rightarrow r)$ premise
2. $(r \rightarrow (s \vee t))$ premise
3. $(\sim s \wedge \sim u)$ premise
4. $\underline{(\sim u \rightarrow \sim t)\}}$ premise

$\therefore$ p conclusion

Proof:

5. $\sim s$ (3), simplification
6. $\sim u$ (3), simplification
7. $\sim t$ (4), (6), modus ponens
8. $\sim s \wedge \sim t$ (5), (7), conjunction
9. $\sim(s \vee t)$ (8), equivalence
10. $\sim r$ (2), (9), modus tollens
11. $\sim(\sim p \vee q)$ (1), (10), modus tollens
12. $p \wedge \sim q$ (11), equivalence
13. p (12), simplification

The argument is valid.

$\therefore$ The given formula is valid.

30. Ans: (a)

Sol: The argument is valid by the rule of constructive dilemma.

31. Ans: (a)

Sol: The given formula is equivalent to the following argument

1. $(\sim p \leftrightarrow q)$ premise
2. $(q \rightarrow r)$ premise
3. $\underline{(\sim r)}$ premise

$\therefore$ p conclusion

Proof:

4. $\sim q$ (2), (3), modus tollens
5. $(\sim p \rightarrow q)$ (1), simplification
6. p (4), (5), modus tollens

The argument is valid.

$\therefore$ The given formula is valid.

32. Ans: (c)

Sol: **S_1**: $(a \wedge b) \vee c$

$= (a.b) + c$

$= (a + c).(b + c)$

$= \{(a + c) + (b.\bar{b})\}.\ \{(a.\bar{a}) + (b + c)\}$

$= (a + b + c).(a + \bar{b} + c).(\bar{a} + b + c)\}$

$= (a \vee b \vee c) \wedge (a \vee \sim b \vee c) \wedge (\sim a \vee b \vee c)$

which is the required conjunctive normal form.

S_2: $a \wedge (b \leftrightarrow c)$

$= a \wedge \{(b \wedge c) \vee (\sim b \wedge \sim c)\}$

$= (a \wedge b \wedge c) \vee (a \wedge \sim b \wedge \sim c)$

Which is the required disjunctive normal form.

33. Ans: (c)

Sol: (I). p: It is not raining

q: Rita has her umbrella.

r: Rita does not get wet.

The given argument in symbolic form is

1. $p \vee q$ premise
2. $\sim q \vee r$ premise
3. $\underline{\sim p \vee r}$ premise

$\therefore$ r conclusion

The argument is valid, by the rule of dilemma.

(II). p: Superman were able to prevent evil

q: Superman were willing to prevent evil

r: Superman would prevent evil

s: Superman would be impotent.

t: Superman would be malevolent.

u: Superman exist.

The given argument in symbolic form is

1. $(p \wedge q) \rightarrow r$	premise
2. $\sim p \rightarrow s$	premise
3. $\sim q \rightarrow t$	premise
4. $\sim r$	premise
5. $u \rightarrow (\sim s \wedge \sim t)$	premise
$\therefore \sim u$	conclusion

Proof:

6. $\sim(p \wedge q)$	(1), (4), modus tollens
7. $\sim p \vee \sim q$	(6), demorgan's law
8. $(s \vee t)$	(2), (3), (7), constructive dilemma
9. $\sim u$	(5), (8), modus tollens

$\therefore$ The argument is valid.

First Order Logic

34. Ans: (c)

Sol: A statement is a predicate if we can replace every variable in the statement by any instance in its domain to form a proposition.

S_1 is false for any real number.

$\therefore$ S_1 is a predicate

S_2 is true for some real numbers which are odd integers.

$\therefore$ S_2 is a predicate

35. Ans: (b)

Sol: $P(x, y) = (x \vee y) \rightarrow z$

$\sim P(x, y) = (x \vee y) \wedge \sim z$

The negation of $\forall x\ \exists y\ P(x, y)$ is $\exists x\ \forall y\ ((x \vee y) \wedge \sim z)$

36. Ans: (d)

Sol: 1. If we choose $y = 17 - x$ then ϕ is true.

2. When $x = 17$, there is no positive integer y which satisfies ϕ

3. When $x = 17$, there is no positive integer y which satisfies ϕ

4. If we choose $y = 17 - x$ then ϕ is true.

37. Ans: (a)

Sol: In general, the universal quantifier take the connective $\rightarrow$ and the existential quantifier take the connective $\wedge$.

The given formula in symbolic form, can be written as

$$\forall n\ [(n > 1) \rightarrow \exists x\ \{p(x) \wedge (n < x < 2n)\}]$$

38. Ans: (a)

Sol: The given statement can be expressed as

$$\forall n\ [(n > 1) \rightarrow \exists x\ \{p(x) \wedge (n < x < 2n)\}]$$

Its negation is

$\sim\{\forall n\ [(n>1) \rightarrow \exists x\{p(x) \wedge (n < x < 2n)\}]$

$\Leftrightarrow \exists n \sim [(n>1) \rightarrow \exists x\ \{p(x) \wedge (n < x < 2n)\}]$

$\Leftrightarrow \exists n\ [(n>1) \wedge \sim \exists x\ \{p(x) \wedge (n<x< 2n)\}]$

$\Leftrightarrow \exists n\ [(n>1) \wedge \forall x \sim\{p(x) \wedge (n<x< 2n)\}]$

$\Leftrightarrow \exists n\ [(n>1) \wedge \forall x\ \{p(x) \rightarrow \sim(n<x< 2n)\}]$

$\Leftrightarrow \exists n\ [(n > 1) \wedge \forall x\ \{p(x) \rightarrow ((x \leq n) \vee (x \geq 2n))\}]$

39. Ans: (d)

Sol: I) Let D (x) : x is a doctor

C (x) : x is a college graduate

G (x) : x is a golfer

The given argument can be written as

1) $\forall_x \{D(x) \rightarrow C(x)\}$

2) $\underline{\exists_x \{D(x) \wedge \sim G(x)\}}$

$\therefore \exists_x \{G(x) \wedge \sim C(x)\}$

3) $\{D(a)\wedge\sim G(a)\}$ — 2), Existential Specification

4) $\{D(a)\rightarrow C(a)\}$ — 1), Universal Specification

5) D (a) — 3), Simplification

6) $\sim$ G (a) — 3), Simplification

7) C (a) — 4), 5), Modus ponens

8) $\{C(a) \wedge \sim G(a)\}$ — 7), 6), Conjunction

9) $\exists_x\{G(x)\wedge\sim C(x)\}$ — 8), Existential Generalization

The argument is not valid

II) Let M (x) x is a mother

N (x) x is a male

P (x) : x is a politician

The given argument is

1) $\forall_x \{M(x) \rightarrow \sim N(x)\}$

2) $\underline{\exists_x \{N(x) \wedge P(x)\}}$

$\therefore \exists_x \{P(x) \wedge \sim M(x)\}$

3) N (a)$\wedge$P (a) — 2), Existential Specification

4) M (a)$\rightarrow\sim$N (a) — 1), Universal Specification

5) N (a) — 3), Simplification

6) P (a) — 3), Simplification

7) $\sim$ M (a) — 4), 5), Modus tollens

8) $\{P(a)\wedge\sim M(a)\}$ — 6), 7), Conjunction

9) $\exists_x\{P(x) \wedge \sim M(x)\}$ — 8), Existential Generalization

$\therefore$ The argument is valid.

40. Ans: (c)

Sol: **S_1:** L.H.S = $\exists x [P(x) \vee Q(x)]$

$\Rightarrow P(a) \vee Q(a)$ [Existential specification]

$\Rightarrow \exists x\, P(x) \vee Q(a)$ [Existential Generalization]

$\Rightarrow(\exists x P(x)\vee\exists x Q(x))$ [Existential Generalization]

$\therefore$ L.H.S. $\Rightarrow$ R.H.S

Now, R.H.S = $(\exists x\, P(x) \vee \exists x\, Q(x))$

$\Rightarrow P(a) \vee \exists x\, Q(x)$ [Existential Specification]

$\Rightarrow P(a) \vee Q(b)$ [Existential Specification]

$\Rightarrow P(a) \vee Q(b) \vee P(b) \vee Q(a)$ [Addition]

$\Rightarrow [P(a) \vee Q(a)] \vee [P(b) \vee Q(b)]$

[By commutative and associative laws]

$\Rightarrow \exists x [P(x) \vee Q(x)] \vee \exists x [P(x) \vee Q(x)]$

[By Existential Generalization]

$\Rightarrow \exists x [P(x) \vee Q(x)]$ [By Idempotent law]

$\therefore$ R.H.S. $\Rightarrow$ L.H.S.

Hence, L.H.S. $\Leftrightarrow$ R.H.S.

S_2: Try your self (Similar to S_1)

41. Ans: (a)

Sol: S_1: Proof by contradiction

1.	$(\forall x\, P(x) \vee \forall x\, Q(x))$	Premise
2.	$\sim\{\forall x[P(x)\vee Q(x)]\}$	New premise for indirect proof
3.	$\exists x [\sim P(x) \wedge \sim Q(x)]$	(2), equivalence
4.	$[\sim P(a) \wedge \sim Q(a)]$	(3),existential generalization
5.	$\sim P(a)$	(4), simplification
6.	$\sim Q(a)$	(4), simplification
7.	$\exists x \sim P(x)$	(5), existential generalization
8.	$\exists x \sim Q(x)$	(6), existential generalization
9.	$\exists x\sim P(x)\wedge\exists x\sim Q(x)$	(7),(8),conjunction
10.	$\sim(\forall x\, P(x) \vee \forall x\, Q(x))$	(9), equivalence

(1) and (10), contradict each other.

$\therefore$ S_1 is valid

S_2: $\forall x[P(x) \vee Q(x)] \Rightarrow (\forall x\, P(x) \vee \forall x\, Q(x))$

We can disprove the above statement by counter example:

Let the universe be {a, b}. Suppose P(a) is true, P(b) is false, Q(a) is false and Q(b) is true.

For these values the given statement is false.

$\therefore$ S_2 is not valid

42. Ans: (a)

Sol: The given statement can be written as

$$\exists x\{S(x) \wedge M(x) \wedge \sim H(x)\}$$

It's negation is

$$\forall x \;\{\sim S(x) \vee \sim M(x) \vee H(x)\}$$

(By demorgan's law)

$$\Leftrightarrow \forall x\; \{\{S(x) \wedge M(x)\} \rightarrow H(x)\}$$

$$(\because (P \vee Q) = (\sim P \rightarrow Q))$$

43. Ans: (d)

Sol: I. Let U = {a, b} be the universe of discourse, such that

P(a) is true, P(b) is false

Q(a) is false, and Q(b) is true

Now, L.H.S of I is true

And R.H.S of I is false

$\therefore$ The statement I is not valid.

II. Let U = {a, b} be the universe of discourse, such that

P(a) is true and P(b) is false

Q(a) is false and Q(b) is true

Now,

The antecedent of II is true and consequent is false

$\therefore$ The statement II is not valid.

44. Ans: (c)

Sol: I. The premises are

1. $\forall x[P(x) \rightarrow \{Q(x) \wedge S(x)\}]$
2. $\forall x\, \{P(x) \wedge R(x)\}$
3. $P(a)\rightarrow\{Q(a)\wedge S(a)\}$ (1) universal specification
4. $P(a) \wedge R(a)$ (2) universal specification
5. $P(a)$ (4), simplification
6. $Q(a) \wedge S(a)$ (3),(5) modus ponens
7. $S(a)$ (6), simplification
8. $R(a)$ (4), simplification
9. $R(a) \wedge S(a)$ (8), (7), Conjunction
10. $\forall x\{R(x) \wedge S(x)\}$ (9) U.G

$\therefore$ Argument I is valid.

II. The given argument contains only universal quantifier. We can drop the quantifiers in the argument.

Now the premises are

1. $\{P(x) \vee Q(x)\}$
2. $\{\sim P(x) \wedge Q(x)\} \rightarrow R(x)$

The conclusion is $\{\sim R(x) \rightarrow P(x)\}$.

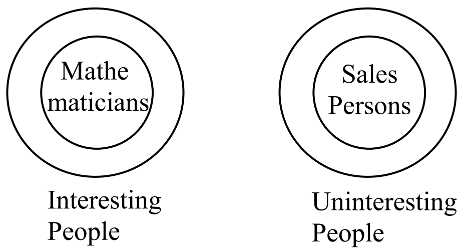
Let us apply conditional proof

3. $\sim R(x)$ new premise
4. $\{P(x) \vee \sim Q(x)\}$ (2),(3), modus tollens
5. $\{P(x) \vee Q(x)\} \wedge \{P(x) \vee \sim Q(x)\}$ (2, (4), conjunction
6. $P(x) \vee \{Q(x) \wedge \sim Q(x)\}$ (5), Dist. Law
7. $P(x) \vee F$ (6)
8. $P(x)$ (7)

$\therefore$ Argument II is valid.

45. Ans: (b)

Sol: The given statements can be represented by the following venn diagram



From venn diagram, option(c) does not follow.

46. Ans: (a)

Sol: The given statements can be expressed as

$$\exists x\ \{D(x) \wedge \sim S(x)\}$$

It's negation is

$$\forall x\ \{(D(x) \rightarrow S(x)\}$$

47. Ans: (c)

Sol: S_1: $\exists x\ \{P(x) \rightarrow Q(x)\}$

$\Leftrightarrow$ $\exists x\ \{\sim P(x) \vee R(x)\}$

$\Leftrightarrow$ $\{\exists x \sim P(x) \vee \exists x\ R(x)\}$

$\Leftrightarrow$ $\{\forall x\ P(x) \rightarrow \exists x\ Q(x)\}$

$\therefore$ S_1 is true

S_2: $\exists x\ \forall y\ P(x, y)$

$\Rightarrow$ $\forall y\ P(a, y)$ for some a

$\Rightarrow$ P(a, b) is true for all b

$\Rightarrow$ $\exists x\ P(x, b)$ is true for all b

$\Rightarrow$ $\forall y\ \exists x\ P(x, y)$ is true

$\therefore$ S_2 is true

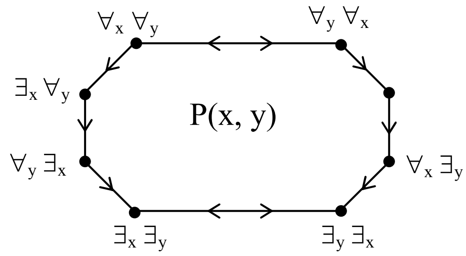
48. Ans: (c)

Sol: $\exists_y \forall_x P(y, x) \rightarrow \forall_y \exists_x P(x, y)$

$\Leftrightarrow \exists_x \forall_y P(x, y) \rightarrow \forall_y \exists_x P(x, y)$

($\because$ x and y are dummy variables)

Which is valid as per the relationship diagram shown below



The remaining options are not true as per the diagram.

49. Ans: (d)

Sol: **S_1 is true**

Once we select any integer n, the integer m = 5 – n does exist and

$n + m = n + (5 - n) = 5$

S_2 is true, because if we choose n=1 the statement nm = m is true for any integer m.

S_3 is false, for example, when m = 0 the statement is false for all n

S_4 is false, here we cannot choose n = – m, because m is fixed.

50. Ans: (a)

Sol: S_1: L.H.S $\Leftrightarrow \exists x\ (A(x) \rightarrow B(x))$

$\Leftrightarrow \exists x\ (\sim A(x) \vee B(x))$, E_{16}

$\Leftrightarrow \exists x \sim A(x) \vee \exists x\ B(x)$, E_{23}

$\Leftrightarrow \forall x\ A(x) \rightarrow \exists x\ B(x)$, E_{16}

= R.H.S

S_2: L.H.S $\Leftrightarrow \{\forall x \sim A(x) \vee \forall x\ B(x))$

$\Rightarrow \forall x\ (\sim A(x) \vee B(x))$

$\Rightarrow \forall x\ (A(x) \rightarrow B(x))$ = R.H.S

But converse is not true

$\therefore$ S_2 is false

S_3 valid equivalence

S_4 is not valid (converse is not true)

51. Ans: (b)

Sol: (a) The given formula is valid by conditional proof, if the following argument is valid.

(1) $\forall_x \{ P(x) \rightarrow Q(x) \}$

$\underline{(2)\ \forall_x P(x)}$ new premise to apply C.P

$\therefore \forall_x Q(x)$

Proof:

(3) $P(a) \rightarrow Q(a)$ (1), U.S

(4) $P(a)$ (2), U.S

(5) $Q(a)$ (3), (4), M.P

(6) $\forall_x Q(x)$ (5), U.S

$\therefore$ The given formula is valid (C.P)

(b) The statement need not be true.

Let c and d are two elements in the universe of discourse, such that P(c) is true and P(d) is false and Q(c) is false and Q(d) is false.

Now, the L.H.S of the given statement is true but R.H.S is false.

$\therefore$ The given statement is not valid.

(c) $\forall_x (P(x) \vee Q(x)) \Rightarrow (\forall_x P(x) \vee \exists_x Q(x))$

Indirect proof:

1) $\forall_x (P(x) \vee Q(x))$ Premise

2) $\sim (\forall_x P(x) \vee \exists_x Q(x))$

New premise to apply Indirect proof

3) $\exists_x \sim P(x) \wedge \forall_x \sim Q(x)$

(2), Demorgan's law

4) $\exists_x \sim P(x)$ (3), Simplification

5) $\forall_x \sim Q(x)$ (3), Simplification

6) $\sim P(a)$ (4), E.S

7) $\sim Q(a)$ (5), U.S

8) $(\sim P(a) \wedge \sim Q(a))$ (6), (7),Conjunction

9) $\sim(P(a) \vee Q(a))$ (8), Demorgan's law

10) $(P(a) \vee Q(a))$ (1), U.S

11) F (9), (10), Conjunction

$\therefore$ valid (Indirect proof)

S_2: The argument is

1) $\forall_x \forall_y (P(x, y) \rightarrow W(x, y))$

$\underline{2) \sim W(a, b)}$

$\therefore \sim P(a, b)$

(d) $\forall_x \{ P(x) \vee Q(x) \}$ follows from $(\forall_x P(x) \vee \forall_x Q(x))$

$\therefore$ The given statement is valid.

52. Ans: (b)

Sol: S_1 is false. For x = 0. There is no integer y such that '0 is a divisor of y'

S_2 is true. If we choose, x = 1, then the statement is true for any integer y

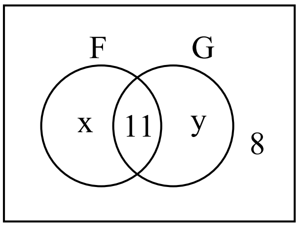
S_3 is true. If we choose, x = 1, then the statement is true for any integer y

S_4 is false, because there is no integer y which is divisible by all integers.

2. Combinatorics

01. Ans: 31

Sol:

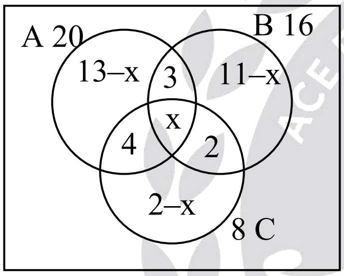


$x + y + 11 + 8 = 50$

$x + y = 31$

02. Ans: 2

Sol:



$13 - x + 11 - x + 2 - x + 3 + 2 + 4 + x = 31$

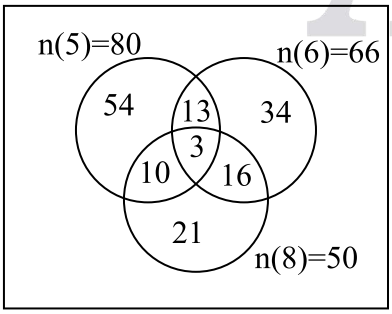
$35 - 2x = 31$

$2x = 4$

$x = 2$

03. Ans: 249

Sol:



$$n(5) = \frac{400}{5} = 80$$

$$n(6) = \frac{400}{6} = 66$$

$$n(8) = \frac{400}{8} = 50$$

$$n(5 \cap 6) = \frac{400}{30} = 13$$

$$n(5 \cap 8) = \frac{400}{40} = 10$$

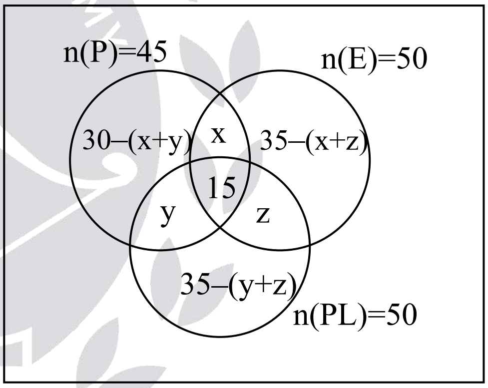
$$n(6 \cap 8) = \frac{400}{24} = 16$$

$$n(5 \cap 6 \cap 8) = \frac{400}{120} = 3$$

$400 - 151 = 249$

04. Ans: (b)

Sol:



$30 - (x + y) + (35 - (x + z)) + (35 - (y + z))$

$+ x + y + z + 15 = 80$

$x + y + z = 35$

$$30 - (x + y) + 35 - (x + z) + 35 - (y + z)$$

$$= 80 - (x + y + z + 15)$$

$= 30$

$30 - (x+y) + 35 - (x+z) + 35 - (y+z) + x + y + z$

$= 30 + 35 = 65$

05. Ans: 86

06. Ans: (c)

Sol: If n is even, then number of bit strings of length n which are palindromes $= 2^{\frac{n}{2}}$.

If n is odd, then number of bit strings of length n which are palindromes $= 2^{\frac{n+1}{2}}$

$\therefore$ Required number of bit strings $= 2^{\left\lceil \frac{n}{2} \right\rceil}$.

07. Ans: 3439

Sol: Number of integers between 1 and 10,000 without digit 7 = $(9^4 - 1) + 1$

Required number of integers = $10,000 - 9^4$

$= 3439$

08. Ans: 64

Sol: In a binary matrix of order 3 × 3 we have '9' elements each element we can choose '2' ways.

By using symmetric relations we have $2^{\frac{n(n-1)}{2}} \times 2^n$ matrices are possible

$\therefore 2^{\frac{3(3-1)}{2}} \times 2^3$

$= 64$

09. Ans: 188

Sol: An English movie and a telugu movie can be selected in (6)(8) = 48 ways

A telugu movie and a hindi movie can be selected in (8).(10) = 80 ways

A hindi movie and an English movie can be selected in (10)(6) = 60 movies

Required number of ways = 48 + 80 + 60

= 188

10. Ans: 262

Sol: The total number of integers 1 through 1000 with atleast one repeated digit

$= 1000 - ({}^9C_1 + {}^9C_1 \times {}^9C_1 + {}^9C_1 \times {}^9C_1 \times {}^8C_1)$

$= 1000 - 738$

$= 262$

11. Ans: 2187

Sol: Number of 4 digit integers with digit '0' appearing exactly once

$= ({}^9C_1 + {}^9C_1 \times {}^9C_1 \times 1) + ({}^9C_1 \times {}^9C_1 \times {}^9C_1 \times 1)$

$+ ({}^9C_1 + {}^9C_1 \times {}^9C_1 \times 1)$

$= 729 + 729 + 729$

$= 2187$

12. Ans: 2940

Sol: Consider an integer with 5 digits.

