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Engineering Mathematics

(Solutions for Volume : I Classroom Practice Questions)

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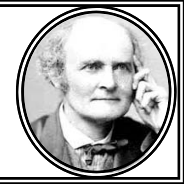
Chapter 1

Linear Algebra



Arthur Cayley
(1821 – 1895)

01. Ans: (d)

Sol: $|A| = \begin{vmatrix} -1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 4 \end{vmatrix}$

$C_4 \to C_4 - 3C_1$

$$|A| = \begin{vmatrix} -1 & -2 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & -2 & 3 & 4 \end{vmatrix} = -12$$

By expanding the above determinant along the 2nd row, we get

$$|A| = (-1)\begin{vmatrix} -2 & 1 & 3 \\ 2 & -1 & 0 \\ -2 & 3 & 4 \end{vmatrix} = -12$$

02. Ans: 1500

Sol: Given that P is 10×5 matrix.

Q is 5×20 matrix

and R is 20×10 matrix

Now PQR is 10×10 matrix. Total number of elements in PQR = 100. Here, we can find the product PQR only in two ways i.e., (PQ)R and P(QR) because PQ ≠ QP.

So, to find the product matrix PQR first we find PQ and then find (PQ)R (or) we, first find QR and then find P(QR)

For the product $(PQ)_{10\times20}$

Number of elements in PQ = 200.

To compute each element of the matrix PQ, we require '5' multiplications.

∴ Number of multiplications = 200×5

= 1000

For the product $[(PQ)R]_{10\times10}$

Number of elements in (PQ) R = 100

To compute each element of the matrix (PQ)R, we require 20 multiplications.

∴ Number of multiplications = 100 × 20

= 2000

Hence, the total number of multiplication operations to find the product $[(PQ)R]_{10\times10}$

= 1000 + 2000

= 3000

Similarly, if we find the product $[P(QR)]_{10\times10}$ by above method, the total number of multiplication operations to find the product $[P(QR)]_{10\times10}$ = 1000 + 500

= 1500

∴ The minimum number of multiplication operations to find PQR = 1500.

03. Ans: 324

Sol: Det $M_r = 2r - 1$

Det M_1 + Det M_2 + ... + Det M_{18}

= 1 + 3 + 5 + ...+ 37

= 324

Arthur Cayley was probably the first mathematician to realize the importance of the notion of a matrix and in 1858 published book, showing the basic operations on matrices. He also discovered a number of important results in matrix theory.

04. Ans: –3

Sol: Given that $|A^{10}| = 1024$

$\Rightarrow |A^{10}| = 2^{10}$

$\Rightarrow |A|^{10} = 2^{10}$

$\Rightarrow |A| = 2$

$\Rightarrow -\alpha^3 - 25 = 2$

$\Rightarrow \alpha^3 + 27 = 0$ or $\alpha^3 + 3^3 = 0$

$\Rightarrow (\alpha + 3)(\alpha^2 + 3\alpha + 3^2) = 0$

$\therefore$ The real value of α is -3

05. Ans: 0.5

Sol: Given that $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

$R_2 - R_1, R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}$$

$= \sin\theta . \cos\theta$

$= \dfrac{\sin 2\theta}{2}$

$\therefore$ The maximum value of $\Delta = \dfrac{1}{2}$

06. Ans: 0

Sol: Given that $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & x \end{vmatrix}$

Applying $\dfrac{R_2}{x}$ and $\dfrac{R_3}{x}$

$$\frac{f(x)}{x^2} = \begin{vmatrix} \cos x & x & 1 \\ \dfrac{2\sin x}{x} & x & 2 \\ \dfrac{\tan x}{x} & 1 & 1 \end{vmatrix}$$

$$\underset{x\to 0}{\text{Lt}} \frac{f(x)}{x^2} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

07. Ans: (b)

Sol: Here, determinant of A = –8

$\because A^{-1} = \dfrac{\text{adj } A}{|A|}$

$\Rightarrow c = \dfrac{-1}{8}$ (cofactor of the element 6 in A)

$\Rightarrow c = \dfrac{-1}{8}.\left[(-1)^{3+1}\right]\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix}$

$\therefore c = -1$

08. Ans: (d)

Sol: Giving that

$(I - A + A^2 - \ldots + (-1)^n A^n) = O$ .... (i)

Multiplying by A^{-1}, we get

$A^{-1} - I + A - A^2 + \ldots + (-1)^{n-1}A^{n-1} = O$ .....(ii)