Digit 3 can appear in 5 ways

Digit 4 can appear in 4 ways

Digit 5 can appear in 3 ways

Each of the remaining digits we can choose in 7 ways.

By product rule,

Required number of integers

= (5)(4)(3)(7)(7) = 2940

13. Ans: (a)

Sol: Since it is a single elimination tournament so we need (n–1) matches to decide winner.

14. Ans: (c)

Sol: Let P, Q are subsets of S so that $P \cap Q = \phi$.

So each element of P, Q are having '3' possibilities

Case (i) : Elements are in P but not in Q

Case (ii) : Elements are in Q but not in P

Case (iii): Elements are not in P and not in Q

$\therefore$ Number of possibilities = 3^n

15. Ans: 151200

Sol: Required number of ways

= Number of ways we can map the 6 persons to 6 of the 10 books

= P(10,6)

= 151200

16. Ans: 2880

Sol: First girls can sit around a circle in $\angle 4$ ways.

Now there are 5 distinct places among the girls, for the 4 boys to sit.

Therefore, the boys can sit in P(5, 4) ways.

By product rule,

Required number of ways = $\angle 4$.P(5, 4)

= 2880

17. Ans: 1152

Sol: Consider 8 positions in a row marked 1, 2, 3,….., 8.

Case 1: Boys can sit in odd numbered positions in $\angle 4$ ways and girls can sit in even numbered positions in $\angle 4$ ways.

Case 2: Boys can sit in even numbered positions in $\angle 4$ ways and girls can sit in odd numbered positions in $\angle 4$ ways.

Required number of ways

= $\angle 4.\angle 4 + \angle 4.\angle 4 = 1152$

18. Ans: 325

Sol: Number of signals we can generate using 1 flag = 5

Number of signals we can generate using two flags = P(5,2) = 5.4 = 20 and so on.

Required number of signals

= 5 + P(5,2) + P(5,3) + P(5,4) + P(5,5)

= 325

19. Ans: (a)

Sol: Each book we can give in 10 ways.

By product rule, required number of ways

= 10^6

20. Ans: 243

Sol: Each digit of the integer we can choose in 3 ways.

By product rule,

Required number of integers = 3^5

= 243

21. Ans: 12600

Sol: Required number of permutations

$$= \frac{\angle 10}{\angle 2.\angle 3.\angle 4} = 12,600$$

22. Ans: 360

Sol: Required number of strings = Number of permutations possible with seven 0's, two 1's and one 2

$$= \frac{\angle 10}{\angle 7.\angle 2.\angle 1} = 360$$

23. Ans: 2520

Sol: Required number of ways

= number of ordered partitions

$$= \frac{\angle 10}{\angle 3 . \angle 2 . \angle 5} = 2520$$

24. Ans: 945

Sol: Required number of ways = Number of unordered partitions of a set into 5 subjects of same size $= \dfrac{\angle 10}{(\angle 2 . \angle 2 . \angle 2 . \angle 2 . \angle 2) . \angle 5}$

= 945

25. Ans: 150

Sol: Required number of ways = Number of onto functions possible from persons to rooms

$= 3^5 - C(3, 1)\ 2^5 + C(3, 2) .\ 1^5$

$= 243 - 3\ (32) + 3$

$= 150$

26. Ans: 5400

Sol: Suppose we are choosing 4 men from 6 men then ${}^6C_4$.

And each men pair with women.

First men can choose any one women of 6 women and second men can choose any one women of 5 women by continuing this process,

Total number of ways= ${}^6C_4 \times (6 \times 5 \times 4 \times 3)$

$= 5400$

27. Ans: 45

Sol: For maximum number of points of intersection, we have to draw 10 lines so that no three lines are concurrent. In that case, each point corresponds to a pair of distinct straight lines.

$\therefore$ Maximum number of points of intersection = number of ways we can choose two straight lines out of 10 straight lines = C (10, 2) = 45

28. Ans: 120

Sol: The 3 zeros can appear in the sequence in C(10,3) ways. The remaining 7 positions of the sequence can be filled with ones in only one way.

Required number of binary sequences

= C(10, 3).1

= 120

29. Ans: 35

Sol: Consider a string of 6 ones in a row. There are 7 positions among the 6 ones for placing 4 zeros. The 4 zeros can be placed in C(7,4) ways.

Required number of binary sequences

= C(7, 4) = C(7, 3)

= 35

30. Ans: 126

Sol: Number of 5 digit integers are possible so that in each of these integers every digit is less than the digit on its right

$= {}^{10}C_5 - {}^{9}C_4$

31. Ans: (b)

Sol: We have 2n persons.

Number of handshakes possible with 2n persons = C(2n,2)

If each person shakes hands with only his/her spouse, then number of handshakes possible = n

Required number of handshakes

= C(2n, 2) – n = 2n(n –1)

32. Ans: 1092

Sol: In a chess board, we have 9 horizontal lines and 9 vertical lines. A rectangle can be formed with any two horizontal lines and any two vertical lines.

Number of rectangles possible

= C(9,2). C(9,2) = (36)(36) = 1296

Number of squares in a chess board

= $1^2 + 2^2 + 3^2 + \ldots + 8^2 = 204$

Every square is also a rectangle.

Required number of rectangles which are not squares = 1296 – 204 = 1092

33. Ans: (A)

Sol: Let a_1, a_2, a_3, a_4, a_5 be the dates of the five days of January that the student will spend in the hospital, in increasing order. Note that the requirement that there are no two consecutive numbers among the a_i, and $1 \le a_1 < a_2 - 1 < a_3 - 2 < a_4 - 3 < a_2 - 4 \le 27$. In other words, there is an obvious bijection between the set of 5 element subsets of {1, 2, ….,31} containing no two consecutive elements and the set of 5 element subsets of {1, 2, …..., 27}.

∴ Required number ways = C(27, 5).

34. Ans: 210

Sol: We can choose 6 persons in C(10, 6) ways

We can distinct 6 similar books among the 6 persons in only one ways

∴ Required number of ways

= C(10, 6). 1

= C(10, 4) = 210

35. Ans: 14656 No range

Sol: Number of committees with all males

= C (12, 5)

Number of committees with all females

= C(8, 5)

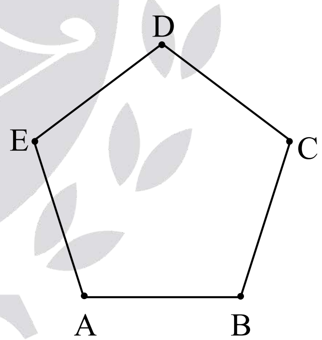
Required number of committees

= C(20, 5) – C(12, 5) – C(8, 5)

= 14656

36. Ans: (c)

Sol:



The number of triangles formed by joining the vertices of n – sided polygon $^nC_3$

Number of triangles having one side common with that of the polygon (n – 4) n

Number of triangles having two sides common with that of polygon = n

The number of triangles having no side common with that polygon = x

Total number of triangles = (n – 4)n + n + x

$\Rightarrow {}^{n}C_3 - (n-4)n - n = x$

$$\Rightarrow x = \frac{n(n-1)(n-2)}{6} - n^2 + 3n$$

$$\Rightarrow x = \frac{n(n-4)(n-5)}{6}$$

37. Ans: 1001

Sol: Required number of ways = V(5,10)

V(n,k) = C(n–1+k, k)

$\Rightarrow$ V(5,10) = C(14,10)

= C(14,4)

= 1001

38. Ans: 455

Sol: To meet the given condition, let us put 1 ball in each box, The remaining 12 balls we can distribute in V(4,12) ways.

Required number of ways = V(4,12).1

= C(15,12) = C(15,3) = 455

39. Ans: 1695

Sol: let $w = x_1 + 12$

$x = x_2 + 12$

$y = x_3 + 12$

$z = x_4 + 12$

where $x_1, x_2, x_3, x_4 \geq 0$

Given $12 \leq w + x + y + z \leq 14$

Let $w + x + y + z + t = 14$ where $t > 0$

$w + x + y + z + t = 13$

$w + x + y + z + t = 12$

So, total number of solutions

$= {}^{18}C_4 + {}^{17}C_4 + {}^{16}C_4$

$= 1695$

40. Ans: 10

Sol: $x_1 + x_2 + x_3 = 8$

$x_1 \geq 3$

$x_2 \geq -2$

$x_3 \geq 4$

Let $x_1 = P + 3$

$x_2 = Q - 2$

$x_3 = R + 4$

$P \geq 0, Q \geq 0, R \geq 0$

$P + 3 + Q - 2 + R + 4 = 8$

$P + Q + R = 3$

Number of solutions = ${}^{5}C_2 = 10$

41. Ans: 63

Sol: Let X_1 = units digit, X_2 = tens digit and X_3= hundreds digit

Number of non negative integer solutions to the equation

$X_1 + X_2 + X_3 = 10$ is

V (3, 10) = C (12, 10) = C (12, 2) = 66

We have to exclude the 3 cases

where $X_i = 10$ (i = 1, 2, 3)

Required number of integers = 66 – 3 = 63

42. Ans: (b)

Sol: We can treat each student and the adjacent empty seat as a single width-2 unit.

Together, these units take up 2k seats, leaving n – 2k extra empty seats to distribute among the students.

The students can sit in alphabetical order in only one way.

Now, there are k + 1 distinct spaces among the students to arrange (n – 2k) empty chairs.

The required number of ways

= V(k + 1, n – 2k)

= C(k + n – 2k, n – 2k)

= C(n – k, k)

43. Ans: 210

Sol: Let $x = x' + 1$, $y = y' + 1$, $z = z' + 1$ and $w = w' + 1$

Then the given inequality becomes

$(x' + y' + z' + w') \le 6$

where x', y', z' and w' are non-negative integers.

The number of solutions to the inequality are same as the number of non-negative integer solutions to equation

$$(x' + y' + z' + w' + v') = 6$$

The required number of solutions

= C (n – 1 + k, k)

Where n = 5 and k = 6

= C(10, 6)

= 210

44. Ans: 10800

Sol: The six symbols can be arranged in $\angle 6$ ways. To meet the given condition,

Let us put 2 blanks between every pair of symbols.

The number of ways we can arrange the remaining two blanks = V(5, 2)

= C(5 – 1 +2, 2) = 15

$\therefore$ Required number of ways = $\angle 6$.(15)

= (720) .(15) = 10,800

45. Ans: (d)

Sol: Average number of letters received by an apartment $= A = \dfrac{410}{50}$

$= 8.2$

Here, $\lceil A \rceil = 9$ and $\lfloor A \rfloor = 8$

By pigeonhole principle, S_1 and S_2 are necessarily true.

S_5 follows from S_1 and S_6 follows from S_2.

S_3 and S_4 need not be true.

46. Ans: (c)

Sol: Average number of passengers per bus

$= \dfrac{2000}{30} = 66.66$

By Pigeon hole principle, some buses contain atleast 67 passengers and some buses contain atmost 66 passengers.

i.e., some buses contain atleast 14 empty seats.

$\therefore$ Both S_1 and S_2 are true.

47. Ans: 97

Sol: If we have n pigeon holes, then minimum number of pigeons required to ensure that atleast (k+1) pigeons belong to same pigeonhole = kn + 1

For the present example, n=12 and k+1= 9

Required number of persons = kn + 1

= 8(12) +1 = 97

48. Ans: 26

Sol: By Pigeonhole principle,

Required number of balls = kn + 1

= 5(5) + 1 = 26

49. Ans: 39

Sol: The favorable colors to draw 9 balls of same color are green, white and yellow.

We have to include all red balls and all green balls in the selection of minimum number of balls. For the favorable colors we can apply pigeonhole principle.

Required number of balls = 6 + 8 + (kn + 1)

Where $k + 1 = 9$

and $n = 3$

$6 + 8 + (8 \times 3 + 1) = 39$

50. Ans: 4

Sol: Suppose $x \geq 6$,

Minimum number of balls required = kn +1=16

where $k + 1 = 6$ and $n = 3$.

$\Rightarrow 5(3) + 1 = 16$

Which is impossible

$\therefore x < 6$

Now, minimum number of balls required

$= x + (kn + 1) = 15$

where $k + 1 = 6$ and $n = 2$

$\Rightarrow x + 5(2) + 1 = 15$

$\Rightarrow x = 4$

51. Ans: 7

Sol: For sum to be 9, the possible 2-element subsets are{0,9}, {1, 8},{2, 7},{3, 6},{4, 5}

If we treat these subsets as pigeon holes, then any subset of S with 6 elements can have at least one of these subsets.

Since we need two such subsets, the required value of k = 7.

52. Ans: 7

Sol: If we divide a number by 10 the possible remainders are 0, 1, 2, …, 9.

Here, we can apply pigeonhole principle. The 6 pigeonholes are

{0}, {5}, {1, 9}, {2, 8}, {3,7}, {4, 6}

In the first two sets both $x + y$ and $x - y$ are divisible by 10. In the remaining sets either $x + y$ or $x - y$ divisible by 10.

$\therefore$ The minimum number of integers we have to choose randomly is 7.

53. Ans: 20

Sol: Let P_i for $1 \leq i \leq 4$ be the set of printers, and C_j for $1 \leq j \leq 8$ be the set of computers. Connect C_k to P_k for $1 \leq k \leq 4$. Again, connect C_k for $5 \leq k \leq 8$ to P_i for $1 \leq i \leq 4$. Clearly, one requires 20 cables. Assume that there are fewer than 20 connections between computers and printers. Hence, some printers would be connected to at most $\left\lfloor \frac{19}{4} \right\rfloor = 4$ computers. Thus, the remaining 3 printers are not enough to allow the other 4 computers to simultaneously access different printers.

Solution for Q54, Q55 and Q56

Sol: Apply Euler's formula $\phi(n)$

If $n = P_1^{\alpha_1} \times P_2^{\alpha_2} \times ....P_n^{\alpha_n}$

Where $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n \in N$

$P_1, P_2 \ldots\ldots\ldots P_n$ are distinct prime.

Then

$$\phi(n) = n\left(1-\frac{1}{P_1}\right)\left(1-\frac{1}{P_2}\right)\ldots\ldots\ldots\left(1-\frac{1}{P_n}\right)$$

57. Ans: (D)

Sol: In this case, m is relatively prime to p^k if and only if m is not divisible by p.

Required number of integers = Euler function of $p^k = \phi(p^k) = \frac{p-1}{p} \cdot p^k = p^{k-1}(p-1)$.

58. Ans: 265

Sol: Required number of 1 – 1 functions

= number of derangements possible with 6 elements

$$= D_6 = \angle 6\left(\frac{1}{\angle 2} - \frac{1}{\angle 3} + \frac{1}{\angle 4} - \frac{1}{\angle 5} + \frac{1}{\angle 6}\right)$$

= 265

59. Ans: (A)

Sol: The required number

=The number of derangements with n objects

$$= D_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$$

60. Ans: (i) 44 (ii) 76 (iii) 20 (iv) 89 (v) 119 (vi) 0

Sol: (i) Number of ways we can put 5 letters, so that no letter is correctly placed

$$= D_5 = \angle 5\left(\frac{1}{\angle 2} - \frac{1}{\angle 3} + \frac{1}{\angle 4} - \frac{1}{\angle 5}\right)$$

= 44

(ii) Number of ways in which we can put 5 letters in 5 envelopes = $\angle 5$

Number of ways we can put the letters so that no letter is correctly placed = D_5

Required number of ways = $\angle 5 - D_5$

= 120 – 44

= 76

(iii) Number of ways we can put the 2 letters correctly = C(5,2) = 10

The remaining 3 letters can be wrongly placed in D_3 ways.

Required number of ways = C(5,2) D_3

= (10) 2

= 20

(iv) Number of ways in which no letter is correctly placed = D_5

Number of ways in which exactly one letter is correctly placed = C(5,1) D_4

Required number of ways

= D_5 + C (5,1)D_4

= 44 +5 (9) = 89

(v) There is only one way in which we can put all 5 letters in correct envelopes.

Required number of ways = $\angle 5$ –1= 119

(vi) It is not possible to put only one letter in wrong envelope.

Required number of ways = 0

61. Ans: (i) 1936 (ii) 14400

Sol: (i) The derangements of first 5 letters in first 5 places = D_5

Similarly, the last 5 letters can be deranged in last 5 places in D_5 ways.

The required number of derangements

$= D_5 D_5 = (44)\ (44)$

$= 1936$

(ii) Any permutation of the sequence in which the first 5 letters are not in first 5 places is a derangement. The first 5 letters can be arranged in last 5 places in $\angle 5$ ways. Similarly, the last 5 letters of the given sequence can be arranged in first 5 places in $\angle 5$ ways.

Required number of derangements

$= \angle 5.\ \angle 5 = 14400$

62. Ans: 216

Sol: First time, the books can be distributed in $\angle 4$ ways.

Second time, we can distribute the books in D_4 ways.

Required number of ways $= \angle 4.D_4 = 216$

63. Ans: (a)

Sol: Let T(n) = Maximum number of pieces form by 'n' cuts

n	0	1	2	3	4	5	6	7
P(n)	1	2	4	7	11	16	22	29

Observe that difference between successive outputs of P(n)

i.e., 1, 2, 4, 7, 11, 16, 22, 29 are

1 + 1 + 2 + 3 + 4 + 5 + 6 + 7

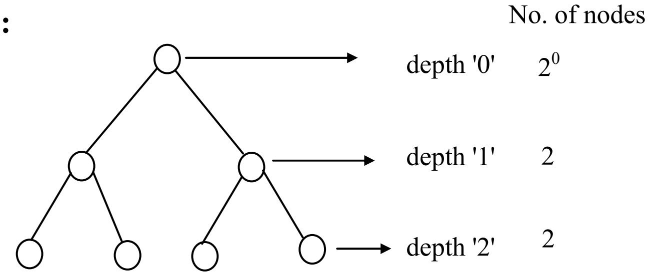
This pattern can be expressed as giving P(n) in terms of P(n – 1)

$\therefore\ P(n) = P(n-1) + n$

n = 1, 2, 3 ……

64. Ans: (b)

Sol:



The number of nodes doubles every time the depth increases by 1

At depth 'd' we have maximum number of nodes $= 2^d$

n(d) = Maximum number of nodes in a binary tree of depth 'd'

$n(d) = n(d-1) + 2^d$

65. Ans: (c)

Sol: Let a_n = number of n-digit quaternary sequences with even number of zeros

Case 1: If the first digit is not 0, then we can choose first digit in 3 ways and the remaining digits we can choose in a_{n-1} ways. By product rule, number of quaternary sequences in this case is $3a_{n-1}$.

Case 2: If the first digit is 0, then the remaining digits should contain odd number of zeros.

Number of quaternary sequences in this case is $(a_{n-1} - 4^{n-1})$

$\therefore$ By sum rule, the recurrence relation is

$\Rightarrow a_n = 3a_{n-1} + (4^{n-1} - a_{n-1})$

$\Rightarrow a_n = 2a_{n-1} + 4^{n-1}$

66. Ans: (a)

Sol: **Case 1:** If the first digit is 1, then number of bit strings possible with 3 consecutive zeros, is a_{n-1}.

Case 2: If the first bit is 0 and second bit is 1, then the number of bit strings possible with 3 consecutive zeros is a_{n-2}.

Case 3: If the first two bits are zeros and third bit is 1, then number of bit strings with 3 consecutive zeros is a_{n-3}

Case 4: If the first 3 bits are zeros, then each of the remaining n–3 bits we can choose in 2 ways. The number of bit strings with 3 consecutive zeros in this case is 2^{n-3}.

$\therefore$ The recurrence relation for a_n is

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}.$$

67. Ans: (a)

Sol: **Case(i):** If the first bit is 1, then the required number of bit strings is a_{n-1}

Case(ii): If the first bit is 0, then all the remaining bits should be zero

The recurrence relation for a_n is

$$a_n = a_{n-1} + 1$$

68. Ans: (a)

Sol: The recurrence relation is

$$a_n - a_{n-1} = 2n - 2 \text{ .... } (1)$$

The characteristic equation is $t - 1 = 0$

Complementary function = $C_1 \cdot 1^n$

Here, 1 is a characteristic root with multiplicity 1.

Let particular solution = $(c\,n^2 + d\,n)$

Substituting in (1),

$(cn^2 + d\,n) - \{c\,(n-1)^2 + d(n-1)\} = 2n - 2$

$n = 1 \Rightarrow c + d = 0$

$n = 0 \Rightarrow -c + d = -2$

$\Rightarrow c = 1$ and $d = -1$

$\therefore$ P. S $= n^2 - n$

The solution is

$$a_n = C_1 + n^2 - n \text{ .... } (1)$$

Using the initial condition, we get $C_1 = 1$

Substituting C_1 value in equation (1), we get

$\therefore a_n = n^2 - n + 2$

69. Ans: 8617

Sol: $a_n = a_{n-1} + 3(n^2)$

$n = 1 \Rightarrow a_1 = a_0 + 3(1^2)$

$n = 2 \Rightarrow a_2 = a_1 + 3(2^2)$

$= a_0 + 3(1^2+2^2)$

$n = 3 \Rightarrow a_3 = a_2 + 3(3^2)$

$= a_0 + 3(1^2+ 2^2 +3^2)$

$a_n = a_0 + 3(1^2+ 2^2 + \ldots + n^2)$

$= 7 + \frac{1}{2} n\,(n+1)\,(2n+1)$

$a_{20} = 7 + \frac{1}{2}\,(20)\,(21)\,(41) = 8617$

70. Ans: (c)

Sol: $a_n = n\,a_{n-1}$

$n=1,\ a_1 = 1.a_0 = 1.1$

$n=2,\ a_2 = 2.a_1 = 2.1$

$n=3,\ a_3 = 3.a_2 = 3.2.1$

In general, $a_n = n.\,(n-1)\ \ldots.\ 3.2.1$

$a_n = n!$

71. Ans: (b)

Sol: $a_n = a_{n-1}+(2n+1)$ where $a_0 = 1$

$n=1,\ a_1 = a_0 + 2(1) +1 =1+2(1) +1 = (1+1)^2$

$n=2,\ a_2 = a_1 + 2(2) +1= 2^2 + 2(2) +1= (2+1)^2$

In general, $a_n = (n+1)^2$

72. Ans: (a)

Sol: $a_n = a_{n-1} + \frac{1}{n(n+1)} = a_{n-1} + \left[\frac{1}{n} - \frac{1}{n+1}\right]$

$n = 1,\ a_1 = a_0 + \left[1 - \frac{1}{2}\right] = 1 + \left[1 - \frac{1}{2}\right]\ [\because a_0 = 1]$

$n = 2,$

$a_2 = a_1 + \left[\frac{1}{2} - \frac{1}{3}\right] = 1 + \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] = 1 + \left[1 - \frac{1}{3}\right]$

In general $a_n = 1 + \left[1 - \frac{1}{n+1}\right]$

$$a_n = 1 + \left[\frac{n}{n+1}\right]$$

$$a_n = \frac{2n+1}{n+1}$$

73. Ans: (c)

Sol: $f(n) = 3f\left(\left\lceil \frac{n}{3} \right\rceil\right)$

$= 3\left(3f\left(\left\lceil \frac{n}{3^2} \right\rceil\right)\right)$

$= 3^2 f\left(\left\lceil \frac{n}{3^2} \right\rceil\right)$

$\vdots$

$= 3^k f\left(\left\lceil \frac{n}{3^k} \right\rceil\right)$ Let $\left\lceil \frac{n}{3^k} \right\rceil = 1$

$= 3^{\log_3 n} f(1)$ $\quad 3^k = n$

$= 3^{\lceil \log_3 n \rceil}$ $\quad k = \lceil \log_3 n \rceil$

$\therefore$ Solution $f(n) = 3^{\lceil \log_3 n \rceil}$

74. Ans: (b)

Sol: The characteristic equation is $t^2 - t - 1 = 0$

$$\Rightarrow t = \frac{1 \pm \sqrt{5}}{2}$$

The solution is

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Using the initial conditions, we get $C_1 = \frac{1}{\sqrt{5}}$

and $C_2 = -\frac{1}{\sqrt{5}}$

75. Ans: (a)

Sol: $a_n - 2a_{n-1} = 32^n$

Replace 'n' by n+1

$(E-2)\, a_n = 3.2^{n+1}$

$C.F = C_1.\ 2^n$

$P.S = 6\left[\frac{1}{(E-2)} 2^n\right]$

$= 6\left[{}^nC_1\, 2^{n-1}\right]$

$= 3n\ 2^{n-1}$

$\therefore$ General solution $a_n = C_1\ 2^n + 3n\ 2^{n-1}$

76. Ans: (a)

Sol: $a_n - 3\,a_{n-1} + 2a_{n-2} = 2^n$

$a_{n+2} - 3a_{n+1} + 2a_n = 2^{n+2}$

$(E^2 - 3\,E + 2)\, a_n = 2^{n+2}$

$\phi(E) = E^2 - 3E + 2$

$= E^2 - 2E - E + 2$

$= (E-2) - 1\,(E-2)$

$= (E-2)\,(E-1)$

$a_n = C_1.\ 2^n + C_1\ 1^n$

$(E^2 - 3E + 2)\, a_n = 2^{n+2}$

$(E-2)(E-1)\, a_n = 2^{n+2}$

C.F $(a_n) = C_1.\, 2^n + C_2.\, 1^n$

$$P.S = \frac{1}{(E-2)(E-1)} 2^{n+2}$$

$$= 2^2\left[\frac{2^n}{(E-2)}\right]$$

$$= 2^2\left({}^nC_1 2^{n-1}\right)$$

$= 2n.\, 2^n$

$a_n = C_1.\, 2^n + C_2 + 2n\, 2^n$

77. Ans: (a)

Sol: $a_n - 6a_{n-1} + 9\, a_{n-2} = 3^n$

$(E^2 - 6E + 9)\, a_n = 3^{n+2}$

$(E-3)^2\, a_n = 3^{n+2}$

C. F $= (C_1 + C_2\, n)\, 3^n$

$$P.S = 3^2\left[\frac{1}{(E-3)^2} 3^n\right]$$

$$= 3^2\left[{}^nC_2\ 3^{n-2}\right]$$

$= {}^nC_2\, 3^n$

$$= \frac{n(n-1)}{2} \times 3^n$$

$$a_n = (C_1 + C_2 n)3^n + \frac{n(n-1)}{2} \times 3^n$$

78. Ans: (d)

Sol: The recurrence relation can be written as

$(E^2 - 2E + 1)\, a_n = 2^{n+2}$

The auxiliary equation is

$t^2 - 2t + 1 = 0$

$t = 1, 1$

C.F. $= (C_1 + C_2 n)$

$$P.S. = \frac{2^{n+2}}{(E-1)^2} = 4\left[\frac{2^n}{(E-1)^2}\right]$$

$$= 4\frac{2^n}{(2-1)^2} = 2^{n+2}$$

$\therefore$ The solution is

$a_n = C_1 + C_2 n + 2^{n+2}$

79. Ans: (a)

Sol: $a_n - 3a_{n-1} = n+3$

$a_{n+1} - 3a_n = n + 4$

C. F $= C_1.\, 3^n$

P.S $= A_0 + A_1\, n$

$A_0 + A_1\, n - 3\,[A_0 + A_1 n - A_1] = n + 3$

Put n = 0

$A_0 - 3A_0 + 3A_1 = 3 \Rightarrow -2A_0 + 3A_1 = 3$ ……(1)

n = 1

$A_0 + A_1 - 3A_0 - 3A_1 + 3A_1 = 4$

$-2A_0 + A_1 = 4$ -------- (2)

$-2A_0 = 4 - A_1$

(1) $\Rightarrow$ $4 - A_1 + 3A_1 = 3$

$4 + 2\, A_1 = 3$

$$A_1 = -\frac{1}{2},$$

$$-2A_0 = 4 - \left(-\frac{1}{2}\right)$$

$$-2A_0 = \frac{9}{2}$$

$$A_0 = -\frac{9}{4}$$

$$C_1 3^n - \frac{n}{2} - \frac{9}{4}$$

80. Ans: (a)

Sol: $a_n - 2a_{n-1} + a_{n-2} = 3n + 5$

$(E-1)^2\, a_n = 3n + 11$

C. F $= C_1 + C_2\, n$

$P.S = (A_0 + A_1 n)\, n^2 = A_0 n^2 + A_1 n^3$

$A_0 n^2 + A_1 n^3 - 2(A_0(n-1)^2 + A_1(n-1)^3)$

$+ A_0 (n-2)^2 + A_1 (n-2)^3 = 3n + 5$

Put n = 0

$-2A_0 + 2A_1 + 4A_0 - 8A_1 = 5$

$2A_0 - 6A_1 = 5$ ---------- (*)

n = 1

$A_0 + A_1 + A_0 - A_1 = 8$

$A_0 = 4$

From (*) $\quad 6A_1 = 8 - 5$

$$A_1 = \frac{1}{2}$$

$$C_1 + C_2 n + 4n^2 + \frac{1}{2} n^3$$

81. Ans: (a)

Sol: Replacing n by n+1, the given relation can be written as

$a_{n+1} = 4a_n + 3(n+1)\, 2^{n+1}$

$\Rightarrow (E-4)\, a_n = 6(n+1)\, 2^n$ ……….(1)

The characteristic equation is

$$t - 4 = 0 \Rightarrow t = 4$$

complementary function = $C_1 4^n$

Let particular solution is

$a_n = 2^n(cn + d)$ where c and d are undetermined coefficients.

Substituting in the given recurrence relation, we have

$2^n (c\,n + d) - 4\; 2^{n-1}\{c(n-1)+d\} = 3n2^n$

$\Rightarrow (c\,n + d) - 2\{c(n-1)+d\} = 3n$

Equating coefficients of n and constants on both sides, we get

$c = -3$ and $d = -6$

$\therefore$ Particular solution = $2^n(-3n-6)$

Hence the solution is

$a_n = C_1 4^n - (3n+6)\, 2^n$ …………(2)

$x = 0 \Rightarrow 4 = C_1 - 6 \Rightarrow C_1 = 10$

$a_n = 10(4^n) - (3n+6)\, 2^n$

82. Ans: (a)

Sol: $a_{n+2} - 2a_{n+1} + a_n = n^2\, 2^n$ ______(*)

$(E^2 - 2E + 1)\, a_n = n^2\, 2^n$

Characterstic roots $t_1 = t_2 = 1$

$\therefore$ C. F = $(C_1 + C_2 n)$

here $F(n) = n^2\, 2^n = b^n . n^k$ where b = 2, k = 2

$\therefore$ Let P. S = $a_n = (A_0 + A_1 n + A_2 n^2) 2^n$

Substitute 'a_n' in equation _________(*)

$\Rightarrow (A_0 + A_1 n + A_2 n^2)\, 2^n - 2^{n+2}$

$(A_0 + A_1(n+1) + A_2(n+1)^2) + 2^{n+2}$

$(A_0 + A_1(n+2) + A_2(n+2)^2) = n^2 2^n$

$\Rightarrow (A_0 + A_1 n + A_2 n^2) - 2^2$

$(A_0 + A_1(n+1) + A_2(n+1)^2) + 2^2$

$(A_0 + A_1(n+2) + A_2(n+2)^2) = n^2\, 2^n$

Put n = 0

$A_0 - \cancel{4A_0} - 4A_1 - 4A_2 + \cancel{4A_0} + 8A_1 + 16A_2 = 0$

$\Rightarrow 12A_2 + 4A_1 + A_0 = 0$ __________(1)

Put n=1

$A_0 - A_1 + A_2 - \cancel{4A_0} + \cancel{4A_0} + 4A_1 + 4A_2 = 1$

$A_0 + 3A_1 + 5A_2 = 1$ __________(2)

Put n = –2

$A_0 - 2A_1 + \cancel{4A_2} - \cancel{4A_0} + 4A_1 - 4A_2 + \cancel{4A_0} = 4$

$2A_1 + A_0 = 4$ _______(3)

By solving (1), (2), (3) we get

$A_0 = 20,\ A_1 = -8,\ A_2 = 1$

$\therefore a_n = C_1 + C_2 n + 2^n(n^2 - 8n + 20)$

83. Ans: (a)

Sol: Let $\sqrt{a_n} = x_n$ (say)

The recurrence relation is

$$x_n - x_{n-1} - 2x_{n-2} = 0$$

Replacing n by n + 2, we have

$$x_{n+2} - x_{n+1} - 2x_n = 0$$

$$(E^2 - E + 2)\, x_n = 0$$

The characteristic equation is

$$t^2 - t + 2 = 0$$

$$t = 2, -1$$

The solution is

$$x_n = \sqrt{a_n} = c_1 2^n + c_2.(-1)^n$$

$n = 0 \Rightarrow 1 = c_1 + c_2$

$n = 1 \Rightarrow 1 = 2c_1 - c_2$

Solving, we get $c_1 = \dfrac{2}{3}$ and $c_2 = \dfrac{1}{3}$

$$\sqrt{a_n} = \frac{2}{3}(2^n) + \frac{1}{3}(-1)^n$$

$$\therefore a_n = \left[\frac{2^{n+1} + (-1)^n}{3}\right]^2$$

84. Ans: (a)

Sol: $T(n) = 7T\left(\dfrac{n}{3}\right) + 2n, \quad T(1) = \dfrac{5}{2}$

$$T(n) = 7T\left(\frac{n}{3}\right) + 2n$$

$$= 7\left(7T\left(\frac{n}{3^2}\right) + 2\frac{n}{3}\right) + 2n$$

$$= 7^2 T\left(\frac{n}{3^2}\right) + \frac{14n}{3} + 2n$$

$$= 7^2\left[7T\left(\frac{n}{3}\right) + \frac{2n}{3^2}\right] + \frac{14n}{3} + 2n$$

$$= 7^3\, T\left(\frac{n}{3^3}\right) + 7^2\frac{2n}{3^2} + \frac{7.2n}{3} + 2n$$

$$\vdots$$

$$= 7^k T\left(\frac{n}{3^k}\right) + \left(\frac{7^{k-1}}{3^{k-1}} + \frac{7^{k-2}}{3^{k-2}} + ... + 1\right)2n$$

Let $\dfrac{n}{3^k} = 1$

$$k = \log_3 n$$

$$= 7^k\left(\frac{5}{2}\right) + 2n\left(\frac{3}{4}\left(\frac{7}{3}\right)^k - 1\right)$$

$$T(n) = \frac{-3n}{2} + 4.7\log_3 n$$

[By substituting k value]

85. Ans: (a)

Sol: {1, –2, 4, –8, 16, ……..}

$= 1 - 2x + 4x^2 - 8x^3 + 16x^4$ ……..

$$= \frac{1}{1-(-2x)}$$

$[\because S_\infty = \dfrac{a}{1-r}$, where a=1, r =–2x]$=(1+2x)^{-1}$

86. Ans: (a)

Sol: Required generating function

$= f(x) = 0 + x + 3x^2 + 9x^3 + 27x^4 + \ldots$..

$= x(1 + 3x + 3^2 x^2 + 3^3 x^3 + \ldots\ \infty)$

$$= x\sum_{n=0}^{\infty} 3^n x^n = x.(1-3x)^{-1}$$

87. Ans: (a)

Sol: Generating function of <$a_0, a_1, a_2, \ldots\ldots$>

$$= \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) x^n \quad [\because a_n = (n+1)(n+2)]$$

$$= 2\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$$

$= 2(1-x)^{-3}$

$$\left[\because \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} = (1-x)^{-3}\right]$$

88. Ans: (d)

Sol: Required generating function

$f(x) = 0 + 0\,x + 1\,x^2 - 2x^3 + 3x^4 - 4x^5 + \ldots$

$= x^2 (1 - 2x + 3x^2 - 4x^3 + \ldots \infty)$

$= x^2(1+x)^{-2}$ (Binomial theorem)

89. Ans: (c)

Sol: The generating function is

$f(x) = 1 + 0.x + 1.x^2 + 0.x^3 + 1.x^4 + \ldots\ldots \infty$

$= 1 + (x^2) + (x^2)^2 + \ldots\ldots \infty$

$= (1-x^2)^{-1}$

90. Ans: (a)

Sol: $(x^4 + 2x^5 + 3x^6 + 4x^7 + \ldots\ldots\infty)^5$

$= x^{20} (1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\infty)^5$

$= x^{20}. [(1-x)^{-2}]^5$

$= x^{20} [1-x]^{-10}$

$$= x^{20} \sum_{n=0}^{\infty} C(n+9, n) x^n$$

Coefficient of $x^{27} = C(16, 7)$

$= C(16, 9)$

91. Ans: (b)

Sol: Required number of ways

= Number of non negative integer solutions to the equation

$x_1 + x_2 + x_3 = 15$ where $1 \le x_1, x_2, x_3 \le 7$

= coefficient of x^{15} in the expansion of f(x)

where, $f(x) = (x + x^2 + \ldots.. + x^7)^3$

$= x^3 (1 + x + x^2 + \ldots.. + x^6)^3$

$$= x^3 \left(\frac{1-x^7}{1-x}\right)^3$$

$= x^3 (1 - 3x^7 + 3x^{14} - x^{21})(1-x)^{-3}$

$$= (x^3 - 3x^{10} + 3x^{17} - x^{24}) \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$$

$$\text{Coefficient of } x^{15} = \frac{(13)(14)}{2} - 3\left(\frac{(6).(7)}{2}\right)$$

$= 91 - 63 = 28$

92. Ans: 60

Sol: If one person chooses 12 books then second person has to take remaining books

Number of ways we choose 12 books can be found by solving $x + y + z = 12$

Where, $0 \le x \le 7$

$0 \le y \le 8$

$0 \le z \le 9$

i. e. coefficient of x^{12} in the following

$= (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)$

$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)$

$(1 + x + x^2 + \ldots. + x^9)$

$$= \left(\frac{1-x^8}{1-x}\right)\left(\frac{1-x^9}{1-x}\right)\left(\frac{1-x^{10}}{1-x}\right)$$

$= (1-x^8)(1-x^9)(1-x^{10}) \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

$= (1-x^9-x^8+x^{17})(1-x^{10}) \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

$= (1 - x^{10} - x^9 + x^{19} - x^8 + x^{18} + x^{17} - x^{27})$

$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

Coefficient of x^{12} in the above expansion

$= 91 - 6 - 10 - 15$

$= 60$

3. Graph Theory

Basic concepts

01. Ans: (b)

Sol: By sum of degrees theorem,

$$\delta(G) \le \frac{2|E|}{|V|} \le \Delta(G)$$

where $\delta(G)$ is minimum of the degrees of all vertices in G and $\Delta(G)$ is maximum of the degrees of all vertices in G.

$\Rightarrow\ 3 \le \frac{2|E|}{|V|} \le 5$

$\Rightarrow\ 33 \le 2|E| \le 55$

$\Rightarrow\ 16.5 \le |E| \le 27.5$

$\Rightarrow\ 17 \le |E| \le 27$ $\quad (\because |E|$ is an integer)

02. Ans: (d)

Sol: Let d be the common degree of the vertices of G, and let v be the number of vertices of G.

Then, by sum of degrees theorem,

v.d = 44

$\Rightarrow\ v = \frac{44}{d} \quad (d = 1, 2, 4, 11, 22, 44)$

$\Rightarrow$ v = 44, 22, 11, 4, 2

As G is simple, the last 3 cases are not possible.

If v = 44 then, v is not a connected graph.

$\therefore$ A possible number of vertices is 11 or 22.

03. Ans: 19

Sol: By sum of degrees theorem,

If degree of each vertex is k, then

k. |V| = 2.|E|

4 |V| = 2(38)

$\therefore$ |V| = 19

04. Ans: (c)

Sol: By Sum of degrees theorem,

k. | V | = 2 |E|

$\Rightarrow\ |E| = \frac{k.|V|}{2}$

Here, |E| is an integer

$\frac{|V|}{2}$ is an integer $\quad (\because$ k is odd)

$\Rightarrow$ |E| = a multiple of k

05. Ans: (e)

Sol: (a) {2, 3, 3, 4, 4, 5}

Here, sum of degrees

= 21, an odd number.

$\therefore$ The given sequence cannot represent a simple non directed graph

(b) $\{2, 3, 4, 4, 5\}$

In a simple graph with 5 vertices, degree of every vertex should be ≤ 4.

$\therefore$ The given sequence cannot represent a simple non directed graph.

(c) $\{1, 3, 3, 4, 5, 6, 6\}$

Here we have two vertices with degree 6. These two vertices are adjacent to all the other vertices. Therefore, a vertex with degree 1 is not possible.

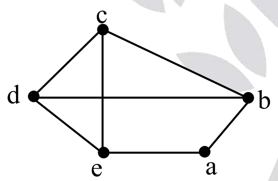
Hence, the given sequence cannot represent a simple non directed graph.

(d) $\{0, 1, 2, \ldots, n-1\}$

Here, we have n vertices, with one vertex having degree n-1. This vertex is adjacent to all the other vertices. Therefore, a vertex with degree 1 is not possible.

Hence, the given sequence, cannot represent a simple non directed graph.

(e) A graph with the degree sequence $\{2,3,3,3, 3\}$ is shown below.



06. Ans: (b)

Sol: S_1: Let us denote the vertices by V_1, V_2, ..., V_8

V_8 is isolated vertex

The vertices V_1, V_2, V_3 and V_4 are adjacent to V_7.

$\therefore$ Degree of V_7 should be atleast 4

$\therefore$ S_1 cannot represent a simple non directed graph.

We can also verify this using Havel-Hakimi result.

$S_2 = \{6, 5, 5, 4, 3, 3, 2, 2, 2\}$

Applying Havel-Hakimi result, S_2 becomes

$\{4, 4, 3, 2, 2, 1, 2, 2\}$

Arranging the vertices in the descending order.

$\{4, 4, 3, 2, 2, 2, 2, 1\}$

Applying Havel-Hakimi result, we have

$\{3, 2, 1, 1, 2, 2, 1\}$

Arranging the vertices in the descending order.

$\{3, 2, 2, 2, 1, 1, 1\}$

Applying Havel-Hakimi result, we have

$\{1, 1, 1, 1, 1, 1\}$

which can be represented by a simple non-directed graph.

$\therefore$ The sequence S_2 also can be represented by a simple non-directed graph.

07. Ans: 8

Sol: By sum of degrees theorem,

$(5 + 2 + 2 + 2 + 1) = 2\,|E|$

$\Rightarrow |E| = 7$

$\therefore$ Number of edges in G = 7

$|E(G)| + |E(\overline{G})| = |E(K_6)|$

$\Rightarrow \quad 7 + |E(\overline{G})| = C(6, 2)$

$\Rightarrow \quad |E(\overline{G})| = 8$

08. Ans: 12

Sol: G is a tree

By sum of degrees theorem,

n.1 + 2 (2) + 4. (3) + 3. (4) = 2 |E|

$\therefore$ n + 28 = 2 (|v| – 1)

= 2 (n + 2 + 4 + 3 – 1)

$\Rightarrow$ n + 28 = 2n + 16

$\Rightarrow$ n = 12

09. Ans: 18

Sol: A simple graph with 10 vertices and minimum number of edges is a tree.

A tree with 10 vertices has 9 edges.

By Sum of degrees theorem, Sum of degrees of all vertices in G = 2

(Number of edges in G) = 2 × 9 = 18

10. Ans: 8

Sol: G has 8 vertices with odd degree.

For any vertex v∈G,

Degree of v in G + degree of v in $\overline{G}$ = 8

If degree of v in G is odd, then degree of v in $\overline{G}$ is also odd. If degree of v in G is even, then degree of v in $\overline{G}$ is also even.

$\therefore$Number of vertices with odd degree in $\overline{G}$ = 8

11. Ans: 27

Sol: By sum of degrees theorem, if degree of each vertex is atmost K,

then K|V| ≥ 2 |E|

$\Rightarrow$ 5 (11) ≥ 2 |E|

$\Rightarrow$ |E| ≤ 27.5

$\Rightarrow$ |E| ≤ 27

12. Ans: (d)

Sol: (a) Let G be any graph of the required type.

Let p be the number of vertices of degree 3.

Thus, (12 – p) vertices are of degree 4.

Hence, according to sum of degrees theorem,

3p – 4(12 – p) = 56.

Thus, p = –8 (Which is impossible)

$\therefore$ Such a graph does not exist.

(b) Maximum number of edges possible in a simple graph with 10 vertices

C(10, 2) = 45

(c) Maximum number of edges possible in a bipartite graph with 9 vertices= $\left\lfloor \frac{9^2}{4} \right\rfloor$

= 20

$\therefore$ Such a graph does not exist.

(d) A connected graph with n vertices and n–1 edges is a tree. A tree is a simple graph.

13. Ans: (b)

Sol: G is a simple graph with 5 vertices.