Adding (i) & (ii), we get

$A^{-1} + (-1)^n A^n = O$

$\therefore A^{-1} = (-1)^{n+1}. A^n$

09. Ans: 0.04

Sol: Now $|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 5$

$\because |adj(A_{n\times n})| = |A|^{n-1}$

$\Rightarrow |adj(A_{3\times3})| = |A|^{3-1} = |A|^2$

$\Rightarrow |adj(A^{-1})| = |A^{-1}|^2 = |A|^{-2} = \dfrac{1}{|A|^2}$

$\therefore |adj(A^{-1})| = \dfrac{1}{25} = 0.04$

10. Ans: (c)

Sol: In a skew symmetric matrix, the diagonal elements are zero and $a_{ij} = -a_{ji}$ for $i \neq j$.

Each element above the principal diagonal, we can choose in 3 ways (0, 1, –1).

Number of elements above the principal diagonal = $\dfrac{n(n-1)}{2}$

$\therefore$ By product rule,

Required number of skew symmetric matrices = $3^{\frac{n(n-1)}{2}}$.

11. Ans: 3

Sol: Given $A = \begin{bmatrix} 1 & -1 & 0 & -2 \\ 2 & 0 & 2 & -2 \\ 4 & 1 & 3 & -8 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 4R_1$

$\sim \begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 2 & 2 & 2 \\ 0 & 5 & 3 & 0 \end{bmatrix}$

$R_3 \to 2R_3 - 5R_2$

$\sim \begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -4 & -10 \end{bmatrix}$

$\therefore \rho(A_{3\times4}) = 3$

12. Ans: (c)

Sol: $\begin{pmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 5 \\ 0 & 0 & a+6 \\ 0 & 0 & a+6 \end{pmatrix}$

$= \begin{pmatrix} -1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ if $a = -6$ and Rank = 1

If $a \neq -6$ then Rank of the matrix is 2

$\therefore$ Option (c) is correct.

13. Ans: 1

Sol: If the vectors are linearly dependent, then

$\begin{vmatrix} 1-t & 0 & 0 \\ 1 & 1-t & 0 \\ 1 & 1 & 1-t \end{vmatrix} = 0$

$\Rightarrow (1-t)^3 = 0$

$\therefore t = 1$

14. Ans: 1

Sol: If the vectors are linearly independent, then

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & t \\ 0 & 0 & 1 & 0 \end{vmatrix} \neq 0$$

Expanding by third column

$$\Rightarrow (-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & t \end{vmatrix} \neq 0$$

$\Rightarrow (-1).\ (1-(t-1)-1) \neq 0$

$\therefore\ t \neq 1$

15. Ans: (b)

Sol: The augmented matrix of the given system is

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_2 - R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_3 + R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Here, Rank of coefficient matrix A = 2 and rank of [A|B] = 3

$\Rightarrow \rho(A) \neq \rho(A|B)$

$\therefore$ The system has no solution

16. Ans: (c)

Sol: Let the given system be AX = B

The augmented matrix of the system

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 - R_1,\ R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

$\therefore$ The system has a unique solution if $\lambda \neq 3$.

17. Ans: (a)

Sol: Given $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$ and n = 3

$R_2 \to R_2 - R_1;\quad R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & -1 \\ 0 & -4 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, $r = \rho(A_{3\times3}) = 2$ and n = number of variables = 3

$\therefore$ The number of linearly independent solutions of AX = O is given by

$p = n - r = 3 - 2 = 1$

18. Ans: (a)

Sol: Consider $(A\,|B) = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 2 & 3 & a & b \end{array}\right]$

$R_2 \to R_2 - R_1;\ R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & a-6 & b-4 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-7 & b-5 \end{array}\right]$$

If $a - 7 = 0$ and $b - 5 = 0$ then

$\rho(A) = 2 = \rho(A|B) < n = 3$

$\therefore$ Many solutions exist if $a = 7$ and $b = 5$

19. Ans: (b)

Sol: Let the given system be $AX = B$

The augmented matrix of the system

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right]$$

$R_2 - 2R_1;\quad R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-b-3a \end{array}\right]$$

The system is inconsistent

if $c - b - 3a \neq 0$

or $3a + b - c \neq 0$

20. Ans: (c)

Sol: If the system has a non trivial solution, then

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = 0$$

$R_2 - R_1,\ R_3 - R_1$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

$\Rightarrow (a+b+c)\,(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\therefore\ a + b + c = 0$ or $a = b = c$