For any vertex v in G,

deg(v) in G + deg(v) in $\overline{G}$ = 4

$\therefore$ The degree sequence $\overline{G}$ is

{4 – 3, 4 – 2, 4 – 2, 4 – 1, 4 – 0}

= {1, 2, 2, 3, 4}= {4, 3, 2, 2, 1}

14. Ans: 455

Sol: Maximum number of edges possible with 6 vertices is C(6, 2) = 15. Out of these edges, we can choose 12 edges in C(15, 2) ways.

$\therefore$ Number of simple graphs possible

$$= C(15, 12) = C(15, 3) = \frac{15.14.13}{1.2.3} = 455$$

15. Ans: (a)

Sol: We know that,

Number of edges in G + Number of edges in $\overline{G}$ = Number of edges in the complete graph K_p.

$$\Rightarrow q + \text{Number of edges in } \overline{G} = \frac{p(p-1)}{2}$$

$$\Rightarrow \text{Number of edges in } \overline{G} = \frac{p(p-1)}{2} - q$$

16. Ans: (c)

Sol: Given that, G is a connected graph.

⇒ Between every pair of vertices in G, a path exists.

∴ By transitivity, there exists an edge between every pair of vertices in G.

⇒ G is a complete graph

∴ Number of edges in G = C(n, 2).

17. Ans: (d)

Sol: The complement of W_n contains an isolated vertex and $\overline{C}_{n-1}$ as components.

Number of edges in

$$\overline{C}_{n-1} = \frac{(n-1)(n-2)}{2} - (n-1)$$

$$= \frac{(n-1)(n-4)}{2}$$

18. Ans: (a)

Sol: Let x = Number of vertices with degree 4

& y = Number of vertices with degree 5

By sum of degrees theorem

4x+5y+14 = 2(n−1) ---------- (1)

Also x+y+14 = n -------------- (2)

Solving (1) & (2), we get y = (40−2n)

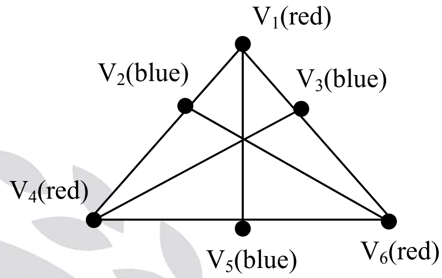
Coloring

19. Ans: 2

Sol: The graph is bipartite,

Therefore, chromatic number = 2

(or apply Welch – powel's algorithm).



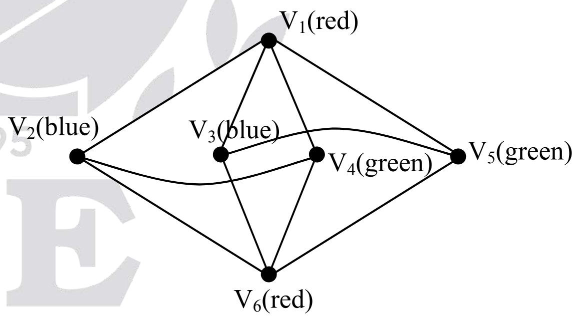
20. Ans: 3

Sol: The graph has cycles of length 3.

$\therefore \chi(G) \geq 3$ ……. (1)

If we apply Welch-powel's algorithm, then 3- coloring is possible

$\therefore \chi(G) = 3$



21. Ans: 4

Sol: The graph is planar,

By 4 – color theorem

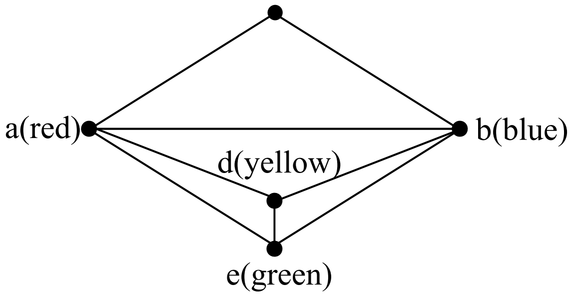
$\chi(G) \leq 4$ …….. (1)

The graph has 4 mutually adjacent vertices {a, b, d, e}

$\therefore \chi(G) \geq 4$ ………. (2)

From (1) and (2), we have

χ (G) = 4



22. Ans: 4

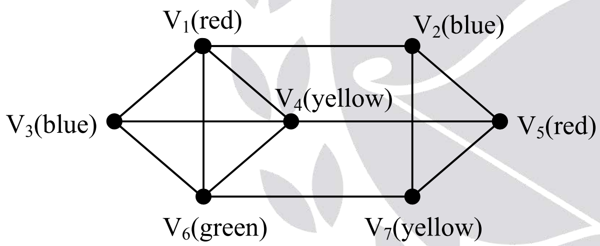
Sol: G is planar graph

By 4 color theorem, $\chi(G) \le 4$ ….. (1)

G has 4 mutually adjacent vertices $\{V_1, V_3, V_4, V_6\}$

$\therefore \chi(G) \ge 4$ …… (2)

Hence, χ (G) = 4

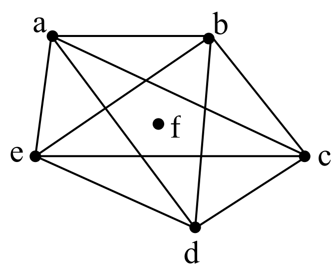


23. Ans: 7

Sol: G is a star graph

$\therefore \chi$ (G) = 2

The graph $\overline{G}$ is shown below



Here, the vertices a, b, c, d, e form a complete graph

$\therefore \chi\left(\overline{G}\right) = 5$

Now, $\chi(G) + \chi\left(\overline{G}\right) = 7$

24. Ans: 3

Sol: Applying welch-powel's algorithm we can see that 3 - colouring is possible

Vertex	d	a	b	c	e	f	g
Color	C_1	C_2	C_3	C_3	C_2	C_2	C_3

$\therefore \chi$ (G) ≤ 3 .... (1)

since, G has cycles of odd length,

χ (G) ≥ 3 .... (2)

From (1) and (2), we have χ (G) = 3.

25. Ans: 2

Sol: In the given graph, all the cycles are of even length.

$\therefore$ G is a bipartite graph and every bipartite graph is 2-colorable

$\therefore$ Chromatic number of G = 2.

26. Ans: 5

Sol: $\overline{G}$ is a disconnected graph with two components, one component is the complete graph K_5 and the other component is the trivial graph with only an isolated vertex

$\therefore$ Chromatic number of $\overline{G} = 5$

27. Ans: (b)

Sol: $\alpha = n - 2\lfloor n/2 \rfloor + 2$

$\beta = n - 2\lceil n/2 \rceil + 4$

$\alpha + \beta = 2n - 2\{\lfloor n/2 \rfloor + \lceil n/2 \rceil\} + 6$

$= 2n - 2n + 6 = 6$

28. Ans: (c)

Sol: Chromatic number of $K_n = n$

If we delete an edge in K_{10}, then for the two vertices connecting that edge we can assign same color.

$\therefore$ Chromatic number = 9

29. Ans: 3

Sol: Here, $G = K_{3,3}$

$\overline{G}$ has two components where each component is a complete graph K_3.

$\therefore$ Chromatic number of $\overline{G} = 3$

30. Ans: (c)

Sol: We know that $\alpha = \left\lfloor \frac{n}{2} \right\rfloor$

$\beta = n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2$

$2\alpha + \beta = 2\left\lfloor \frac{n}{2} \right\rfloor + n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2 = n + 2$

31. Ans: (d)

Sol: The chromatic number of any bipartite graph (with atleast one edge) is 2.

$\therefore$ Option (d) is false.

(a) K_n has n mutually adjacent vertices.

$\therefore$ K_n requires atleast n colours

$\therefore$ The vertex chromatic number of complete graph $K_n = n$

(b) The vertex chromatic number of cycle graph $C_n = 2$ if n is even

= 3 if n is odd

$= n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2$

(c) The vertex chromatic number of wheel graph $W_n = 3$ if n is odd

= 4 if n is even

$= n - 2\left\lceil \frac{n}{2} \right\rceil + 4$

32. Ans: (d)

Sol: (a) The chromatic number of any bipartite graph (with atleast one edge) is 2.

(b) A star graph with n vertices is a bipartite graph $K_{1, n-1}$

$\therefore$ Chromatic number = 2

(c) A tree is a bipartite graph

$\therefore$ Chromatic number = 2

(d) If G is a simple graph in which all the cycles are of even length, then G is a bipartite graph

$\therefore$ The vertex chromatic number of G = 2

Hence, option (d) is false.

Matchings

33. Ans: (d)

Sol: (a) In K_{2n}, each vertex is adjacent to remaining 2n – 1 vertices.

One vertex we can match in 2n – 1 ways.

Next vertex in 2n – 3 ways.

And another vertex in 2n – 5 ways, and so on.

Number of perfect matchings in

K_{2n}=(2n–1).(2n–3).(2n–5)..........(5)(3)(1)

$= \frac{2n\,!}{2^n\; n!}$

(b) In $K_{n,n}$, we divide the vertices into two groups such that each vertex of a group is adjacent to all the vertices of the other group.

One vertex of a group we can match in n ways, next vertex in n – 1 ways and so on.

$\therefore$ Number of perfect matchings in $K_{n,n} = n(n-1)(n-2)$ 1
$= n!$

(c) If n is even then the possible perfect matchings are

$V_1 - V_2, V_2 - V_3, \ldots\ldots, V_{n-1} - V_n$

And $V_2 - V_3, V_3 - V_4, \ldots\ldots, V_n - V_1$

$\therefore$ Number of perfect matchings in C_n (n is even) = 2

(d) In W_{2n} there is a vertex (Hub) which is adjacent to all the other vertices.

This vertex we can match in 2n–1 ways.

$\therefore$ Number of perfect matchings in $W_{2n} = 2n - 1$

Hence, option (d) is false.

34. Ans: (d)

Sol: (a) A tree can have atmost one perfect matching.

$\therefore$ Option (a) is false

(b) In a star graph, perfect matching is not possible if n > 2.

$\therefore$ Option (b) is false

(c) In a complete bipartite graph $K_{m,n}$, a perfect matching exists iff m = n

$\therefore$ Option (c) is false

(d) Number of perfect matchings in $K_{3,3} = 3!$
$= 6$

35. Ans: (c)

Sol: S_1 is true (By definition of $K_{m,n}$)

S_2 is not true because,

A graph G has a perfect matching
$\Rightarrow$ Number of vertices in G is even

But converse is not true.

S_3 is not true because,

A bipartite graph G with vertex partition $\{V_1, V_2\}$ has a complete matching $\Rightarrow |V_1| < |V_2|$.

But converse is not true.

36. Ans: (d)

Sol: (a) If n is even, then K_n has a perfect matching. Therefore, matching number is $\frac{n}{2}$. If n is odd number, then we can match only (n–1) vertices. Therefore matching number is $\left(\frac{n-1}{2}\right)$.

Hence, matching number = $\left\lfloor \frac{n}{2} \right\rfloor$

(b) If n is even, then C_n has a perfect matching. Therefore, matching number is $\frac{n}{2}$. If n is odd number, then we can match only (n–1) vertices. Therefore matching number is $\left(\frac{n-1}{2}\right)$.

Hence, matching number = $\left\lfloor \frac{n}{2} \right\rfloor$

(c) If n is even, then W_n has a perfect matching. Therefore, matching number is $\frac{n}{2}$. If n is odd number, then we can match only (n–1) vertices. Therefore matching number is $\left(\frac{n-1}{2}\right)$.

Hence, matching number = $\left\lfloor \frac{n}{2} \right\rfloor$

(d) In a complete bipartite graph, the vertices are partitioned into two groups so that no two vertices in the same group are adjacent.

$\therefore$ Matching number of $K_{m,n}$ = minimum of {m, n}

Hence, option (d) is false.

37. Ans: (d)

Sol: (a) Every star Graph with n vertices is a complete bipartite graph of the form $K_{1,n-1}$.

$\therefore$ Matching number = 1

(b) In a complete bipartite graph, the vertices are partitioned into two groups so that no two vertices in the same group are adjacent.

$\therefore$ Matching number of $K_{m,n}$ = minimum of {m, n}

Hence, Matching number of $K_{m,m}$ = m

(c) Refer, option (b)

(d) Matching number of a tree with n vertices ≥ 1

$\therefore$ Option (d) is false.

38. Ans: (a)

Sol: If G is a complete bipartite graph with n vertices (n ≥ 2) and minimum number of edges, then

G = $K_{1,\,n-1}$ (star graph)

$\therefore$ Matching number = 1

39. Ans: 13

Sol: A disconnected graph with 10 vertices and maximum number of edges has two components K_9 and an isolated vertex.

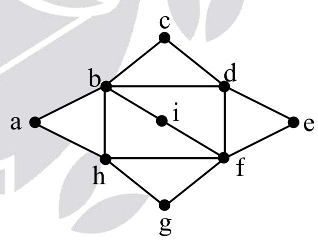
Matching number of $K_9 = \left\lfloor \frac{9}{2} \right\rfloor = 4$

$\therefore$ Matching number of G = 4

Chromatic number of G = 9

40. Ans: 4

Sol:



The graph has 9 vertices. The maximum number of vertices we can match is 8.

A matching in which we can match 8 vertices is { a – b, c – d, e – f, g – h}

$\therefore$ Matching number of the graph = 4

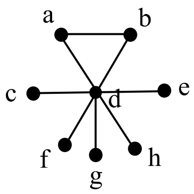
41. Ans: 2

Sol: The given graph is $K_{2,4}$

$\therefore$ Matching number = 2

42. Ans: 2

Sol: The given graph is

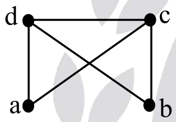


If we delete the edge {a,b} then the graph is a star graph. If we match a with b, then in the remaining vertices we can match only two vertices.

$\therefore$ Matching number = 2

43. Ans: 3

Sol: Let us label the vertices of the graph as shown below

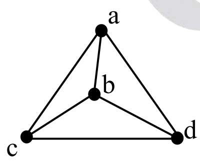


There are 3 maximal matchings as given below

{a – d, b – c}, {a – c, b – d} and {c – d}

44. Ans: 3

Sol: The given graph is



The maximal matchings are

{a – b, c – d}, {a – c, b – d}, {a – d, b – c}

45. Ans: 10

Sol: The graph has 3 maximal matching's 6 matching's with one edge and a matching with no edges.

$\therefore$ Number of matching's = 10

46. Ans: (a)

Sol: If n is even, then a bipartite graph with maximum number of edges is $k_{n/2,n/2}$

$\therefore$ Matching number of G = $\frac{n}{2}$

If n is odd, then a bipartite graph with maximum number of edges = $k_{m,n}$

Where $m = \frac{n-1}{2}$ and $n = \frac{n+1}{2}$

$\therefore$ Matching number of G

$$= \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ Matching number of G = $\left\lfloor \frac{n}{2} \right\rfloor$

Connectivity

47. Ans: (a)

Sol: If G has n vertices and k components, then

$$(n-k) \le |E| \le \frac{(n-k)(n-k+1)}{2}$$

$$\Rightarrow 7 \le |E| \le 28$$

48. Ans: 4

Sol: Here, G = $K_{4,5}$

Vertex connectivity of G = Minimum of {4, 5} = 4

49. Ans: (c)

Sol: If G is a simple graph with maximum number of edges, then G should have two components K_{n-1} and an isolated vertex.

$\therefore$ Number of edges in $K_{n-1} = \frac{(n-1)(n-2)}{2}$.

50. Ans: 0

Sol: Here, G is a cycle graph.

Every edge of G is part of a cycle in 'G'.

$\therefore$ 'G' has no cut edge

51. Ans: (b)

Sol: In a connected graph G, if all vertices are of even degree then G has Euler circuit.

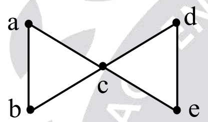
$\Rightarrow$ Every edge is part of a cycle in G.

$\Rightarrow$ G has no cut edge

$\therefore$ S_2 is true.

S_1 is false.

We have the following counter example.



Here, all vertices in the graph are of even degree.

But c is a cut vertex of the graph.

52. Ans: (d)

Sol: If $|E| < (n - 1)$, then G is disconnected

If $|E| > \frac{(n-1)(n-2)}{2}$, then G is connected.

then G may or may not be connected.

53. Ans: (d)

Sol: The given graph is a complete graph K_6, with 6 vertices of odd degree.

$\therefore$ G is not traversable

54. Ans: 3

Sol: d is the cut vertex of G

$\Rightarrow$ vertex connectivity of G = 1

G has no cut edge.

$\Rightarrow \lambda(G) \geq 2$ .... (1)

By deleting the edges d – e and d – f, we can disconnect G.

$\therefore$ Edge connectivity = $\lambda(G) = 2$

55. Ans: 105

Sol: If G is a simple graph with n vertices and k components then $|E| \leq \frac{(n-k)(n-k+1)}{2}$

Here n = 20 and k = 5

$\therefore$ Maximum number of edges possible

$= \frac{(20-5)(20-6)}{2} = 105$

56. Ans: (a)

Sol: If G is any graph having p vertices and $\delta(G) \geq \frac{p-1}{2}$, then G is connected.

57. Ans: (b)

Sol: If a component has n vertices, then maximum number of edges possible in that component = C(n, 2)

$\therefore$ The maximum number of edges possible in G = C(5,2) + C(6,2) + C(7,2) + C(8,2)

= 10 + 15 + 21 + 28

= 74

58. Ans: 9

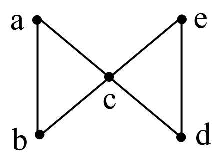
Sol: In a tree, each edge is a cut set.

Number of edges in a tree with 10 vertices = 9

$\therefore$ Number of cut sets possible on a tree with 10 vertices = 9

59. Ans: 1, 2

Sol: The graph can be labeled as



c is a cut vertex of the graph G.

$\therefore$ vertex connectivity of G = K(G) = 1

G has no cut edge.

$\Rightarrow$ Edge connectivity = λ (G) $\geq$ 2 ….. (1)

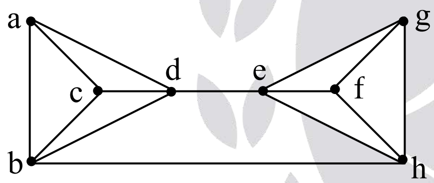
We have, λ (G) $\leq$ δ (G) = 2 …… (2)

From (1) and (2), we have

λ (G) = 2

60. Ans: 2, 2

Sol: The graph G can be labeled as



G has no cut edge and no cut vertex. By deleting the edges {d, e} and {b, h} we can disconnect G.

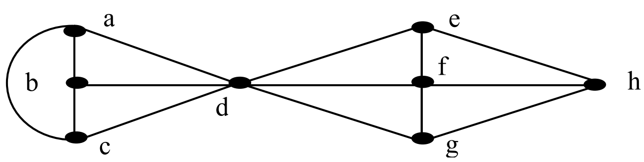
$\therefore$ λ (G) = 2

By deleting the vertices b and d, we can disconnect G.

$\therefore$ K (G) = 2

61. Ans: 1, 3

Sol: The graph G can be labeled as



The vertex d is a cut vertex of G.

$\therefore$ K (G) = 1

We have λ(G) $\leq$ δ (G) = 3 ...... (1)

G has no cut edge and by deleting any two edges of G we cannot disconnect G.

$\therefore$ λ (G) = 3

62. Ans: (a)

Sol: If G is disconnected then $\overline{G}$ is always connected. (Theorem)

If G is connected then $\overline{G}$ may or may not be connected (we can prove this by counter examples).

$\therefore$ Option (a) is true.

63. Ans: (c)

Sol: **S_1:** This statement is true.

Proof:

Suppose G is not connected G has atleast 2 connected components.

Let G_1 and G_2 are two components of G.

Let u and v are any two vertices in G

We can prove that there exists a path between u and v in G.

Case1: u and v are in different components of G.

Now u and v are not adjacent in G.

$\therefore$ u and v are adjacent in $\overline{G}$

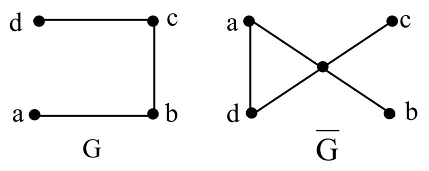
Case2: u and v are in same components G_1 of G. Take any vertex w$\in G_2$.

Now u and v are adjacent to w in G.

$\therefore$ There exists a path between u and v in G. Hence, $\overline{G}$ is connected.

S_2: The statement is false.

we can give a counter example.



Here, G is connected and $\overline{G}$ is also connected.

S_3: Suppose G is not connected

Let G_1 and G_2 are two connected components of G.

Let $v \in G_1$

$$\Rightarrow \deg(v) \geq \frac{n-1}{2} \quad \left(\because \delta(G) = \frac{n-1}{2}\right)$$

Now $|V(G_1)| \geq \left(\frac{n-1}{2}+1\right)$

Similarly, $|V(G_2)| \geq \frac{n+1}{2}$

Now, $|V(G)| = |V(G_1)| + |V(G_2)|$

$\Rightarrow |V(G)| \geq n + 1$

which is a contradiction

$\therefore$ G is connected.

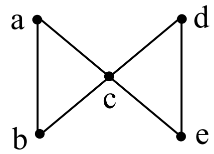
S_4: If G is connected, then the statement is true. If G is not connected, then the two vertices of odd degree should lie in the same component.

By the sum of degrees of vertices theorem.

$\therefore$ There exists a path between the 2 vertices.

64. Ans: (c)

Sol: The graph G can be labeled as



The number of vertices with odd degree is 0.

$\therefore$ S_1 and S_2 are true

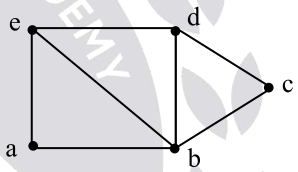
C is a cut vertex of G.

$\therefore$ Hamiltonian cycle does not exists.

By deleting the edges {a, c} and {c, e}, there exists a Hamiltonian path a–b–c –d – e

65. Ans: (a)

Sol: The graph G can be labeled as



The number of vertices with odd degree = 2

$\therefore$ Euler path exists but Euler circuit does not exist.

There exists a cycle passing through all the vertices of G.

a – b – c – d – e – a is the Hamiltonian cycle of G. The Hamiltonian path is a–b–c–d – e.

66. Ans: (b)

Sol: The number of vertices with odd degree = 0

$\therefore$ S_1 and S_2 are true.

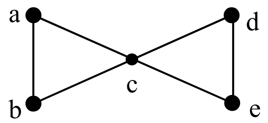
To construct Hamiltonian cycle, we have to delete two edges at each of the vertices a and f. Then, we are left with 4 edges and 6 vertices.

$\therefore$ G has neither Hamiltonian cycle nor Hamiltonian path.

67. Ans: (b)

Sol: S_1 is false. We can prove it by giving a counter example.

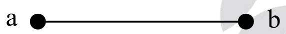
Consider the graph G shown below



'e' is a cut vertex of G. But, G has no cut edge

S_2 is false. We can prove it by giving a counter example.

For the graph K_2 shown below,



The edge {a, b} is a cut edge. But K_2 has no cut vertex.

68. Ans: 33

Sol: If G has K components, then

$$|E| = |V| - K$$

$$\Rightarrow \quad 26 = |V| - 7$$

$$\Rightarrow |V| = 33$$

69. Ans: (c)

Sol: The forest F can be converted into a tree by adding (k – 1) edges to F.

$\therefore$ Number of edges in F = (n – 1) – (k – 1)
= (n – k)

70. Ans: (b)

Sol: A 2-regular graph G has a perfect matching iff every component of G is an even cycle.

$\therefore S_2$ and S_4 are true.

S_1 need not be true. For example the complete graph K_2 has a perfect matching but K_2 has no cycle.

S_3 need not be true. For example G can have two components where each component is K_2.

71. Ans: 36

Sol: The maximum number of edges possible in

$$G = \frac{(n-k)(n-k+1)}{2}$$

Where, n = 12 and k = 4 = 36

72. Ans: (b)

Sol: G has exactly two vertices of odd degree. Therefore, Euler path exists in G but Euler circuit does not exist.

In Hamiltonian cycle, degree of each vertex is 2. So, we have to delete 2 edges at vertex 'd' and one edge at each of the vertices 'a' and 'g'. Then we are left with 8 vertices and 6 edges. Therefore, neither Hamiltonian cycle exists nor Hamiltonian path exists.

73. Ans: (B)

Sol: S1 is not true. A triangle is a counter example.

A triangle contains Euler circuit and the number of edges is 3 (odd)

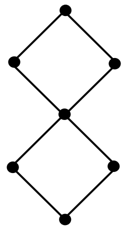
S2 is true. The some of all degrees is even.

$\therefore$ The some degrees is atleast 28.

The statement S2 follows by Pigeonhole principle. $\left(\left\lceil \frac{28}{9} \right\rceil = 4\right)$

74. Ans: (d)

Sol: S1 is false. A counter example is shown below.



The above graph has Euler circuit (because all the vertices are of even degree) but the graph has no Hamiltonian cycle (because a cut vertex exists).

S2 is false. A counter example is a complete graph on 2n vertices ($n \geq 2$).

75. Ans: (b)

Sol: G has cycles of odd length

$\therefore$ Chromatic number of

$G = \chi(G) \geq 3$ ……(1)

For the vertices c and h we can use same color C_1

The remaining vertices from a cycle of length 6.

A cycle of even length require only two colors for its vertex coloring.

For vertices a, d and f we can apply same color C_2

For the vertices {b, e, g} we can use same color C_3

$\therefore \chi(G) = 3$

A perfect matching of the graph is

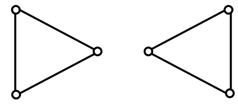
a–b, c – d, e –f, g – h

$\therefore$ Matching number = 4

Hence, Chromatic number of G + Matching number of G = 3 + 4 = 7

76. Ans: (c)

Sol: S_1 need not be true. Consider the graph



Here, we have 6 vertices with degree 2, but the graph is not connected.

S_2 need not be true. For the graph given above, Euler circuit does not exist, because it is not a connected graph.

A simple graph G with n vertices is necessarily connected if $\delta(G) \geq \frac{n-1}{2}$.

$\therefore$ S_3 is true.

77. Ans: (a)

Sol: Vertex connectivity of $G = k(G) \leq \delta(G)$

$\Rightarrow \delta(G) \geq 3$

By sum of degrees theorem

$3|V| \leq 2|E|$

$\Rightarrow |E| \geq 15$

$\therefore$ Minimum number of edges necessary = 15

78. Ans: (a)

Sol: Because, G is connected and every vertex has even degree.

Euler-Circuit exists in G.

Fix some particular circuit and consider a partition of V into two sets S and T.

There must be atleast one edge between S and T, since G is connected.

But if there is only one edge, then euler path can't return to S or T once it leaves.

$\therefore$ It follows that there are atleast two edges between S and T.

79. Ans: (d)

Sol: In a connected graph, Euler circuit exists iff all vertices are of even degree

(a) If n odd then all vertices in K_n are of even degree (n – 1 is even)

$\therefore$ In a complete graph K_n ($n \geq 3$), Euler circuit exists $\Leftrightarrow$ n is odd

(b) If m and n are even, then all vertices in $K_{m,n}$ are of even degree

$\therefore$ In a complete bipartite graph $K_{m,n}$ ($m \geq 2$ and $n \geq 2$), Euler circuit exists $\Leftrightarrow$ m and n are even

(c) In cycle graph degree of each vertex is 2 (even)

$\therefore$ In a cycle graph C_n ($n \geq 3$), Euler circuit exists for all n

(d) In wheel graph W_n, we have n – 1 vertices with degree 3 (odd).

$\therefore$ In a wheel graph W_n ($n \geq 4$), Euler circuit does not exist.

80. Ans: All options are true

Sol: (a) The complete graph K_n can be considered as a polygon with n vertices with all internal diagonals.

The polygon is a Hamiltonian cycle.

$\therefore$ In a complete graph K_n ($n \geq 3$), Hamiltonian cycle exists for all n

(b) If m = n, then we can construct Hamiltonian cycle in $K_{m,n}$.

$\therefore$ In a complete bipartite graph $K_{m,n}$ ($m \geq 2$ and $n \geq 2$), Hamiltonian cycle exists $\Leftrightarrow$ m = n

(c) The cycle graph C_n has a Hamiltonian cycle which is C_n itself.

$\therefore$ In a cycle graph C_n ($n\geq3$), Hamiltonian cycle exists for all n

(d) In a wheel graph W_n ($n\geq4$), Hamiltonian cycle exists $\Leftrightarrow$ n is even.

$\therefore$ All the options are true.

81. Ans: (d)

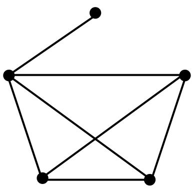
Sol: (a) Number of edge disjoint Hamiltonian cycles in $K_n = \dfrac{n-1}{2}$ (Result)

(b) If G is a simple graph with n vertices and degree of each vertex is atleast $\dfrac{n}{2}$, then Hamiltonian cycle exists in G (Dirac's theorem)

(c) Number of Hamiltonian cycles in $K_{n,n} = \dfrac{n!(n-1)!}{2}$.

Number of Hamiltonian cycles in $K_{4,4} = \dfrac{4!(4-1)!}{2} = 72$

(d) The statement is false, for example,



Here, the above G is a simple graph with 5 vertices and 7 edges, but Hamiltonian cycle does not exist.

4. Set Theory

01. Ans: (a)

Sol: Let $|X| = m$

$\Rightarrow n = 2^m$

Number of elements in $Y = m + 2$

Number of subsets in $Y = 2^{m+2}$

$= 4 \times 2^m$

$= 4n$

02. Ans: (d)

Sol: $\mathbf{S_1}$ **:** Let $A = \{1\}$ and $B = \{A\}$ and $C = B$

Now, $A \in B$ and $B \subseteq C$

But $A \notin C$

$\therefore S_1$ is false

$\mathbf{S_2}$ **:** Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{B\}$

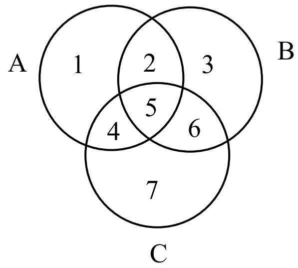
Now, $A \subseteq B$ and $B \in C$

But $A \notin C$

$\therefore S_2$ is false

03. Ans: (c)

Sol: Consider the venn diagram, with seven regions 1, 2, 3, 4, 5, 6, 7.



S1: $A \cup (B - C) = \{1, 2, 4, 5\} \cup \{2, 3\}$

$= \{1, 2, 3, 4, 5\}$

$(A \cup B) - (C - A) = \{1, 2, 3, 4, 5, 6\} - \{6, 7\}$

$= \{1, 2, 3, 4, 5\}$

$\therefore A \cup (B - C) = (A \cup B) - (C - A)$

S2: $A \cap (B - C) = \{1, 2, 4, 5\} \cap \{2, 3\}$

$= \{2\}$

$(A \cap B) - (A \cap C) = \{2, 5\} - \{4, 5\}$

$= \{2\}$

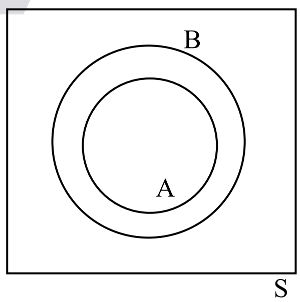
$\therefore A \cap (B - C) = (A \cap B) - (A \cap C)$

04. Ans: (b)

Sol: Given that

$A \subseteq B \subseteq S$

The venn-diagram is shown here



Here, each element of S can appear in 3 ways.

i.e., $x \in A$ or $x \in (B - A)$ or $x \in (S - B)$

In all 3 cases, $A \subseteq B \subseteq S$.

By product rule, the n elements of S can appear in 3^n ways.

$\therefore$ Required number of ordered pairs = 3^n

05. Ans: (d)

Sol: We can show that each element of X can appear in A and B in two ways.

Let $x \in X$

Case 1:

If x is even number then it can appear in two ways i.e., either $x \in (A-B)$ or $x \in (B-A)$

Case 2:

If x is odd number then it can appear in two ways i.e., $x \in (A \cap B)$ or $x \in \left(\overline{A \cup B}\right)$

$\therefore$ By product rule, required number of subsets = 2^{2n}

06. Ans: (d)

Sol: (a) Let $A \oplus B = A$

$\Rightarrow A \oplus B = A \oplus \phi$

$\Rightarrow \quad B = \phi$ (Cancellation law)

(b) $(A \oplus B) \oplus B$

$= A \oplus (B \oplus B)$ (Associative law)

$= A \oplus \phi$

$= A$

(c) $A \oplus C = B \oplus C$

$\Rightarrow A = B$ (Cancellation law)

(d) LHS = $A \oplus B = (A \cup B) - (A \cap B)$

RHS = $(A \cup B) \cap (A - B)$

$= (A - B)$

$\therefore$ L.H.S $\neq$ R.H.S

07. Ans: (c)

Sol: (a) Let A = {1}, B = {2}, C = {3}

Now $(A \cap B) = (B \cap A) = \phi$

But $A \neq B$

$\therefore$ (a) is not true

(b) Let A = {1}, B = {2}, C = {1, 2}

Now $A \cup C = B \cup C = C$

But $A \neq B$

$\therefore$ (b) is not true

(c) Let $x \in A$.

Consider the two cases

Case1: $x \in C$

$\Rightarrow x \notin (A \Delta C)$ $\quad (\because x \in (A \Delta C))$

$\Rightarrow x \notin (B \Delta C)$ $\quad (\because A \Delta C = B \Delta C)$

$\Rightarrow x \in B$ ……….(1)

Case2: $x \notin C$

$\Rightarrow x \in (A \Delta C)$

$\Rightarrow x \in (B \Delta C)$

$\Rightarrow x \in B$ $\quad (\because x \notin C)$ ………(2)

$\therefore A \subseteq B$ (Form (1) and (2))

Similarly we can show that $B \subseteq A$.

$\therefore A = B$

Hence, (c) is true

(d) Let A = {1, 2}

B = {2, 3}

C = {1, 3}

Here, A – C = {2} = B – C

But, $A \neq B$

$\therefore$ Option (d) is not true

08. Ans: (c)

Sol: U = {1, 2, ... n}

A = {(x, X) | x ∈ X and X ⊆ U}

Number of non empty subsets of U = C(n, 1) + C(n, 2) +.......+ C(n, n)

Number of elements in A = $\sum_{k=1}^{n} k\ C(n,k)$

Using Binomial Theorem, we have

$$\sum_{k=1}^{n} k\ C(n,k) = n.\ 2^{n-1}$$

∴ Both I and II are true.

09. Ans: 3 No range

Sol: The elements related to 1 are 1 and 5.

Hence, equivalence class of 1 = [1] = {1, 5}

We pick an element which does not belong to [1] say 2. The elements related to 2 are 2, 3 and 6, hence [2] = {2, 3, 6}

The only element which does not belong to [1] or [2] is 4. The only element related to 4 is 4.

Thus [4] = {4}

Hence, required number of equivalence classes = 3

10. Ans: (d)

Sol: We know that, if R is anti-symmetric relation then any subset of R is also anti-symmetric.

Further (R∩ S) and (R – S) are subsets of R.

Hence, (R ∩ S) and (R – S) are always anti-symmetric.

If (a, b) ∈ R then only (b, a) ∈ R^{-1}

∴ R^{-1} is always anti-symmetric.

Let A = {1, 2, 3} and R={(1, 1), (2, 2), (3,3), (1, 2), (2, 3)}

Then $\overline{R}$ = {(2, 1), (3, 2), (1, 3), (3, 1)}.

Here, R is anti-symmetric but $\overline{R}$ is not anti-symmetric.

∴ Option (d) is correct.

11. Ans: 10

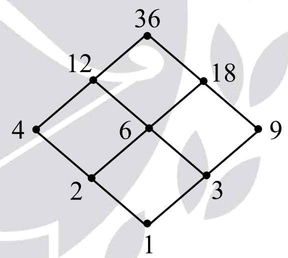
Sol: Symmetric closure of R = {(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (2, 4), (4, 4)}

Transitive symmetric closure of R = {(1, 1), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}.

∴ Required number of ordered pairs = 10

12. Ans: 12

Sol: The Hasse diagram is shown below.



∴ Required number of edges = 12

13. Ans: (b)

Sol: S1 need not be true. We have the following counter example.

R = {(1, 2), (2, 1)}

The transitive closure of R = {(1, 2), (2, 1), (1, 1), (2, 2)}

Which is not irreflexive.

S2 is true. Suppose that (a, b) $\in R^*$; then there is a path from a to b in (the digraph for) R. Given such a path, if R is symmetric, then the reverse of every edge in the path is also in R; Therefore there is a path from b to a in R (following the given path backwards). This means that (b, a) is in R^* whenever (a, b) is, exactly what we needed to prove.

14. Ans: (a)

Sol: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in R$.

R is symmetric, for if xRy, where x and y are real numbers, then $|x - y| < 1$.

$\Rightarrow |y - x| = |x - y| < 1$,

$\Rightarrow$ yRx.

However, R is not an equivalence relation because it is not transitive.

For example, x = 2.8, y = 1.9, and z = 1.1,

Here, $|x - y| = |2.8 - 1.9| = 0.9 < 1$,

$|y - z| = |1.9 - 1.1| = 0.8 < 1$

but $|x - z| = |2.8 - 1.1| = 1.7 > 1$.

i.e., $2.8 \,^R\, 1.9$, $1.9 \,^R\, 1.1$, but 2.8 is not related to 1.1.

15. Ans: (a)

Sol: Let S = {1, 2, ..., n}

If a relation R on S is symmetric and anti-symmetric then R is any subset of the diagonal relation

$\Delta_A = \{(1, 1), (2, 2), ..., (n, n)\}$.

Any subset of Δ_A is also transitive.

$\therefore$ The required number of relations

= Number of subset of Δ_A

$= 2^n$

Relations

16. Ans: (d)

Sol: R_2 is reflexive because for all

$a \in N$, $\frac{a}{a} = 1 = 2^0$, this (a, a) $\in$ R.

R_2 is not symmetric because if (a, b) $\in R_2$,

then $\frac{a}{b} = 2^i$, where $i \geq 0$.

But $\frac{b}{a} = 2^{-i}$, where $-i \leq 0$.

$\therefore$ (b, a) $\notin$ R

17. Ans: (c)

Sol: R can be represented by a square matrix of order n with all the diagonal elements as 1.

Since, R is symmetric,

number of elements above the principal diagonal = number of elements below the principal diagonal.

$\therefore$ Number of elements in R = 2k + n

where k is number of elements above the diagonal

Hence, if n is even then number of elements in R is even and

if n is odd then number of elements in R is odd

18. Ans: (a)

Sol: S_1: Suppose both R and S are reflexive

Let $a \in A$

If {(a, a) $\in$ R and (a, a) $\in$ S} then (a, a) $\in (R \cup S)$.

$\therefore (R \cup S)$ is reflexive

S_2: Suppose both R and S are symmetric

Let $(x, y) \in (R \cup S)$

$\Rightarrow (x, y) \in R$ or $(x, y) \in S$

$\Rightarrow (y, x) \in R$ or $(y, x) \in S$

$\Rightarrow (y, x) \in (R \cup S)$

$\therefore (R \cup S)$ is symmetric

S_3: Suppose both R and S are transitive

Let $R = \{(a, b)\}$ and $S = \{(b, a)\}$

Here, R and S are transitive but $(R \cup S)$ is not transitive

19. Ans: (b)

Sol: If S is any set, then a sub division $\{S_1, S_2, ..., S_n\}$ of S is called a partition of S if $S_1 \cup S_2 \cup .... \cup S_n = S$ and $S_1, S_2, ..., S_n$ are non-empty disjoint sub sets of S.

P_2 is the only one that is not a partition of S, because in which

$\{7, 4, 3, 8\} \cap \{1, 5, 10, 3\} \neq \phi$

20. Ans: (d)

Sol: P_4 is a refinement of both P_1 and P_3, because P_4 itself is a partition of S and every element of P_4 is a subset of one of the elements in P_1 and P_3.

21. Ans: 10

Sol: The number of refinements of a partition P is the number of the ways to further partition cells in P. The cell {1, 2, 3} has 5 ways, {4, 5} has 2 ways, and {6} has one way. Therefore, the total number of refinements of P is $5 \times 2 \times 1 = 10$.

22. Ans: (c)

Sol: S_1 is true, by definition of anti-symmetric relation.

S_2 is true, by definition of transitive relation.

23. Ans: (b)

Sol: $R = \{(a, b) \mid a \text{ divides } b\}$

$R^{-1} = \{(a, b) \mid b \text{ divides } a\}$

Symmetric closure of $R = R \cup R^{-1}$

$= \{(a, b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\}$

24. Ans: (b)

Sol: The smallest relation containing R and $S = R \cup S$

$= \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

Here, $R \cup S$ is reflexive, symmetric and transitive.

The smallest equivalence relation containing R and $S = R \cup S$

The partition corresponding to $R \cup S = \{ \{1, 2\}, \{3, 4, 5\} \}$

25. Ans: (d)

Sol: For any two elements $x, y \in A$, the corresponding equivalence classes are either disjoint or identical.

i.e. if $x \,^{R}\, y$ then $[x] = [y]$

and if x is not related to y, then $[x] \cap [y] = \{\}$.

26. Ans: 1

Sol: The only relation on A, which is both equivalence and partial order is the diagonal relation on A.

i.e., $R = \{(1, 1), (2, 2), (3, 3)\}$

27. Ans: (a)

Sol: R is reflexive, because

(x – x) is an even integer

$\Rightarrow \quad x \, R \, x \quad \forall x \in Z$

Let $x \, R \, y$

$\Rightarrow$ (x – y) is an even integer

$\Rightarrow$ (y – x) is an even integer

$\Rightarrow y \, R \, x \quad \forall x, y \in Z$

$\therefore$ R is symmetric

Let $x \, R \, y$ and $y \, R \, z$

$\Rightarrow$ (x – y) and (y – z) are even integers

Now, (x – z) = (x – y) + (y – z) = an even integer

$\Rightarrow$ R is transitive

$\therefore$ R is an equivalence relation

R is not a partial order, because R is not anti-symmetric.

For example, $2 \, R \, 4$ and $4 \, R \, 2$

28. Ans: 48

Sol: If a relation is neither reflexive nor irreflexive then diagonal pairs can appear in $(2^3 - 2)$ ways.

If the relation is symmetric then non diagonal pairs can appear in 2^3 ways.

By product rule

Required number of relations = $(2^n - 2).\,2^{\frac{n(n-1)}{2}}$, where n = 3

= 6.(8)

= 48

29. Ans: (c)

Sol: The diagonal relation on A is $\Delta_A = \{(1, 1), (2, 2), (3, 3)\}$.

Δ_A is an equivalence relation as well as a partial order on A.

The relation is not a total order.

For example, the elements 2 and 3 are not comparable.

30. Ans: (b)

Sol: S_1 need not be true.

We can give the following counter example.

Let A = {1, 2} and R = {(1, 1), (2, 2), (1, 2)}

and S = {(1, 1), (2, 2), (2, 1)}

Here, R and S are partial orders, but $R \cup S$ is not a partial order.

S_2 is true.

If R and S are any two reflexive relations on a set A, then $(R \cap S)$ is also reflexive.

If R and S are any two anti-symmetric relations on a set A, then $(R \cap S)$ is also anti-symmetric.

If R and S are any two transitive relations on a set A, then $(R \cap S)$ is also transitive.

Hence, If R and S are any two partial orders on a set A, then $(R \cap S)$ is also partial order

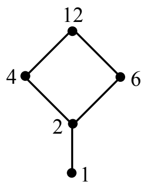
31. Ans: 0

Sol: On a set with 2 elements, if a relation is reflexive and symmetric then it is also transitive.

$\therefore$ There is no relation which is reflexive and symmetric but not transitive.

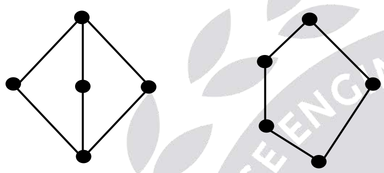
32. Ans: (b)

Sol: The Hasse diagram is shown below.



The poset is a bounded lattice with upper bound 12 and lower bound 1.

The poset is a distributive lattice because it has no sub lattice isomorphic to L_1^* or L_2^* shown below.



L_1^* $\quad$ L_2^*

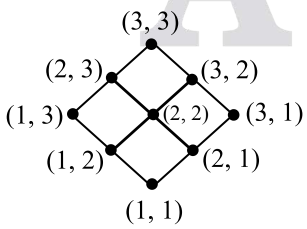
The poset is not a complemented lattice because complements do not exist for the element 2, 4 and 6.

33. Ans: (c)

Sol: The relation R is reflexive, anti-symmetric and transitive.

$\therefore$ R is a partial order.

The Hasse diagram of the poset $[A \times A; R]$ is shown below.



From the Hasse diagram we can see that LUB and GLB exist for every pair of ordered pairs.

$\therefore$ The poset is a lattice.