21. Ans: (d)

Sol: $A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$

$R_2 - 5R_1;\quad R_3 + 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -3 & -9 \\ 0 & 1 & 3 \end{pmatrix}$$

$3R_3 + R_2$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -3 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

Here, $\rho[A] = 2$

If B is a linear combination of columns of A,

then $\rho[A] = \rho[A|B]$

$\therefore$ The system has infinitely many solutions

22. Ans: (a)

Sol: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

The eigen values of A are 1, 2

The eigen vectors for $\lambda = 1$ are given by

$[A - I]\, X = O$

$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y = 0$

$\therefore X_1 = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The eigen vectors for $\lambda = 2$ are given by

$[A - 2I]\, X = O$

$\Rightarrow \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x + 2y = 0$

$\therefore X_2 = c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore$ The eigen vector pair is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

23. Ans: 1

Sol: Product of eigen values = $|A| = 0$

$$\Rightarrow \begin{vmatrix} 3 & 4 & 2 \\ 9 & 13 & 7 \\ -6 & -9+x & -4 \end{vmatrix} = 0$$

$R_2 - 3R_1,\ R_3 + 2R_1$

$$\Rightarrow \begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & x-1 & 0 \end{vmatrix} = 0$$

$\Rightarrow 3(1 - x) = 0$

$\therefore x = 1$

24. Ans: (d)

Sol: The given characteristic equation is

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\Rightarrow$ $|A|$ = product of the roots of the characteristic equation = 4 and

Trace of A = sum of the roots of characteristic equation = 6

$\therefore$ option (d) is correct

25. Ans: 3

Sol: Sum of the eigen values = Trace of A = 14

$\Rightarrow a + b + 7 = 14$ ...... (i)

product of eigen values = $|A| = 100$

$\Rightarrow 10ab = 100$

$\Rightarrow ab = 10$ .....(ii)

solving (i) & (ii), we have

$\Rightarrow a = 5$ and $b = 2$

$\therefore |a - b| = 3$

26. Ans: (c)

Sol: The characteristic equation is

$$\begin{vmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a - \lambda)\, [\{(a - \lambda)^2 - 1\} - (a - \lambda)] = 0$

$\therefore \lambda = a,\ a \pm \sqrt{2}$

27. Ans: –6

Sol: The given matrix has rank 2

$\therefore$ There are only 2 non zero eigen values

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & -1-\lambda & -1 & -1 & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 0 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_5$ and

$R_2 \to R_2 + R_3 + R_4$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 & 0 & 2-\lambda \\ 0 & -3-\lambda & -3-\lambda & -3-\lambda & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 0 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda).(-3-\lambda). \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 0 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 2, -3$

$\therefore$ Product of the non zero eigen values = –6

28. Ans: (b)

Sol: A is a real symmetric matrix.

$\Rightarrow$ The eigen vectors of A are orthogonal.

For the given eigen vector, only the vector given in option (b) is orthogonal.

$\therefore$ option (b) is correct.

29. Ans: 0

Sol: Let $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = \pm 5$

The eigen vectors for $\lambda = 5$ are given by

$[A - 5I]\, X = O$

$$\Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - 2y = 0$

$\therefore X_1 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The eigen vectors for $\lambda = -5$ are given by

$[A + 5I]\, X = 0$

$$\Rightarrow \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x + y = 0$

$\therefore X_2 = c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Hence, a + b = 0

30. Ans: 3

Sol: If λ is an eigen value of a matrix A then $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ 17 & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$$

$\Rightarrow$ $6 + 2k = \lambda$ .... (1)

$21 + k = 2\lambda$ .... (2)

Solving (1) & (2), we get

$\lambda = 12$ and $k = 3$

31. Ans: (c)

Sol: If A is singular then 0 is an eigen value of A.

$\therefore$ The minimum eigen value of A is 0.

The eigen vectors corresponding to the eigen value $\lambda = 0$ is given by

$$[A - 0I]X = O$$

$$\Rightarrow \begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying cross multiplication rule for first and second rows of A, we have

$$\Rightarrow \frac{x}{11} = \frac{y}{-11} = \frac{z}{11}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore$ The eigen vectors are $X = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

32. Ans: 3

Sol: Given matrix is $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ The eigen values of A are $\lambda = 2, 0, 1$ which are different.