34. Ans: (c)

Sol: As per the Hasse diagram given in the above example,

The upper bound = I = 36

The lower bound = O = 1

In a lattice, 2 elements a and b are complements of each other if

least upper bound (LUB) of a and b = I and

greatest lower bound (GLB) of a and b = O.

(A) LUB of 2 & 18=LCM of 2 and 18=18$\neq$ I

$\therefore$ Complement of 2 is not 18.

(B) The LUB of 3 and 12 = LCM of 3 and 12 = 12 $\neq$ I

$\therefore$ Complement of 3 is not 12.

(C) The LUB of 4 and 9 = LCM of 4 and 9 = 36 = I

The GLB of 4 and 9 = GCD of 4 and 9 = 1 = O

$\therefore$ Complement of 4 = 9.

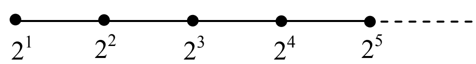
(D) LUB of 6 & 1 = LCM of 6 and 1 = 6 $\neq$ I

$\therefore$ Complement of 6 is not 1.

35. Ans: (c)

Sol: The set A with respect to R is a totally ordered set and therefore a distributive lattice.

The Hasse diagram is shown below.



The upper bound of the lattice does not exist.

$\therefore$ Option (C) is true.

36. Ans: (b)

Sol: In the lattice $[P(A); \subseteq]$,

Complement of $X = A - X \quad \forall X \in P(A)$

$B = \{2, 3, 5, 7\}$

$\therefore$ Complement of $B = A - B$

$= \{1, 4, 6, 8, 9, 10\}$

37. Ans: (a)

Sol: Let x and y be any two elements of S.

Then, the set {x, y} is a subset of S.

So, it has a minimum element z.

if z = x then x R y

if z = y then y R x

$\therefore$ x and y are comparable.

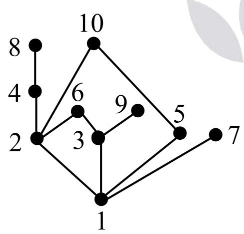
$\Rightarrow$ S is a totally ordered set.

The maximum element of S may not exist.

$\therefore$ Other options need not be true.

38. Ans: 11

Sol: The Hasse diagram is shown below.



$\therefore$ The number of edges in the diagram = 11.

39. Ans: (a)

Sol: If $R \cup R^{-1} = A \times A$, then the given relation R is a total order (linear order).

$\therefore$ The poset [A; R] is a totally ordered set.

Every totally ordered set is a distributive lattice.

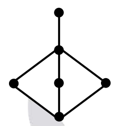
The poset [A; R] is not a complemented lattice, because in a totally ordered set, complements exists only for upper bound and lower bounds.

40. Ans: (d)

Sol: S_1 is false

Proof by counter example:

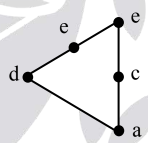
For the lattice shown below



Each element has atmost one complement, but the lattice is not distributive.

$\therefore$ S_1 is false.

For the lattice shown below.



The lattice is complemented. But the sub lattice {a, c, e} is not complemented.

$\therefore$ S_2 is false

41. Ans: (b)

Sol: The given expression is an upper bound of y, so it is at least y.

On the other hand, y is a common upper bound for y and $x \wedge y$, so it is indeed their least upper bound.

42. Ans: (d)

Sol: S_1: L.H.S = $x \vee (y \wedge z)$

$= x \vee O$

$= x$

R.H.S = $(x \vee y) \wedge z$

$= I \wedge z$

$= z$

$\therefore$ L.H.S $\neq$ R.H.S

S_2 : L.H.S = $x \vee (y \wedge z)$

$= x \vee O$

$= x$

R.H.S = $(x \vee y) \wedge (x \vee z)$

$= I \wedge I$

$= I$

$\therefore$ L.H.S $\neq$ R.H.S

43. Ans: (d)

Sol: (a) f is not 1-1 and therefore not a bijection.

For example, $f(1) = f(-1) = 1$

(b) g(x) is not 1-1 and hence not a bijection.

For example, $g(1) = g(-1) = 1$

(c) h(x) is not 1-1 and hence not a bijection.

For example, $h(1.1) = h(1.2) = 1$

(d) Let $\phi(a) = \phi(b)$

$\Rightarrow a^3 = b^3$

$\Rightarrow a = b$

$\Rightarrow \phi$ is one-to-one

Let $\phi(x) = x^3 = y$

$\Rightarrow x = y^{\frac{1}{3}}$

For each real number y, there exists a real number x such that $x = y^{\frac{1}{3}}$.

$\Rightarrow \phi$ is on-to

$\therefore \phi$ is a bijection.

44. Ans: (c)

Sol: Let $f(A) = g(B) = h(C) = D$

We can choose D in C(n, k) ways.

Now, there are k! injections for each of the sets A, B and C.

By productive rule,

Required number of triples of functions $= C(n, k). (k!)^3$

45. Ans: (a)

Sol: S_1: Let $f(x) = x$

Then $f(x) = f(y)$

$\Rightarrow x = y$

$\Rightarrow$ f is one to one

S_2: Let $A = \phi$, then there is not function at all from B to A, surjection or not.

46. Ans: (c)

Sol: Here, A and B are finite sets and $|A| = |B|$

$\therefore$ Every one-to-one function from A to B is on-to, and hence a bijection.

For every bijection f, f^{-1} exists.

$\therefore$ S_1 and S_2 are true

47. Ans: (c)

Sol: **S1:** Let $x \in f^{-1}(S \cup T)$

$\Rightarrow f(x) \in (S \cup T)$

$\Rightarrow f(x) \in S$ or $f(x) \in T$

$\Rightarrow x \in f^{-1}(S)$ or $x \in f^{-1}(T)$

$\Rightarrow x \in \{f^{-1}(S) \cup f^{-1}(T)\}$

$\Rightarrow f^{-1}(S \cup T) \subseteq \{f^{-1}(S) \cup f^{-1}(T)\}$

By retracing the steps, we can show that

$\{f^{-1}(S) \cup f^{-1}(T)\} \subseteq f^{-1}(S \cup T)$

$\therefore f^{-1}(S \cup T) = f^{-1}(S) \cup f^{1}(T)$

Hence, S1 is true.

S2: The proof is similar to that of S1. Please try yourself.

48. Ans: (d)

Sol: Let f(x,y) = (2x–y, x–2y) = (u,v)

$\Rightarrow$ 2x–y = u & x–2y = v

By solving $u = \left(\frac{2x-y}{3}\right)$ & $v = \left(\frac{x-2y}{3}\right)$

$\therefore\ f^{-1}(x,y) = \left(\frac{2x-y}{3}, \frac{x-2y}{3}\right)$

49. Ans: (b)

Sol: (i) If S is a bit string with all ones, then f(S) does not exists.

$\therefore$ f is not a function.

(ii) The number of 1 bits in a bit string is a non negative integer.

$\therefore$ For each bit string S we can assign only one non negative integer in the codomain.

$\therefore$ f is a function.

50. Ans: (a)

Sol: S_1: Let f(a) = f(b)

Where a and b are integers.

$\Rightarrow a^3 = b^3$

$\Rightarrow$ a = b

$\Rightarrow$ f is 1 –1

$f(x) = x^3$ is not on – to. For example, the integer 2 in the codomain is not mapped by any integer of the domain.

$\therefore\ f(x) = \left\lceil \frac{n}{2} \right\rceil$ is not 1-1.

For ex. f(1) = f(2)

However, f(x) is on-to function, because each integer in the co-domain is mapped by atleast one element of the domain.

51. Ans: (a)

Sol: Let f(a) = f(b)

$\Rightarrow \frac{a-2}{a-3} = \frac{b-2}{b-3}$

$\Rightarrow (a-2)(b-3) = (a-3)(b-2)$

$\Rightarrow$ a = b

$\therefore$ f is 1 – 1

Let $f(x) = \frac{x-2}{x-3} = y$

$\Rightarrow x - 2 = (x-3)\,y$

$\Rightarrow x - xy = 2 - 3y$

$\Rightarrow x = \frac{2-3y}{1-y} \in A$

$\therefore$ For each $y \in B$, there exists an element $x \in A$,such that f(x) = y.

$\therefore$ f is on –to

Hence, f is a bijection.

52. Ans: (a)

Sol: (fog)x = f{g(x)}

$= f\left(\frac{x}{1-x}\right)$

$= \frac{\left(\frac{x}{1-x}\right)}{\left(\frac{x}{1-x}\right)+1} = x$

$\Rightarrow$ (fog)x = x

$\Rightarrow$ (fog) is an identity function

$\Rightarrow (fog)^{-1}\,x = (fog)x = x$

53. Ans: (d)

Sol: (d) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

Case 1: when $x \geq 0$

$|x| = x$

$\therefore\ |x| - x = 0$

$\therefore$ f(x) is not defined when $x \geq 0$.

Case 2: when $x < 0$

$|x| = -x$

$\therefore\ |x| - x = -2x > 0$

$\therefore$ Domain of $f(x) = (-\infty, 0)$

54. Ans: (a)

Sol: (a) If $f: A \to B$ then $f^{-1}: B \to A$

$fof^{-1} : B \to B$

$\therefore\ fof^{-1} = I_B$

$\therefore$ Option (a) is false.

55. Ans: (d)

Sol: Let us show that f is injective.

Let x, y be elements of A such that f(x)=f(y)

Then, $x = I_A(x) = g(f(x)) = g(f(y)) = I_A(y) = y$

$\therefore$ f is one to one function

Let us show that g is surjective

Let x be any element of A

Then, f(x) is an element of B

Such that $g(f(x)) = I_A(x) = x$

$\Rightarrow$ g is a on-to function

56. Ans: (c)

Sol: The order of element a = the smallest positive integer n such that $a^n = e$ (identity).

(a) The element 1 is identity element of the group

$\therefore$ order of 1 = 1

(b) $2^1 = 2, 2^2 = 4, 2^3 = 1$

$\therefore$ order of 2 = 3

(c) $3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 4, 3^5 = 5, 3^6 = 1$

$\therefore$ order of 3 = 6

Hence, option (C) is not true

(d) $4^1 = 4, 4^2 = 2, 4^3 = 1$

$\therefore$ order of 4 = 3

57. Ans: (c)

Sol: $A \oplus B = (A-B) \cup (B-A)$

we have $A \oplus B \in P(S), \quad \forall A, B \in P(S)$

$\therefore$ * is a closed operation

We have $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

$\therefore$ * is associative on P(S)

we have, $A \oplus \phi = A, \ \forall A \in P(S)$

$\therefore\ \phi$ is identity element in P(S) w.r.t. *.

We have, $A \oplus A = \phi, \forall A \in P(S)$

$\therefore$ For each element of P(S), inverse exists, because inverse of A=A, $\forall A \in P(S)$.

$\therefore$ (P(S), *) is a group.

58. Ans: 1

Sol: Let e be the identity element.

$a * e = a$

$\Rightarrow \quad \dfrac{ae}{2} = a$

$\Rightarrow \quad e = 2$

Let a^{-1} is inverse of a

$a * a^{-1} = e$

$\Rightarrow \quad \dfrac{aa^{-1}}{2} = 2$

$\Rightarrow \quad a^{-1} = \dfrac{4}{a}$

Inverse of 4 = $\dfrac{4}{4} = 1$

59. Ans: (c)

Sol: Let e be the identity element.

Now a * e = a

$\Rightarrow$ 2 a e = a

$\Rightarrow e = \dfrac{1}{2}$

Let inverse of $\dfrac{2}{3}$ is x

$\dfrac{2}{3} * x = \dfrac{1}{2}$

$\Rightarrow 2\left(\dfrac{2}{3}.x\right) = \dfrac{1}{2}$

$\Rightarrow x = \dfrac{3}{8}$

60. Ans: (b)

Sol: Let e be the identity element.

$\therefore$ a * e = a

$\Rightarrow$ a + e + a.e = a

$\Rightarrow$ e = 0

Let a^{-1} = inverse of a

$a*a^{-1} = e$

$\Rightarrow a+a^{-1}+aa^{-1} = 0$ ($\because$ 0 is identity element)

$\Rightarrow a^{-1} = \dfrac{-a}{a+1}$

$\therefore$ Inverse of –1 does not exist.

Hence, option (b) is false.

61. Ans: (d)

Sol: (d) G = {1,–1, i,–i}

(i) G is closed with respect to multiplication.

(ii) Multiplication is associative on G.

(iii) 1 is identity element in G with respect to multiplication.

(iv) The inverse elements of 1,–1,i,–i are 1, –1, –i, i respectively.

$\therefore$ G is group with respect to multiplication.

62. Ans: (d)

Sol: (d) The cube roots of unity, G= $\{1, \omega, \omega^2\}$ is a group with respect to multiplication.

The inverse of $\omega = \omega^2$

$\therefore$ The statement is false.

63. Ans: (c)

Sol: $5 \oplus_6 2 = 1$

$\Rightarrow$ Inverse of 5 is not 2.

64. Ans: (c)

Sol: Order of (–i) = 4, because the smallest integer n such that $(-i)^n = 1$ is n = 4

65. Ans: (a)

Sol: (a) G = {1, 3, 5, 7} is a group with respect to $\otimes_8$.

$H_1 = \{1, 3\}$ and $H_2 = \{1, 5\}$

$H_1 \cup H_2 = \{1, 3, 5\}$

Here, H_1 and H_2 are subgroups of G,

but $H_1 \cup H_2$ is not a subgroup of G.

66. Ans: (d)

Sol: (d) Every subgroup of a cyclic group is cyclic (theorem)

67. Ans: (d)

Sol: (d) $2^2 = 2 \otimes_7 2 = 4$

$2^3 = 4 \otimes_7 2 = 1$

2 is not a generator of G, because we can not generate 3, 5 and 6 with 2.

68. Ans: (c)

Sol: The identity element of G is 0. In the sets given in options (b) and (d), the identity element is missing.

The set {0, 4} is not closed w.r.t $\oplus$.

The set {0, 2, 4} is closed w.r.t $\oplus$.

$\therefore$ The set in option (c) is a subgroup of G.

69. Ans: 4

Sol: Number of generators in G = $\phi(10) = 4$ where ϕ is Euler function.

70. Ans: (d)

71. Ans: (a)

Sol: Any group with 4 elements is abelian.

$\Rightarrow$ The rows and columns of the table are identical

$\Rightarrow$ First column is $[b \quad d \quad a \quad c]^T$ and second column is $[d \quad c \quad b \quad a]^T$.

Now, the modified table is

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	×	×
d	c	a	×	×

In the composition table of a group, one of the rows of entries should coincide with the top row.

$\therefore$ The third row is a b c d

Hence, the identity element is c.

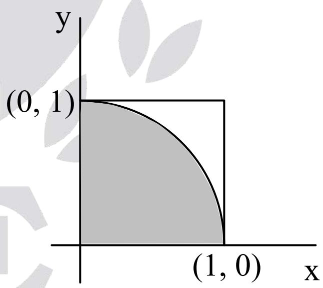
Further, we can show that fourth row is c a d b and

$a^{-1} = d$, $b^{-1} = b$, $c^{-1} = c$ and $d^{-1} = a$.

5. Probability and Statistics

01. Ans: (a)

Sol:



Let x and y are two numbers in the interval (0, 1)

We have to choose x and y such that $x^2 + y^2 < 1$.

$$\text{Required probability} = \frac{\text{Area of the shaded region}}{\text{Area of the square}}$$

$$= \frac{\pi/4}{1} = \frac{\pi}{4}$$

02. Ans: (a)

Sol: A non-decreasing sequence can be described by a partition $n = n_0 + n_1 + n_2$

where n_i is number of times the digit i appear in the sequence.

There are (n + 1) choices for n_0 and given n_0 there are $n - n_0 + 1$ choices for n_1.

So, the total number of possibilities is

$$\sum_{n_0=0}^{n}(n - n_0 + 1) = (n+1).(n+1) - \sum_{n_0=0}^{n} n_0$$

$$= (n + 1) . (n + 1) - \frac{n^2 + n}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

Required probability $= \dfrac{n^2 + 3n + 2}{2(3^n)}$

03. Ans: (d)

Sol: Number of ways, we can choose R = C(n, 3)

We have to count number of ways we can choose R, so that median (R) = median (S).

Each such set R contains median S, one of the $\left(\frac{n-1}{2}\right)$ elements of S less than median (S), and one of the $\left(\frac{n-1}{2}\right)$ elements of S greater than median (S).

So, there are $\left(\frac{n-1}{2}\right)^2$ choices for R.

Required probability $= \dfrac{\left(\frac{n-1}{2}\right)^2}{C(n,3)}$

$$= \frac{3(n-1)}{2n(n-2)}$$

04. Ans: (a)

Sol: For each i ∈ {1, 2, ..., n},

let A_i heads be the event that the coin comes up heads for the first time and continues to come up heads there after.

Then, the desired event is the disjoint union of A_i.

Since, each A_i occurs with probability 2^{-n}.

The required probability = n. 2^{-n}

05. Ans: (b)

Sol: Probability of the event that we never get the consecutive heads or tails

= P(HT HT HT ....) + P(TH TH TH ....)

$$= \left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n$$

$$= 2\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n$$

The required probability $= 1 - 2\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n$

$$= \frac{3^n - 2^{n+1}}{3^{2n}}$$

06. Ans: (c)

Sol: Number of ways of selecting three integers $= {}^{20}C_3$

We know that, product of three integers is even, if atleast one of the number is even.

Number of ways of selecting 3 odd integers $= {}^{10}C_3$

∴ Required probability

$$= 1 - \frac{{}^{10}C_3}{{}^{20}C_3} = 1 - \frac{2}{19} = \frac{17}{19}$$

Conditional probability

07. Ans: (c)

Sol: Given that P(A|B) = 1

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = 1$$

$\Rightarrow P(A \cap B) = P(B)$ .... (1)

$$P(B^C \mid A^C) = \frac{P(B^C \cap A^C)}{P(A^C)} = \frac{1 - P(A \cup B)}{1 - P(A)}$$

$$= \frac{1 - \{P(A) + P(B) - P(A \cap B)\}}{1 - P(A)}$$

$$= \frac{1 - P(A)}{1 - P(A)} \quad \text{[from (1)]}$$

$= 1$

08. Ans: (a)

Sol: Let A = Getting electric contract and B = Getting plumbing contract

$P(A) = \frac{2}{5}$; $P(\overline{B}) = \frac{4}{7}$; $P(B) = \frac{3}{7}$

$P(A \cup B) = \frac{2}{3}$;

$P(A \cap B) = \frac{2}{5} + \frac{3}{7} - \frac{2}{3} = \frac{17}{105}$

09. Ans: (d)

Sol: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A) = \frac{33}{100}$

$P(B) = \frac{14}{100}$

$P(A \cap B) = \frac{4}{100}$

$(A \cap B)$ is not empty set.

Therefore, A and B are not mutually exclusive.

$P(A \cap B) \neq P(A) \,.\, P(B)$

Therefore, A and B are not independent.

10. Ans: 0.2

Sol: To find the number of favourable cases consider the following partition of the given set {1, 2, ..., 100}

$S_1 = \{1, 6, 11, ..., 96\}$

$S_2 = \{2, 7, 12, ..., 97\}$

$S_3 = \{3, 8, 13, ..., 98\}$

$S_4 = \{4, 9, 14, ..., 99\}$

$S_5 = \{5, 10, 15, ..., 100\}$

Each of the above sets has 20 elements. If one of the two numbers selected from S_1 then the other must be chosen from S_4.

If one of the two numbers selected from S_2 then the other must be chosen from S_3.

Number of favourable cases

= C(20,1).C(20,1)+C(20,1).C(20,1)+C(20,2)

= 400 + 400 + 190 = 990

$\therefore$ Required probability $= \frac{990}{C(100,2)}$

$= \frac{990}{50 \times 99} = 0.2$

11. Ans: 0.66 Range 0.65 to 0.67

Sol: Let N = the number of families

Total No. of children $= \left(\frac{N}{2} \times 1\right) + \left(\frac{N}{2} \times 2\right)$

$= \frac{3N}{2}$

$$\therefore \text{The Required Probability} = \frac{\left(\frac{N}{2}\times 2\right)}{\frac{3N}{2}}$$

$$= \frac{2}{3} = 0.66$$

12. Ans: 0.125

Sol: Total number of outcomes = 6^3

Number of outcomes in which sum of the numbers is 10 = Number of non-negative integer solutions to the equation a+b+ c =10 where $1 \le a, b, c \le 6$

= Co-efficient of x^{10} in the function

$(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$

$(x+x^2+x^3+x^4+x^5+x^6)^3 = x^3(1+x+x^2+x^3+x^4+x^5)^3$

$= x^3(1 - x^6)^3 (1 - x)^{-3}$

$$= x^3(1-3x^6 + 3x^{12} - x^{18}) \sum_0^{\infty} \frac{(n+1)(n+2)}{2}.x^n$$

$$= (x^3 - 3x^9 + 3^{18} - x^{21}) \sum_0^{\infty} \frac{(n+1)(n+2)}{2}.x^n$$

Co-efficient of $x^{10} = 36 - 3 \times 3 = 27$

$$\therefore \text{ Required probability} = \frac{27}{216} = 0.125$$

13. Ans: (a)

Sol: If A and B be disjoint events then $A\cap B=\{\ \}$

Probability of $A \cap B = 0$ .... (1)

If A and B are independent then

$P(A \cap B) = P(A).P(B)$ .... (2)

From (1) and (2)

$P(A).P(B) = 0$

$\Rightarrow Pr(A) = 0$ or $Pr(B) = 0$

14. Ans: 2.916 range 2.9 to 2.92

Sol: $E(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$

$$E(X^2) = \frac{1}{6}(1^2+2^2+3^3+4^4+5^2+6^2) = \frac{91}{6}$$

$\therefore$ Variance = $E(X^2) - \{E(X)\}^2$

$$= \frac{91}{6} - (3.5)^2 = 2.916$$

15. Ans: (c)

Sol: Total number of counters

$$= 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

Probability of choosing counter k and winning $k^2 = \frac{2k}{n(n+1)}$

$$\text{Expectation} = \sum_{k=1}^{n}\left\{k^2.\frac{2k}{n(n+1)}\right\}$$

$$= \frac{2}{n(n+1)}.\frac{n^2(n+1)^2}{4} = \frac{n(n+1)}{2}$$

16. Ans: (b)

Sol: The probability that she gives birth between 8 am and 4 pm in a day = $\frac{1}{3}$

By Total theorem of probability,

The required probability

$$= \left(\frac{1}{3}\times\frac{3}{4}\right)+\left(\frac{2}{3}\times\frac{1}{4}\right) = \frac{5}{12}$$

Random Variables

17. Ans: 0.75 (No range)

Sol: Total probability = $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{0}^{2} cx \ dx = 1$$

$$\Rightarrow c = \frac{1}{2}$$

$$P(X > 1) = \int_{1}^{\infty} f(x)dx = \int_{1}^{2} \frac{1}{2} x(dx) = \frac{3}{4} = 0.75$$

18. Ans: 1.944 range 1.94 to 1.95

Sol: The probability distribution for Z is

Z	0	1	2	3	4	5
P(Z)	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

E(Z) = ΣZ . P(Z)

$$= \frac{1}{36}(0(6) + 1(10) + 2(8) + 3(6) + 4(4) + 5(2))$$

$$= \frac{70}{36} = \frac{35}{18} = 1.944$$

19. Ans: (c)

Sol: $E(a^x) = \sum_{k=0}^{n} a^k . P(X = k)$

$$= \sum_{k=0}^{n} a^k C(n,k) \left(\frac{1}{2}\right)^k . \left(\frac{1}{2}\right)^{n-k}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} a^k C(n,k) a^k . (1)^{n-k}$$

$$= \left(\frac{a+1}{2}\right)^n$$

20. Ans: (d)

Sol: Given that mean = E(X) = 1

and Variance = V(X) = 5

$E((2 + X)^2) = E[X^2 + 4X + 4]$

$= E(X^2) + 4\,E(X) + 4$

Given V(X) = 5

$\Rightarrow E(X^2) - (E(X))^2 = 5$

$\Rightarrow E(X^2) = 5 + 1 = 6$

$E((2 + X)^2) = 6 + 4(1) + 4 = 14$

21. Ans: (a)

Sol: Total Probability = $\sum_{x=1}^{\infty} P(X = x) = 1$

$$\Rightarrow \sum_{x=1}^{\infty} K(1-\beta)^{x-1} = 1$$

$$\Rightarrow K(1 + (1-\beta) + (1-\beta)^2 + \ldots\ldots\ \infty) = 1$$

$$\Rightarrow \frac{K}{1-(1-\beta)} = 1$$

$$\Rightarrow K = \beta$$

22. Ans : 209

Sol:

x	2	–3	4	–5	6	–7	8	–9	10	–11	12
P(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(X) = \sum x\,P(x) = (-3)\times\frac{1}{6}+6\times\frac{1}{2}+9\times\frac{1}{3}=\frac{11}{2}$$

$$E(X^2) = \sum x^2\,P(x) = 9\times\frac{1}{6}+36\times\frac{1}{2}+81\times\frac{1}{3}=\frac{93}{2}$$

$$\therefore\ E(2X+1)^2 = E(4X^2+4X+1)$$

$$= 4E(X^2)+4E(X)+1$$

$$= 4\times\frac{93}{2}+4\times\frac{11}{2}+1$$

$$= 209$$

23. Ans: (d)

Sol: Let X = Amount your win in rupees

The probability distribution of X is shown below.

X	1	–2	3	–4	5	–6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The required expectation

$= E(X) = \sum[X.\,P(X)]$

$$= \frac{1}{6}\ (1-2+3-4+5-6) = \frac{-1}{2}$$

24. Ans: 0.1

Sol: $E(W) = \int_0^{10} 0.003\,V^2\,f(V)\ dV$

$$= \int_0^{10} 0.003\,V^2\,\frac{1}{10}\,dV$$

$= 0.1$ *lb/ft*^{2}

Where f(V)= probability density function of V

25. Ans: (b)

Sol: By Chebyshev inequality

$$\Pr(\mu - k\sigma < X < \mu + k\sigma) \ge 1-\frac{1}{k^2}$$

26. Ans: 0

Sol: Let X = Number of rupees you win on each throw. The probability distribution of X is

$E(X) = \sum X.P(X) = 0$

27. Ans: (c)

Sol: Let X = Number of rupees you win on each throw.

The probability distribution of X is

X	0	1	2	3	4	5
P(X)	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

$E(X) = \sum X.P(X) = \frac{35}{18}$

28. Ans: 0.23 **range 0.22 to 0.24**

Sol: Let X = number of ones in the sequence

n = 5

p = probability for digit 1 = 0.6

q = 0.4

Required probability = P(X = 2)

$= C(5, 2).\ (0.6)^2.\ (0.4)^3$

$= 0.23$

29. Ans: 0.25 **range 0.24 to 0.26**

Sol: Given that, mean = 2(variance)

$\Rightarrow$ np = 2(npq) .... (1)

further, np + npq = 3 .... (2)

Solving, n = 4, $p = q = \frac{1}{2}$

$$P(X=3) = C(4,3).\left(\frac{1}{2}\right)^3.\left(\frac{1}{2}\right) = \frac{1}{4} = 0.25$$

30. Ans: (d)

Sol: Let X = Number of times we get negative values.

By using Binomial Distribution,

$$P(X = k) = C(n, k)\, p^k q^{n-k}$$

Where $p = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 5$

Required probability = $P(X \le 1)$

$= P(X = 0) + P(X = 1)$

$$= {}^5C_0 \times \left(\frac{1}{2}\right)^5 + {}^5C_1 \times \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)$$

$$= \frac{1+5}{32} = \frac{6}{32}$$

31. Ans: (d)

Sol: We can choose four out of six winning in C(6, 4) different ways and if the probability of winning a game is p, then the probability of winning four out of six games

$= C(6, 4)\, p^4(1 - p)^2$

$= 15(p^4 - 2p^5 + p^6)$

32. Ans: 0.5706

Sol: The odds that the program will run is 2 : 1.

Therefore, Pr(a program will run) = $\frac{2}{3}$. Let B denote the event that four or more programs will run and A_j denote that exactly j program will run. Then,

$Pr(B) = Pr(A_4 \cup A_5 \cup A_6)$

$= Pr(A_4) + Pr(A_5) + Pr(A_6)$

$$= C(6,4)\left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + C(6,5)\left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) + C(6,6)\left(\frac{2}{3}\right)^5$$

= 0.5706

33. Ans: 0.224 range 0.2 to 0.3

Sol: Average calls per minute = $\frac{180}{60} = 3$

Here, we can use poisson distribution with λ=3.

Required Probability = $P(X = 2) = \frac{e^{-3}.3^2}{2\,!}$

$$= \frac{e^{-3}.9}{2} = 4.5\, e^{-3} = 0.224$$

34. Ans: 0.168

Sol: λ = average number of cars pass that point in a 12 min period = $\frac{15}{60/12} = 3$

Using the Poisson distribution,

$$Pr(k) = e^{-\lambda} \frac{\lambda^k}{k\,!}$$

$\therefore$ Required probability $Pr(4) = e^{-3}\frac{3^4}{4\,!} = 0.168$

35. Ans: 0.7 range 0.65 to 0.75

Sol: The probability density function of

$$X = f(x) = \begin{cases} \frac{1}{10} & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P\left\{\left(X + \frac{10}{X}\right) \ge 7\right\} = \{P\left(X^2 + 10 \ge 7X\right)$$

$= P(X^2 - 7X + 10 \ge 0)$

$= P\{(X - 5)(X - 2) \ge 0\}$

$= P(X \le 2 \text{ or } X \ge 5)$

$= 1 - P(2 \le X \le 5)$

$$= 1 - \int_2^5 f(x)\, dx$$

$$= 1 - \int_2^5 \frac{1}{10}\, dx$$

$$= 1 - \frac{3}{10} = 0.7$$

36. Ans: (a)

Sol: We can use Exponential Distribution with mean $\mu = 5$

Let X is waiting time in minutes.

Probability Density function of X is

$$f(x) = 0.2\, e^{-(0.2)x} \quad \text{if } x \ge 0$$
$$= 0 \quad \text{if } x < 0$$

The required probability = $P(0 < X < 1)$

$$= \int_0^1 0.2\, e^{-(0.2)x}\, dx = 0.1813$$

37. Ans: (a)

Sol: $\sum_{r=1}^{\infty} P(X = r) = 1$

$\Rightarrow k(1 + (1-\beta) + (1-\beta)^2 + \ldots\ldots \infty) = 1$

$\Rightarrow k\left\{\frac{1}{1-(1-\beta)}\right\} = 1$

$\Rightarrow k = \beta$

$\therefore P(X = r) = \beta(1-\beta)^{r-1}$

This function is maximum when r = 1.

$\therefore$ mode = 1

38. Ans: Mean = 34, Median = 35, Modes = 35, 36 & SD = 4.14

Sol: Mean $= \frac{\sum x_i}{n} = 34$

Median is the middle most value of the data by keeping the data points in increasing order or decreasing order.

Mode = 36

S.D = 4.14

39. Ans: 1.095

Sol: $\mu = \text{Mean} = \sum_{k=1}^{5}\{x_k . P(X = k)\}$

$= 1(0.1) + 2(0.2) + 3(0.4) + 4(0.2) + 5(0.1)$

$= 3$

$P(X \le 2) = 0.1 + 0.2 = 0.3$

$P(X \le 3) = 0.1 + 0.2 + 0.4 = 0.7$

$\therefore \text{Median} = \frac{2+3}{2} = 2.5$

Mode = The value of X at which P(X) is maximum = 3

$\text{Variance} = \sum_{k=1}^{5} x_k^2 . P(X = k) - \mu^2$

$= 10.2 - 9 = 1.2$

Standard deviation $= \sqrt{1.2} = 1.095$

40. Ans: k = 6, Mean = $\frac{1}{2}$, Median = $\frac{1}{2}$, Mode = $\frac{1}{2}$ and S.D = $\frac{1}{2\sqrt{5}}$

Sol: We have $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^1 k(x - x^2)\, dx = 1$$

$$\Rightarrow k\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1\right] = 1$$

$$\Rightarrow k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$$

$$\Rightarrow k\left(\frac{3-2}{6}\right) = 1 \Rightarrow k = 6$$

Mean = $\int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{1} 6(x^2 - x^3)\,dx$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{2}$$

Median is that value 'a' for which

$$P(X \le a) = \frac{1}{2} \quad \int_0^a 6(x - x^2)\,dx = \frac{1}{2}$$

$$\Rightarrow 6\left(\frac{a^2}{2} - \frac{a^3}{3}\right) = \frac{1}{2}$$

$$\Rightarrow 3a^2 - 2a^3 = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{2}$$

Mode a that value at which f(x) is max/min

$\therefore$ $f(x) = 6x - 6x^2$

$f^{1}(x) = 6 - 12x$

For max or min $f^{1}(x) = 0 \Rightarrow 6 - 12x = 0$

$\Rightarrow x = \frac{1}{2}$ $f^{11}(x) = -12$ $\quad f^{11}\left(\frac{1}{2}\right) = -12 < 0$

$\therefore$ maximum at $x = 1/2$

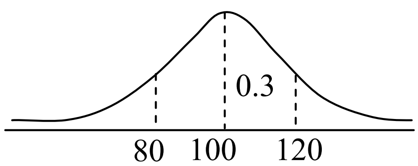
$\therefore$ mode is 1/2

$$S.D = \sqrt{E(x^2) - (E(x))^2}$$

$$= \frac{1}{2\sqrt{5}}$$

41. Ans: 0.2

Sol: The area under normal curve is 1 and the curve is symmetric about mean.



$\therefore$ $P(100 < X < 120) = P(80 < X < 120)$

$= 0.3$

Now, $P(X < 80) = 0.5 - P(80 < X < 120)$

$= 0.5 - 0.3 = 0.2$

42. Ans: 4

Sol: If n missiles are fired then probability of not hitting the target = $[1 - (0.3)]^n = (0.7)^n$

$\Rightarrow$ Probability of hitting the target atleast once = $1 - (0.7)^n$

We have to fired the smallest +ve integer n so that, $\{1 - (0.7)^n\} > \frac{75}{100}$

$\Rightarrow \{1 - (0.7)^n\} > 0.75$

The smallest +ve integer satisfying this inequality is n = 4

43. Ans: 0.865 range 0.86 to 0.87

Sol: Let X = number of cashew nuts per biscuit.

We can use Poisson distribution with mean

$$= \lambda = \frac{2000}{1000} = 2$$

$$P(X = k) = \frac{e^{-\lambda}.\lambda^k}{\angle k} \quad (k = 0,1,2...)$$

Probability that the biscuit contains no cashew nut = P (X = 0)

$= e^{-\lambda} = e^{-2} = 0.135$

Required probability = 1 – 0.135 = 0.865

44. Ans: (b)

Sol: Let A = getting red marble both times

B = getting both marbles of same color

$$P(A \cap B) = \frac{3}{10}.\frac{2}{10}$$

$$P(B) = \frac{7}{10}.\frac{6}{10} + \frac{3}{10}.\frac{2}{10}$$

Required probability $= \frac{P(A \cap B)}{P(B)} = \frac{6}{48} = \frac{1}{8}$

45. Ans: (d)

Sol: Let E_1 = The item selected is produced machine C and E_2 = Item selected is defective

$$P(E_1 \wedge E_2) = \frac{20}{100}.\frac{5}{100}$$

$$P(E_2) = \frac{50}{100}.\left(\frac{3}{100}\right) + \frac{30}{100}.\left(\frac{4}{100}\right) + \frac{20}{100}.\left(\frac{5}{100}\right)$$

Required probability

$$= P(E_1/E_2) = \frac{P(E_1 \wedge E_2)}{P(E_2)} = \frac{100}{370} = \frac{10}{37}$$

6. Linear Algebra

01. Ans: 3

Sol: If rank of A is 1, then A has only one independent row.

The elements in R_1 and R_2 are proportional

$$\Rightarrow \frac{3}{P} = \frac{P}{3} = \frac{P}{P}$$

$$\Rightarrow P = 3$$

02. Ans: 25

Sol: Let $A = \begin{pmatrix} x & y \\ y & 10-x \end{pmatrix}$

Det A = $x(10 - x) - y^2$

For maximum value of Det A, y = 0

Now, $A = \begin{pmatrix} x & 0 \\ 0 & 10-x \end{pmatrix}$

$\Rightarrow |A| = x(10 - x) = 10x - x^2$

Let $f(x) = 10x - x^2$

$\Rightarrow f'(x) = 10 - 2x$

$\Rightarrow f''(x) = -2$

Consider, $f'(x) = 0$

$\Rightarrow x = 5$

At x = 5, $f''(x) = -2 < 0$

$\therefore$ At x=5, the function f(x) has a maximum and is equal to 25.

03. Ans: (c)

Sol: Given $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = A^{-1}$.

The element in the second row and third column of B

= Cofactor of the element in the third row second column of A

$$= (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$\therefore$ Required element $= \frac{1}{|A|}(-2) = \frac{-1}{2}$

04. Ans: (a)

Sol: Here, A^n is a zero matrix. [Property]

$\therefore$ rank of $A^n = 0$

05. Ans: 46

Sol: Here, $|\text{adj } A| = |A|^2$

$\Rightarrow 2116 = |A|^2$

$\Rightarrow |A| = \pm 46$

$\Rightarrow$ Absolute value of $|A| = 46$

06. Ans: (b)

Sol: S_1) If A and B are symmetric then AB need not be equal to BA

for example, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

then A and B are symmetric but AB is not equal to BA.

$\therefore$ S_1 is false.

S_2) If A and B are symmetric then AB – BA is a skew-symmetric matrix of order 3.

$\therefore$ $|AB - BA| = 0$

($\because$ determinant of a skew-symmetric matrix of odd order is 0) Hence, S_2 is true.

07. Ans: (a)

Sol: Each element of the matrix in the principal diagonal and above the diagonal, we can choose in q ways.

Number of elements in the principal diagonal = n

Number of elements above the principal diagonal = $n\left(\frac{n-1}{2}\right)$

By product rule,

number of ways we can choose these elements = $q^n . q^{n\left(\frac{n-1}{2}\right)}$

Required number of symmetric matrices= $q^{n\left(\frac{n+1}{2}\right)}$

08. Ans: (b)

Sol: $A = \begin{bmatrix} n-1 & -1 & ..... & -1 \\ -1 & n-1 & ..... & -1 \\ .... & ..... & .... & .... \\ -1 & -1 & ..... & n-1 \end{bmatrix}$

$R_1 \to R_1 + R_2 + ... + R_{n-1}$

$A = \begin{bmatrix} 1 & 1 & ..... & 1 \\ -1 & n-1 & ..... & -1 \\ .... & ..... & .... & .... \\ -1 & -1 & ..... & n-1 \end{bmatrix}$

$R_2 \to R_2 + R_1, R_3 \to R_3 + R_1, ..., R_{n-1} \to R_{n-1} + R_1,$

$A = \begin{bmatrix} 1 & 1 & ..... & 1 \\ 0 & n & ..... & 0 \\ .... & ..... & .... & .... \\ 0 & 0 & ..... & n \end{bmatrix}$

$= n^{n-2}$

09. Ans: (a)

Sol: S1 is true because, any subset of four linearly independent sequence of vectors is always linearly independent.

S2 is not necessarily true,

For example, x_1, x_2 and x_3 can be linearly independent and x_4 is linear combination of x_1, x_2 and x_3.

10. Ans: (c)

Sol: The given matrix is skew-symmetric.

Determinant of a skew symmetric matrix of odd order is 0.

$\therefore$ Rank of A < 3.

Determinant of a non-zero skew symmetric matrix is ≥ 2

$\therefore$ Rank of A = 2

11. Ans: (a)

Sol: Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}$

For the system of linear equations to have a unique solution, det(A) $\neq 0$.

$\Rightarrow (0 - 2\alpha) + 2(2\alpha + 2\alpha) + (4 - 0) \neq 0$

$\Rightarrow -2\alpha + 8\alpha + 4 \neq 0$

$\Rightarrow 6\alpha + 4 \neq 0$

$\Rightarrow 6\alpha \neq -4$

$\Rightarrow \alpha \neq \dfrac{-2}{3}$

$\therefore$ Option (a) is correct.

12. Ans: (c)

Sol: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$

Applying $R_2 - 2R_1$, $R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & -5 & -2 \end{bmatrix}$$

Applying $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

which is an echelon matrix with two non-zero rows.

$\therefore$ Rank of A = 2

If rank of A is less than number of variables, then the system AX = O has infinitely many non-zero solutions.

If rank of A is less than number of variables, then the system AX = B cannot have unique solution.

Hence, option (c) is not true.

If rank of A is less than order of A, then the matrix A is singular.

$\therefore$ A^{-1} does not exist

13. Ans: (b)

Sol: $D = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix}$

$= k^3 + 1 + 1 - k - k - k$

$= (k - 1)^2 (k + 2)$

Thus, the system has a unique solution when

$(k - 1)^2 (k + 2) \neq 0$

$\Rightarrow k \neq 1$ and $k \neq -2$

14. Ans: (c)

Sol: The augmented matrix is

$$(A \mid B) = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix} \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \to 5R_3 + R_2 \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho(A \mid B) = 2$ (< number of variables).

$\therefore$ The system has infinitely many solutions.

15. Ans: (c)

Sol: Given AX = B

$$\Rightarrow \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 4 & k\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\4\\6\end{bmatrix}$$

$$[A \mid B] = \left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\1 & 2 & 3 & 4\\1 & 4 & k & 6\end{array}\right]$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\0 & 1 & 2 & 1\\0 & 3 & k-1 & 3\end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\0 & 1 & 2 & 1\\0 & 0 & k-7 & 0\end{array}\right]$$

If $k - 7 \neq 0$ then the system will have unique solution.

If k = 7, we have rank of A = rank of [A | B] = 2 (< number of variables)

$\therefore$ The system has infinitely many solutions.

16. Ans: –1 or 0

Sol: $A = \begin{bmatrix}k & k & k\\0 & k-1 & k-1\\0 & 0 & k^2-1\end{bmatrix}$

Given that AX = 0 has only one independent solution

$\Rightarrow$ Rank of A = 2 $\quad (\because\ n - r = 3 - r = 1)$

$\Rightarrow k = -1$ or $k = 0$

17. Ans: (b)

Sol: Given AX = B

B = linear combination of independent columns of A

$\Rightarrow \rho(A) = \rho(A \mid B) = 3$

$\therefore$ The system has infinitely many solutions.

18. Ans: (c)

Sol: The augmented matrix of the given system

$$\text{is } (AB) = \begin{pmatrix}3 & 2 & 0 & 1\\4 & 0 & 7 & 1\\1 & 1 & 1 & 3\\1 & -2 & 7 & 0\end{pmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix}1 & 1 & 1 & 3\\4 & 0 & 7 & 1\\3 & 2 & 0 & 1\\1 & -2 & 7 & 0\end{pmatrix}$$

$R_2 - 4R_1$, $R_3 - 3R_1$, $R_4 - R_1$

$$\sim \begin{pmatrix}1 & 1 & 1 & 3\\0 & -4 & 3 & -11\\0 & -1 & -3 & -8\\0 & -3 & 6 & -3\end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix}1 & 1 & 1 & 3\\0 & -1 & -3 & -8\\0 & -4 & 3 & -11\\0 & -3 & 6 & -3\end{pmatrix}$$

$R_3 - 4R_2$, $R_4 - 3R_2$

$$\sim \begin{pmatrix}1 & 1 & 1 & 3\\0 & -1 & -3 & -8\\0 & 0 & 15 & 21\\0 & 0 & 15 & 21\end{pmatrix}$$

$R_4 - R_3$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore \rho(A) = \rho(AB) = 3$

$= \text{no. of variables}$

Hence, there exists only one solution.

19. Ans: (d)

Sol: If $A_{n\times n}$ has n distinct eigen values, then A has n linearly independent eigen vectors.

If zero is one of the eigen values of A, then A is singular and A^{-1} does not exist.

If A is singular then rank of A < 3 and A cannot have 3 linearly independent rows.

$\therefore$ Only option (d) is correct.

20. Ans: (b)

Sol: $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equations is

$$d^3 - 18d^2 + 45d = 0$$

$\Rightarrow d = 0, 3, 15$ are eigen values of A.

21. Ans: (a)

Sol: Since, A is singular, $\lambda = 0$ is an eigen value.

Also, rank of A = 1.

The root $\lambda = 0$ is repeated $n - 1$ times.

trace of A = $n = 0 + 0 + ... + \lambda_n$.

$\Rightarrow \lambda_n = n$

$\therefore$ The distinct eigen values are 0 and n.

22. Ans: (c)

Sol: The characteristic equation is

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

By Caley Hamilton's theorem,

$$A^3 - 6A^2 + 11A - 6\,I = 0$$

Multiplying by A^{-1},

$$(A^2 - 6A + 11\,I) = 6A^{-1}$$

23. Ans: (b)

Sol: Let $A = \begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$

Consider $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - (-2)\lambda + (-120 + 72) = 0$

$\Rightarrow \lambda^2 + 2\lambda - 48 = 0$

$\therefore \lambda = 6, -8$ are eigen values of A.