By the property of eigen values and eigen vectors, if the n^{th} order matrix A has 'n' different eigen values $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ then n^{th} order matrix A will have 'n' linearly independent eigen vectors $X_1, X_2, \ldots\ldots X_n$.

$\therefore$ The given matrix $A_{3\times3}$ has '3' linearly independent eigen vectors corresponding to three different eigen values $\lambda_1 = 0$, $\lambda_2 = 2$ & $\lambda_3 = 1$.

33. Ans: 3

Sol: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Here, A is an upper triangular matrix

$\Rightarrow$ The eigen values are $\lambda = 2, 2, 3$

The eigen vectors for $\lambda = 2$ are given by

$[A - 2I]X = O$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Here Rank of $[A - 2I] = 1$

$\therefore$ Number of Linearly independent eigen vectors for $\lambda = 2$ is $p = n - r = 3 - 1 = 2$

For since, $\lambda = 3$ is not a repeated eigen value, there will be only one independent eigen vector for $\lambda = 3$.

$\therefore$ The number of linearly independent eigen vectors of A is 3.

34. Ans: 2

Sol: Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \lambda = 1, 1, 1$ which are repeated eigen values

Consider the characteristic matrix of $A_{3\times3}$ matrix.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for } \lambda = 1$$

$\Rightarrow r = \rho(A - \lambda I) = 1$

$\Rightarrow P = n - r = (\text{order of matrix } A_{3\times3}) - r$

$\Rightarrow P = 3 - 1 = 2$

$\therefore$ The number of linearly independent eigen vectors of the given matrix $A_{3\times3}$ corresponding to 3 repeated eigen values $\lambda = 1, 1, 1$ is 2.

35. Ans: (c)

Sol: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the required matrix.

The given eigen vectors of a matrix $A_{2\times2}$ corresponding to eigen values $\lambda_1 = -2$ and $\lambda_2 = 5$ and $X_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

Consider $Ax_1 = \lambda_1 x_1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = -2\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$-4a + 3b = 8$…….(1)

$-4c + 3d = -6$……(2)

Consider $Ax_2 = \lambda_2 x_2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$a + b = 5$…….(3)

$c + d = 5$…….(4)

Solving (1) and (3), we get

$a = 1$ & $b = 4$.

Again Solving (2) and (4), we get

$c = 3$ & $d = 2$

$\therefore$ The required matrix is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

36. Ans: (c)

Sol: The characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow (\lambda^2 - 4)(\lambda^2 + 4) = 0$

$\Rightarrow \lambda^4 = 16$

By Caley Hamilton's theorem, we have

$A^4 = 16I$

37. Ans: (d)

Sol: The characteristic equation is a matrix A is $\lambda^4 = \lambda$

$\Rightarrow \lambda^4 - \lambda = 0$

$\Rightarrow \lambda(\lambda^3 - 1) = 0$

$\Rightarrow \lambda = 0, 1, -0.5 \pm \sqrt{3}\, i$

$\therefore \lambda = 0, 1, -0.5 \pm (0.866)i$

38. Ans: (a)

Sol: Let the given vectors $X_1 = [2, 2, 0]$, $X_2 = [3, 0, 2]$ and $X_3 = [2, -2, 2]$

Suppose $X_1 = a\,X_2 + b\,X_3$

$\Rightarrow [2, 2, 0] = a[3, 0, 2] + b[2, -2, 2]$

$\Rightarrow$ $2 = 3a + 2b$ .... (i)

$2 = -2b$ .... (ii)

$0 = 2a + 2b$ ....(iii)

From (i) and (ii), we get

a = 0 and b = –1

But, equation (iii) is not satisfied for these values.

$\therefore$ The given vectors are linearly independent

39. Ans: $k \neq 0$

Sol: If the given vectors form a basis, then they are linearly independent

$$\Rightarrow \begin{vmatrix} k & 1 & 1 \\ 0 & 1 & 1 \\ k & 0 & k \end{vmatrix} \neq 0$$

$$\Rightarrow k^2 + k - k \neq 0$$

$$\therefore k \neq 0$$

Chapter **2**

Calculus

(With Vector Calculus & Fouries Series)

Sir Isaac Newton (1643 – 1727)

G. W. Von Leibniz (1646 – 1716)

01. Ans: (d)

Sol: By the definition of modulus function, we have $y = f(x) = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

$$\text{L.H.L} = \underset{x\to 0^-}{\text{Lt}} \frac{|x|}{x} = \underset{x\to 0}{\text{Lt}} \frac{-x}{x} = \underset{x\to 0}{\text{Lt}} -1 = -1$$

$$\text{R.H.L} = \underset{x\to 0^+}{\text{Lt}} \frac{|x|}{x} = \underset{x\to 0}{\text{Lt}} \frac{x}{x} = \underset{x\to 0}{\text{Lt}} 1 = 1$$

Here, L.H.L $= -1 \ne$ R.H.L $= 1$

$\therefore \underset{x\to 0}{\text{Lt}} \dfrac{|x|}{x}$ does not exist

02. Ans: (d)

Sol: By the definition of step function, we have

$$y = f(x) = [x] = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$$

Now, $[x] = \begin{cases} 5, & 5 \le x < 6 \\ 6, & 6 \le x < 7 \end{cases}$

$$\text{L.H.L} = \underset{x\to 6^-}{\text{Lt}} [x] = \underset{x\to 6}{\text{Lt}} 5 = 5$$

$$\text{R.H.L} = \underset{x\to 6^+}{\text{Lt}} [x] = \underset{x\to 6}{\text{Lt}} 6 = 6$$

Here, L.H.L $= 5 \ne$ R.H.L $= 6$

$\therefore \underset{x\to 6}{\text{Lt}} [x]$ does not exist

03. Ans: (a)

Sol: $\underset{x\to 0}{\text{Lt}} \dfrac{x - \sin x}{1 - \cos(x)}$ $\quad \left[\dfrac{0}{0} \text{ form}\right]$

Now, we have to evaluate it by using the L-Hospital's rule.

$$\underset{x\to 0}{\text{Lt}} \frac{x - \sin(x)}{1 - \cos(x)} = \underset{x\to 0}{\text{Lt}} \frac{1 - \cos(x)}{0 + \sin(x)} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$\Rightarrow \underset{x\to 0}{\text{Lt}} \frac{x - \sin(x)}{1 - \cos(x)} = \underset{x\to 0}{\text{Lt}} \frac{0 + \sin(x)}{\cos(x)}$$

$$\therefore \underset{x\to 0}{\text{Lt}} \frac{x - \sin(x)}{1 - \cos(x)} = \frac{0}{1} = 0$$

04. Ans: (b)

Sol: $\underset{x\to \frac{\pi}{2}}{\text{Lt}} \left(\sec x - \dfrac{1}{1 - \sin x}\right) = \underset{x\to 0}{\text{Lt}} \left(\dfrac{1}{\cos x} - \dfrac{1}{1 - \sin x}\right)$

$$= \underset{x\to \frac{\pi}{2}}{\text{Lt}} \left[\frac{1 - \sin x - \cos x}{\cos x(1 - \sin x)}\right] \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{x\to \frac{\pi}{2}}{\text{Lt}} \left[\frac{-\cos x + \sin x}{-\sin x - \cos 2x}\right]$$

(by L' Hospital's Rule)

$= \infty$

05. Ans: (c)

Sol: $\underset{x\to \infty}{\text{Lt}} \left(1 + x^2\right)^{e^{-x}} = \infty^0$ form

Let $y = \underset{x\to \infty}{\text{Lt}} \left(1 + x^2\right)^{e^{-x}} \to \infty^0$

Then $\log y = \underset{x\to \infty}{\text{Lt}} e^{-x} \log\left(1 + x^2\right) \to 0.\infty$

Issac Newton(most influential scientist) and Leibniz (universal genius) independently developed calculus which leads to the development of differential and integral equations of mathematical physics

$\Rightarrow \quad \log y = \underset{x\to\infty}{Lt} \dfrac{\log(1+x^2)}{e^x} \quad \to \dfrac{\infty}{\infty}$

By L-Hospitals rule, we have

$\Rightarrow \quad \log y = \underset{x\to\infty}{Lt} \dfrac{\frac{1}{1+x^2}2x}{e^x}$

$\Rightarrow \quad \log y = \underset{x\to\infty}{Lt} \dfrac{2x}{(1+x^2)e^x} \quad \to \dfrac{\infty}{\infty}$

$\Rightarrow \quad \log y = \underset{x\to\infty}{Lt} \dfrac{2}{(1+x^2)e^x + e^x(2x)} = \dfrac{2}{\infty}$