For $\lambda = 6$, the eigen vectors are given by

$$[A - 6I]\,X = O$$

$$\Rightarrow \begin{bmatrix} 4 & -4 \\ 18 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

The eigen vectors are of the form

$$X_1 = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -8$, the eigen vectors are given by

$$[A+8I]\,X = O$$

$$\Rightarrow \begin{bmatrix} 18 & -4 \\ 18 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 18x - 4y = 0$

$\Rightarrow \quad 9x - 2y = 0$

The eigen vectors are of the form

$$X_1 = k_2\begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

24. Ans: (c)

Sol: The given matrix is upper triangular. The eigen values are same as the diagonal elements 1, 2, –1 and 0.

The smallest eigen value is λ = –1. The eigen vectors for λ = –1 is given by

$(A - \lambda I)X = 0$

$\Rightarrow (A + I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$\Rightarrow w = 0, y = 0, 2x - z = 0$

$\Rightarrow X = k[1\ \ 0\ \ 2\ \ 0]^T$

25. Ans: (b)

Sol: Let λ be the third eigen value.

Sum of the eigen values of A = Trace (A)

$\Rightarrow (-3) + (-3) + \lambda = -2 + 1 + 0$

$\Rightarrow \lambda = 5$

The eigen vector for λ = 5 is given by

$[A - 5I]X = O$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$$\therefore \text{ The third eigen vector} = k\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

26. Ans: 7

Sol: Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}$

eigen vector $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

We know that $AX = \lambda X$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ -16 - 2x \\ 15 \end{bmatrix} = \begin{bmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{bmatrix}$$

Clearly eigen value λ = 15

$\Rightarrow -16 - 2x = -30$

$\therefore -2x = -14$

$x = 7$

27. Ans: 2

Sol: If λ is an Eigen values of A, then

$\lambda^4 - 3\lambda^3$ is an Eigen value of $(A^4 - 3A^3)$

Putting λ = 1, –1 and 3 in $(\lambda^4 - 3\lambda^3)$,

we get the eigen values of $(A^4 - 3A^3)$

are –2, 4, 0

Trace of $(A^4 - 3A^3)$ = Sum of eigen values of $(A^4 - 3A^3) = 2$

28. Ans: 8

Sol: Given $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

The characteristic equation is $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

By Caley-Hamilton's theorem,

$A^3 - A^2 - 4A + 4I = O$

adding 2I on both sides

$A^3 - A^2 - 4A + 6I = 2I$

Let $B = A^3 - A^2 - 4A + 6I$

Now $B = 2I$

$\therefore |B| = |2I| = 8$

29. Ans: 2

Sol: $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Clearly $\lambda = 2$

30. Ans: (d)

Sol: We have, $A^T = -A$ ($\because$ A is skew-symmetric)

$\Rightarrow A + A^T = (A - A) = O$

Rank of $(A + A^T) = 0$

$\therefore$ Number of linearly independent eigen vectors = n – rank of $(A + A^T) = n$

31. Ans: (a)

Sol: For upper triangular matrix the eigen values are same as the elements in the principal diagonal.

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$$

$$(I + A) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$|I + A| = 1$

$\therefore$ I + A is non-singular and hence invertible.

32. Ans: 8

Sol: The characteristic equation of M is

$\lambda^3 - 12\lambda^2 + a\lambda - 32 = 0$ ……… (1)

Substituting $\lambda = 2$ in (1), we get a = 36

Now, the characteristic equation is

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$

$\Rightarrow \lambda = 2, 2, 8$

$\therefore$ The largest among the absolute values of the eigen values of M = 8.

33. Ans: (b)

Sol: $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix}$

Applying $R_2 + 3R_1, R_3 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & -3 & 1 \end{bmatrix}$$

Applying $R_3 + \frac{3}{2}R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\therefore\ U = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & \frac{-3}{2} & 1 \end{bmatrix}$$

[The corresponding coefficients in the elementary operations]

7. Calculus

01. Ans: 0.8165 range 0.81 to 0.82

Sol: Let $f(x) = \dfrac{1+x}{2+x}$

and $g(x) = \dfrac{1-\sqrt{x}}{1-x}$

$\underset{x\to 1}{\text{Lt}}\ f(x) = \dfrac{2}{3}$ (finite)

$\underset{x\to 1}{\text{Lt}}\ g(x) = \dfrac{1}{2}$ (finite)

Since, both the limits are finite.

The given limit $= \left(\dfrac{2}{3}\right)^{\frac{1}{2}} = \sqrt{\dfrac{2}{3}} = 0.8165$

02. Ans: (a)

Sol: Require formula $= \underset{R\to 0}{\text{Lt}}\ \dfrac{E}{R}\left(1-e^{-Rt/L}\right)$

$$= \underset{R\to 0}{\text{Lt}}\ \frac{E\left(e^{-Rt/L}\right)\frac{t}{L}}{1}$$

(By L.Hospital's Rule)

$$= \frac{Et}{L}$$

03. Ans: (c)

Sol: f(x) is in $\dfrac{0}{0}$ form

By L-Hospital's rule

$$\underset{x\to a}{\text{Lt}}\ f(x) = \underset{x\to a}{\text{Lt}}\left[\frac{\sqrt{2a^3x - x^4} - a\left(a^2x\right)^{\frac{1}{3}}}{a - \left(ax^3\right)^{\frac{1}{4}}}\right]$$

By L-Hospital's rule

$$= \underset{x\to a}{\text{Lt}}\left[\frac{\dfrac{2a^3 - 4x^3}{2\sqrt{2a^3x - x^4}} - \dfrac{1}{3}a^{\frac{5}{3}}x^{\frac{-2}{3}}}{-a^{\frac{1}{4}}\dfrac{3}{4}x^{\frac{-1}{4}}}\right]$$

$$= \frac{\left(\frac{-4a}{3}\right)}{\left(\frac{-3}{4}\right)} = \frac{16a}{9}$$

04. Ans: (c)

Sol: (a) f(x) =|x| is not differentiable at x = 0

(b) f(x) = cot x is neither continuous nor differentiable at x = 0

(c) f(x) = sec x is differentiable in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and hence in the interval [–1, 1]

(d) f(x) = cosec x is neither continuous nor differentiable at x = 0

05. Ans: (a)

Sol: we have $\underset{x\to 1-}{\text{Lt}}\ f(x) = \underset{x\to 1+}{\text{Lt}}\ f(x) = f(1)$

$\therefore$ f(x) is continuous at x = 1

$$f^{1}(1-) = \underset{h\to 0-}{\text{Lt}}\left[\frac{f(1+h) - f(1)}{h}\right]$$

$$= \underset{h\to 0-}{\mathrm{Lt}}\left[\frac{(1+h)-1}{h}\right] = 1$$

$$f^1(1+) = \underset{h\to 0+}{\mathrm{Lt}}\left[\frac{f(1+h)-f(1)}{h}\right]$$

$$= \underset{h\to 0+}{\mathrm{Lt}}\left[\frac{\{2(1+h)-1\}-1}{h}\right] = 2$$

$\therefore f^1(1-) \neq f^1(1+)$

Hence, f(x) is not differentiable at x = 1.

06. Ans: (a)

Sol: Since, f is differentiable at x = 2,

$f'(2-) = f'(2+)$

$\Rightarrow (2x)_{x=2} = m$

$\Rightarrow m = 4$

Since, f is continuous at x = 2

$(x^2)_{x=2} = (mx + b)_{x=2}$

$\Rightarrow 4 = 2m + b$

$\Rightarrow b = -4$

Hence, option (a) is correct.

07. Ans: (c)

Sol: By Lagrange's theorem,

$$f'(C) = \frac{f(8)-f(1)}{8-1}$$

$$1-\frac{4}{C^2} = \frac{8.5-5}{7}$$

$$C = \pm 2\sqrt{2}$$

But only, $C = 2\sqrt{2} \in (1, 8)$

08. Ans: (a)

Sol: Given $f(x) = 3x^2 + 4x - 5$

$f'(x) = 6x + 4$

By Lagrange's Mean Value Theorem, there exist a value $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

$$= \frac{32}{2} = 16$$

09. Ans: 2.5 range 2.49 to 2.51

Sol: By Cauchy's mean value theorem,

$$\frac{f'(c)}{g'(c)} = \frac{f(3)-f(2)}{g(3)-g(2)}$$

$$\Rightarrow -e^{2c} = \frac{e^3-e^2}{e^{-3}-e^{-2}}$$

$\Rightarrow c = 2.5$

10. Ans: (a)

Sol: The conditions of Cauchy's theorem hold good for f(x) and g(x).

By Cauchy's theorem, there exists a value c such that

$$\frac{f'(c)}{g'(c)} = \frac{f(3)-f(2)}{g(3)-g(2)}$$

$$\frac{\left(\frac{-1}{c^2}\right)}{\left(\frac{-2}{c^3}\right)} = \frac{\left(\frac{1}{3}-\frac{1}{2}\right)}{\left(\frac{1}{9}-\frac{1}{4}\right)}$$

$\Rightarrow c = 2.4$

11. Ans: (a)

Sol: $f(x) = \cosh x + \cos x$

$f'(x) = \sinh x - \sin x \Rightarrow f'(0) = 0$

$f''(x) = \cosh x - \cos x \Rightarrow f''(0) = 0$

$f'''(x) = \sinh x + \sin x \Rightarrow f'''(0) = 0$

$f''''(x) = \cosh x + \cos x \Rightarrow f''''(0) = 2 > 0$

$\therefore$ f(x) has a minimum at x = 0

12. Ans: (b)

Sol: $y' = 0 \Rightarrow 4x^3 - 6x^2 + 2x = 0$

$\Rightarrow x = 0, \frac{1}{2}, 1$ are stationary points

$y'' = 12x^2 - 12x + 2$

$\Rightarrow$ y(x) has minimum at x = 0 & x = 1

$\therefore$ Required Area

$$= \int_0^1 \left(x^4 - 2x^3 + x^2 + 3\right)dx = \frac{91}{30}$$

13. Ans: 0.785 range 0.78 to 0.79

Sol: $\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\cos^4 x + \sin^4 x}\, dx$

$$= 2\int_0^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x\left(1 + \tan^4 x\right)}\, dx$$

$$= \int_0^1 \frac{2t}{1+t^4}\, dt \quad \text{(by putting tan x = t)}$$

$$= \frac{\pi}{4} = 0.785$$

14. Ans: 0.53 range 0.52 to 0.54

Sol: The curve is symmetric about x-axis and intersect x-axis at x = 0 and x = 1.

$\therefore$ Required area

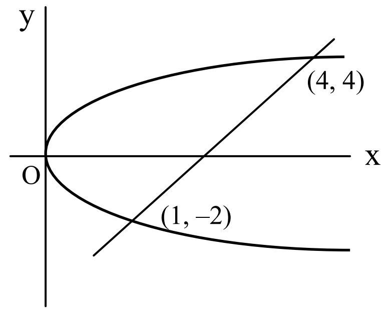
$$= 2\int_0^1 y\, dx = 2\int_0^1 \sqrt{x}\left(x-1\right) dx = \frac{8}{15}$$

$$= 0.53$$

15. Ans: 9

Sol: The required area

$$= \int x\, dy = \int_{-2}^{4}\left(\frac{1}{2}\left(y+4\right) - \frac{1}{4}y^2\right)dy = 9$$



16. Ans: (c)

Sol: Let $f\left(\alpha\right) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx$..................(i)

Differentiating with respect to α, partially

$$f'\left(\alpha\right) = \int_0^1 \frac{1}{\log x}\left(x^\alpha \log x\right)dx = \int_0^1 x^\alpha\, dx = \frac{1}{1+\alpha}$$

Integrating, f(α) = log(1 + α) + C……… (ii)

From (i), f(0) = 0

From (ii), f(0) = log(1) + C

$\Rightarrow$ 0 = log(1) + C

$\Rightarrow$ C = 0

$\therefore$ f(α) = log(1 + α)

17. Ans: (a)

Sol: Given that, $x \sin(\pi x) = \int_0^{x^2} f\left(t\right)dt$

differentiating both sides

$x \cos(\pi x).\pi + \sin(\pi x) = f(x).2x$

Putting x = 4

$4\pi \cos(4\pi) = f(4).8$

$\Rightarrow f(4) = \frac{\pi}{2}$

18. Ans: (c)

Sol: First of all note that, the integrand $f(x) = x^m (\ln x)^n$ has no meaning at x = 0. It can be made continuous on the interval [0, 1] for any $m > 0$ and $n > 0$, by putting $f(0) = 0$.

Indeed $\underset{x \to +0}{Lt}\ x^m (\ln x)^n = \underset{x \to +0}{Lt}\left(x^{\frac{m}{n}} \ln x\right)^n = 0$

Hence, in particular, it follows that the integral I_n exists at $m > 0$, $n > 0$. To compute it we integrate by parts, putting

$u = (\ln x)^n$, $\quad dv = x^m\, dx$,

$du = \dfrac{n(\ln x)^{n-1}}{x}\, dx$, $\quad v = \dfrac{x^{m+1}}{m+1}$.

Hence,

$$\int_0^1 x^m (\ln x)^n\, dx = \left.\frac{x^{m+1}(\ln x)^n}{m+1}\right|_0^1 - \frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1}\, dx$$

$$= -\frac{n}{m+1} I_{n-1}$$

The formula obtained reduces I_n to I_{n-1}. In particular, with a natural n, taking into account that

$$I_0 = \int_0^1 x^m\, dx = \frac{1}{m+1}$$

we get,

$$I_n = (-1)^n \frac{n!}{(m+1)^{n+1}}.$$

19. Ans: (c)

Sol: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+a^2+x^2)^{\frac{3}{2}}}$

$$= 2\int_0^{\infty} \frac{dx}{(1+a^2+x^2)^{\frac{3}{2}}}$$

[∵ Integrand is even function]

$$= 2\int_0^{\infty} \frac{dx}{(b^2+x^2)^{\frac{3}{2}}} \quad \text{Put } x = b\tan\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{b\sec^2\theta\, d\theta}{b^3\sec^3\theta}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{b^2} = \frac{2}{b^2} = \frac{2}{1+a^2}$$

20. Ans: (d)

Sol: $\displaystyle\int_{-\infty}^{0} e^{x+e^x} dx$

$$= \int_{-\infty}^{0} e^x e^{e^x} dx$$

Put $e^x = t$

$$= \int_0^1 e^t\, dt = e - 1$$

21. Ans: (a)

Sol: Let $I = \displaystyle\int_0^{\pi} x\sin^2 x\, dx$ .... (1)

$$I = \int_0^{\pi} (\pi - x)\sin^2(\pi - x)\, dx$$

[By property of definite integrals]

$$I = \int_0^{\pi} (\pi - x)\sin^2 x\, dx \quad .... (2)$$

Adding (1) and (2)

$$2I = \int_0^{\pi} \pi \sin^2 x\, dx$$

$$I = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$$

$$I = \pi\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

22. Ans: (a)

Sol: Given $(f \circ g)(x) = f[g(x)]$

In $(-\infty, 0)$, $g(x) = -x$

$\Rightarrow f[g(x)] = f(-x)$

$\Rightarrow f[g(x)] = x^2$

$\therefore$ f[g(x)] has no points of discontinuities in $(-\infty, 0)$.

23. Ans: (c)

Sol: $y = \underset{x\to\infty}{\text{Lt}} \left(1+x^2\right)^{e^{-x}}$ $\quad \left(\infty^0 \text{ form}\right)$

Taking logarithms

$$\log y = \underset{x\to\infty}{\text{Lt}}\ e^{-x}.\log\left(1+x^2\right) \quad (0.\infty \text{ form})$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{\log\left(1+x^2\right)}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{\left(\frac{2x}{1+x^2}\right)}{e^x} \quad \text{(By L Hospital's rule)}$$

$$= \underset{x\to\infty}{\text{Lt}} \left[\frac{2x}{\left(1+x^2\right)e^x}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x\to\infty}{\text{Lt}} \left[\frac{2}{\left(1+x^2e^x+2xe^x\right)}\right]$$

[$\because$ By L Hospital's rule]

$= 0$

$\therefore \quad y = e^0 = 1$

24. Ans: (a)

Sol: $f(x) = x(x-1)(x-2)$

$= x^3 - 3x^2 + 2x$

$f'(x) = 3x^2 - 6x + 2$

Consider $f'(c) = 0$

$\Rightarrow 3c^2 - 6c + 2 = 0$

$$\Rightarrow c = \frac{6 \pm \sqrt{36-24}}{6} = 1 \pm \frac{1}{\sqrt{3}}$$

$$\therefore\ c = \left(1 + \frac{1}{\sqrt{3}}\right) \in (1, 2)$$

$= 1.577$

25. Ans: (b)

Sol: $$\underset{x\to 0}{\text{Lt}} \frac{\int_0^{x^2} \sin\sqrt{x}\ dx}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

Applying L-Hospital rule,

$$= \underset{x\to 0}{\text{Lt}} \frac{\sin x\ (2x)}{3x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2\sin x}{3x} = \frac{2}{3}$$

26. Ans: (b)

Sol: Given $f(x) = x^3 - 3x^2 - 24x + 100$ in $[-3, 3]$

$\Rightarrow \quad f'(x) = 3x^2 - 6x - 24$, $f''(x) = 6x - 6$

Consider $f'(x) = 0$

$\Rightarrow 3x^2 - 6x - 24 = 0$

$\Rightarrow x = -2, 4$ are stationary points

At $x = -2$, $f''(-2) < 0$

$\Rightarrow$ f(x) has a maximum at $x = -2$

At $x = 4$, $f''(4) > 0$

$\Rightarrow$ f(x) has a minimum at $x = 4$

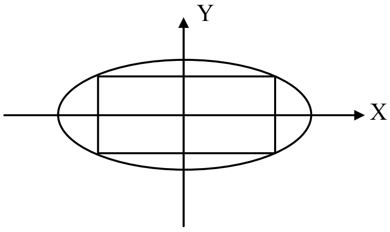
But $x = 4 \notin [-3, 3]$

$\therefore$ Global minimum of f(x) = min{f(–3), f(3)}

= min{118, 28}=28

27. Ans: 1

Sol: Let 2x & 2y be the length & breadth of the rectangle.



Let A = 2x×2y = 4xy be the area of the rectangle.

Then $A^2 = 4x^2y^2 = x^2(1-x^2) = x^2 - x^4$

Let $f(x) = x^2 - x^4$

Then $f'(x) = 2x - 4x^3$ and $f''(x) = 2 - 12x^2$

For maximum, we have

$$f'(x) = 0$$

$$\Rightarrow \; 2x(1-2x^2) = 0$$

$$\Rightarrow x = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

Here $f''(0) > 0$, $\quad f''\left(\frac{1}{\sqrt{2}}\right) < 0$

$\therefore$ Area $A = 4xy = 4x \times \frac{\sqrt{1-x^2}}{2}$

$$= 2x\sqrt{1-x^2}$$

$$= 2 \times \frac{1}{\sqrt{2}} \times \sqrt{1-\frac{1}{2}} = 1$$

28. Ans: –13

Sol: Given $f(x) = 2x^3 - x^4 - 10$

$\Rightarrow f'(x) = 6x^2 - 4x^3$, $f''(x) = 12x - 12x^2$ and $f'''(x) = 12 - 24x$.

Consider $f'(x) = 0$

$\Rightarrow 6x^2 - 4x^3 = 0$

$\Rightarrow x = 0, 1.5$ are stationary points

But x = 1.5 lies outside of [–1, 1]

At x = 0, $f''(0) = 0$ and $f'''(0) = 12 > 0$

$\Rightarrow$ f(x) has a minimum T x = 0

$\therefore$ The minimum value of

f(x) in [–1, 1] = min{f(–1), f(1), f(0)}

= min{–13, –9, –10}

= –13

29. Ans: (c)

Sol: Given $f(x) = (k^2 - 4)x^2 + 6x^3 + 8x^4$

$\Rightarrow \quad f'(x) = 32x^3 + 18x^2 + 2(k^2-4)x$

and $\quad f''(x) = 96x^2 + 36x + 2(k^2-4)$

f(x) has local maxima at x = 0

$\Rightarrow f''(0) < 0$

$\Rightarrow 2(k^2 - 4) < 0$

$\Rightarrow k^2 - 4 < 0$ (or) $(k-2)(k+2) < 0$

$\therefore -2 < k < 2$

30. Ans: (c)

Sol: $f(x) = \int_0^x \frac{\sin t}{t}\, dt$

$f'(x) = \frac{\sin x}{x}$

$f'(x) = 0 \Rightarrow x = n\pi$

where n = 1, 2, 3,.......

$f''(x) = \frac{x\cos x - \sin x}{x^2}$

Here $f''(x)$ is negative when n is odd.

$\therefore$ f(x) has a maximum at $x = n\pi$, where n is odd

31. Ans: (c)

Sol: $f(x) = \dfrac{50}{3x^4 + 8x^3 - 18x^2 + 60}$

Let F(x) = $3x^4 + 8x^3 - 18x^2 + 60$

$F'(x) = 12x^3 + 24x^2 - 36x$

$F'(x) = 0$

$\Rightarrow x = 0, 1, -3$

$F''(x) = 36x^2 + 48x - 36$

$F''(1) = 48 > 0$

$\therefore$ F(x) has a local minimum at x = 1

$\Rightarrow$ f(x) has a local maximum at x = 1

32. Ans: (a)

Sol: $I = \int_0^{\pi} x \ \sin^4 x \ \cos^6 x \, dx$ .... (1)

$= \int_0^{\pi} (\pi - x) \ \sin^4 (\pi - x) \ \cos^6 (\pi - x) \, dx$

$I = \int_0^{\pi} (\pi - x) \ \sin^4 x \ \cos^6 x \, dx$ .... (2)

Adding (1) and (2)

$2I = \int_0^{\pi} \pi \ \sin^4 x \ \cos^6 x \, dx$

$I = \dfrac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin^4 x \ \cos^6 x \ dx$

$= \dfrac{\pi}{2} \dfrac{(3\times 1)(5\times 3\times 1)}{10\times 8\times 6\times 4\times 2} \dfrac{\pi}{2}$

$= 3\pi^2 / 512$

33. Ans: 4

Sol: $\int_0^{2\pi} |x \sin x| \, dx = k\pi$

$\Rightarrow \int_0^{\pi} |x \sin x| \, dx + \int_{\pi}^{2\pi} |x \sin x| \, dx = k\pi$

$\Rightarrow \int_0^{\pi} x \sin x \ dx - \int_{\pi}^{2\pi} x \sin x \ dx = k\pi$

$\Rightarrow \left[x(-\cos x) + \sin x\right]_0^{\pi} + \left[x(\cos x) + \sin x\right]_{\pi}^{2\pi}$

$= k\pi$

$\Rightarrow \pi + 3\pi = k\pi$

$\therefore k = 4$