$\Rightarrow \quad \log y = 0$

$\therefore\ y = e^0 = 1 = \underset{x\to\infty}{Lt} (1+x^2)e^{-x}$

06. Ans: (a)

Sol: $\underset{n\to\infty}{Lt}(7^n+5^n) = \underset{n\to\infty}{Lt}\ 7\left(1+\left(\dfrac{5}{7}\right)^n\right)^{\frac{1}{n}}$

$= 7 \quad \left[\because \underset{n\to\infty}{Lt}\ r^n = 0,\ |r|<1\right]$

07. Ans: (a)

Sol: $\underset{x\to0}{Lt} \dfrac{\sin 2x + a\sin x}{x^3} = b$

By L' Hospital's rule

$\underset{x\to0}{Lt} \dfrac{2\cos 2x + a\cos x}{3x^2} = b$

$\Rightarrow 2 + a = 0 \quad (\because$ b is finite)

$\therefore\ a = -2$

By L' Hospital's rule,

$\underset{x\to0}{Lt} \dfrac{2\cos 2x - 2\cos x}{3x^2} = b$

$\underset{x\to0}{Lt} \dfrac{-4\sin 2x + 2\sin x}{6x} = b$

Again, by L' Hospital's rule

$\underset{x\to0}{Lt} \dfrac{-8\cos 2x + 2\cos x}{6} = b$

$\Rightarrow b = -1$

$\therefore\ a = -2\ \&\ b = -1$

08. Ans: (a)

Sol: If f(x) is continuous at x = 0, then

$\underset{x\to0}{Lt} f(x) = f(0)$

$\Rightarrow \underset{x\to0}{Lt}\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{x}} = f(0)$

$\Rightarrow \underset{x\to0}{Lt}\left[\dfrac{(1-x)^{\frac{1}{x}}}{(1+x)^{\frac{1}{x}}}\right] = f(0)$

$\Rightarrow \dfrac{e^{-1}}{e} = f(0)$

$\therefore\ f(0) = e^{-2}$

09. Ans: (a)

Sol: Given $f(x) = \begin{cases} 5, & \text{if } x = 3 \\ 2x-1, & \text{if } x>3 \\ \dfrac{x+7}{2}, & \text{if } x<3 \end{cases}$

L.H.L $= \underset{x\to3^-}{Lt} f(x) = \underset{x\to3}{Lt} \dfrac{x+7}{2} = 5$

R.H.L $= \underset{x\to3}{Lt} f(x) = \underset{x\to3}{Lt}\ 2x-1 = 5$

and also f(3) = 5

Here, L.H.L = 5 = R.H.L = f(3) = 5

$\therefore$ f(x) is continuous at x = 3

Hence, option (A) is correct.

10. Ans: (d)

Sol: Given $f(x) = x^2 \quad, x \geq 0$

$= -x^2 \ , x < 0$

$$\text{LHD} = \underset{x\to 0^-}{\text{Lt}} \frac{f(x)-f(0)}{x-0}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{-x^2-0}{x-0} = \underset{x\to 0}{\text{Lt}} -x = 0$$

$$\text{RHD} = \underset{x\to 0^+}{\text{Lt}} \frac{f(x)-f(0)}{x-0}$$

$$= \underset{x\to 0^+}{\text{Lt}} \frac{x^2-0}{x-0} = \underset{x\to 0}{\text{Lt}}\ x = 0$$

Here, LHD = RHD for f(x) at x = 0

$\therefore$ f(x) is differentiable once at x = 0

Let $g(x) = f^1(x) = 2x \quad , x \geq 0$

$= -2x \ , x < 0$

Test for second derivative

$$\text{LHD} = \underset{x\to 0^-}{\text{Lt}} \frac{g(x)-g(0)}{x-0} = \underset{x\to 0}{\text{Lt}} \frac{-2x-0}{x-0} = -2$$

$$\text{RHD} = \underset{x\to 0^+}{\text{Lt}} \frac{g(x)-g(0)}{x-0} = \underset{x\to 0}{\text{Lt}} \frac{2x-0}{x-0} = 2$$

Here, LHD $\neq$ RHD for g(x) at x = 0

$\Rightarrow$ g(x) is not differentiable at x = 0,

i.e., $f^1(x)$ is not differentiable at x = 0

$\therefore$ f(x) is differentiable once at x = 0, but not twice.

11. Ans: (c)

Sol: Now, $f^1(x) = 2ax, x \leq 1$

$= 2x + a, x > 1$

Consider $f^1(1^-) = f'(1^+)$

($\because$ since f(x) is differentiable at x = 1)

$2a = a + 2 \Rightarrow a = 2$

Consider $f(x) = \begin{cases} ax^2+1 & , x \leq 1 \\ x^2+ax+b, & x > 1 \end{cases}$

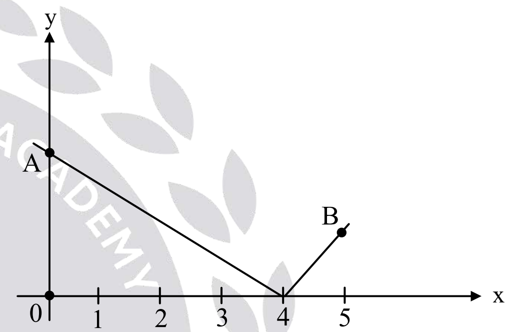
$f(1^-) = f(1^+)$ ($\because$ f(x) is continuous at x = 1)

$\Rightarrow a + 1 = 1 + a + b$

$\therefore b = 0$

12. Ans: (d)

Sol: Given f(x) = |x – 4| on [0, 5]



From the above graph of the function y = f(x) = |x – 4|, the given function f(x) is continuous from a point A to point B and the graph of the function f(x) has a sharp corner at x = 4 in [0, 5].

$\therefore$ The function f(x) = |x – 4| is continuous in [0, 5] but not differentiable in [0, 5].

Hence, option (D) is correct.

13. And: (b)

Sol: Let

$$f'(x) = \sin(x) + 2.\sin(2x) + 3.\sin(3x) - \frac{8}{\pi} = 0$$

be the given equation.

Then,

$$f(x) = -\cos(x) - \cos(2x) - \cos(3x) - \frac{8}{\pi}(x) + k$$

Here, if the function f(x) satisfies the all the three conditions of the Rolle's theorem in [a, b].

Then, $f'(x) = 0$ has at least one real root in (a, b).

As cos(ax) is continuous & differentiable function and $a_0 + a_1x$ is continuous & differentiable function for all x, the function f(x) is continuous and differentiable for all x.

Here, (i) f(x) is continuous on $\left[0, \frac{\pi}{2}\right]$

(ii) f(x) is differentiable on $\left(0, \frac{\pi}{2}\right)$

(iii) $f(0) = -3 + k = f\left(\frac{\pi}{2}\right)$

$\therefore$ By a Rolle's theorem, the given equation has at least one root in $\left(0, \frac{\pi}{2}\right)$.

Hence, option (B) is correct.

14. Ans: (d)

Sol: Here, the given function $f(x) = x^{\frac{-2}{3}}$ is not defined at x = 0 in [–1, 1].

$\Rightarrow$ f(x) is neither continuous nor differentiable in the interval [–1, +1].

$\therefore$ Option (d) is correct.

15. Ans: (b)

Sol: Let f(x) be defined on

$[a, b] = [0, 2] \ni f^1(x) = \frac{1}{1+x^2} \forall x$.

Then by Lagrange’s mean value theorem,

$\exists\, c \in (0,2) \ni f^1(c) = \frac{f(2)-f(0)}{2-0}$

$\Rightarrow \frac{1}{1+c^2} = \frac{f(2)}{2}$

$\because c \in (0, 2)$

$\Rightarrow 0 < c < 2$

$\Rightarrow 0^2 < c^2 < 2^2$

$\Rightarrow 1 + 0^2 < 1 + c^2 < 1 + 2^2$

$\Rightarrow \frac{1}{1+0^2} > \frac{1}{1+c^2} > \frac{1}{1+2^2}$

$\Rightarrow \frac{1}{1+2^2} < \frac{1}{1+c^2} < \frac{1}{1+0^2}$

$\Rightarrow \frac{1}{5} < \frac{f(2)}{2} < \frac{1}{1}$

$\therefore \frac{2}{5} < f(2) < 2$ (or) $f(2) \in (0.4, 2)$

16. Ans: (a)

Sol: By Cauchy's mean value theorem

$\frac{f'(c)}{g'(c)} = \frac{f(3)-f(1)}{g(3)-g(1)}$

$\Rightarrow -c = \left[\frac{\sqrt{3}-1}{\frac{1}{\sqrt{3}}-1}\right]$

$\therefore c = \sqrt{3}$

17. Ans: (c)

Sol: Let f(x) = logx & a = 2

Then the coefficient of $(x-a)^n$ in the Taylor series expansion of f(x) about x = a is given by $a_n = \dfrac{f^{(n)}(a)}{n!}$

$\therefore$ The coefficient of $(x-2)^3$ is given by

$$a_3 = \frac{f^{111}(2)}{3!} = \frac{1}{3!}\left(\frac{2}{x^3}\right)_{x=2} = \frac{1}{24}$$

18. Ans: (d)

Sol: Let $x - \pi = t$

Then $x = \pi + t$

Now, $\dfrac{\sin x}{x-\pi} = \dfrac{\sin(\pi+t)}{t} = \dfrac{-\sin t}{t}$

$$= \frac{-1}{t}\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\ldots\ldots\right\}$$

$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \ldots\ldots.$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + \ldots\ldots.$$

19. Ans: (c)

Sol: $$e^{x+x^2} = 1 + \frac{(x+x^2)}{1!} + \frac{(x+x^2)^2}{2!} + \frac{(x+x^2)^3}{3!} + \ldots.$$

$$\therefore\ e^{x+x^2} = 1 + x + \frac{3x^2}{2} + \frac{7}{6}x^3$$

20. Ans: (b)

Sol: Let $u = \tan^{-1}\left(\dfrac{x^3y^3}{x^4+y^4}\right)$

Then $f(u) = \tan u = \dfrac{x^3\,y^3}{x^4+y^4}$

Here, tan(u) is a homogeneous function of degree 2.

By Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{f(u)}{f'(u)} = 2\left(\frac{\tan u}{\sec^2 u}\right) = \sin 2u$$

21. Ans: (c)

Sol: Given $u(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$

Then, u(x, y) is a homogenous function of degree 2

By Euler's theorem, we have

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = 2(2-1)u$$

$$= 2\,u$$

22. Ans: (b)

Sol: Let $u = \sqrt{x^2+y^2+z^2}$ or $u^2 = x^2+y^2+z^2$

Then, $u_x = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{u}$

$\Rightarrow u_{xx} = \dfrac{u^2-x^2}{u^3}$

Similarly, $u_{yy} = \dfrac{u^2-y^2}{u^3}$ and $u_{zz} = \dfrac{u^2-z^2}{u^3}$

Now, $u_{xx} + u_{yy} + u_{zz} = \dfrac{3u^2-(x^2+y^2+z^2)}{u^3}$

$\therefore\ u_{xx} + u_{yy} + u_{zz} = \dfrac{2}{u}$

23. Ans: 2.718

Sol: Given $u = x\,e^y\,z$, where $y = \sqrt{1-x^2}$

and $z = \sin^2 x$

Now, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}.\dfrac{dy}{dx} + \dfrac{\partial u}{\partial z}.\dfrac{dz}{dx}$

$$= e^y z + x\, e^y z\,.\left(\frac{-x}{\sqrt{1-x^2}}\right) + x\, e^y \sin 2x$$

$$\therefore \left(\frac{dy}{dx}\right)_{(0,1,1)} = e = 2.718$$

24. Ans: –1

Sol: Let $f(x, y) = x^y + y^x = c$

Then $\dfrac{dy}{dx} = -\dfrac{\left(\dfrac{\partial f}{\partial x}\right)}{\left(\dfrac{\partial f}{\partial y}\right)} = -\dfrac{y\,x^{y-1} + y^x.\log y}{x^y.\log x + x\,y^{x-1}}$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

25. Ans: (c)

Sol: Given $u = \sin(x^2 + y^2)$ …….(1)

$a^2x^2 + b^2y^2 + c^2$ ……..(2)

Now, $\dfrac{dy}{dx} = \dfrac{\partial u}{dx}\dfrac{dx}{dx} + \dfrac{\partial u}{dy}\dfrac{dy}{dx}$ ………(3)

Differentiating (2) w.r.t 'x', we get

$$2a^2x + 2b^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2a^2x}{2b^2y} \text{ …….(4)}$$

From (1) & (4) , (3) becomes

$$\frac{du}{dx} = 2x.\cos\left(x^2+y^2\right).1 + 2y\cos\left(x^2+y^2\right)\left(\frac{-2a^2x}{2b^2y}\right)$$

$$\therefore \frac{du}{dx} = \left(1 - \frac{a^2}{b^2}\right)2x.\cos\left(x^2+y^2\right)$$

26. Ans: (b)

Sol: Given $f'(x) = (x-2)^2 (x-1)$

$\Rightarrow f''(x) = (x-2)^2 + 2(x-1)(x-2)$

Consider $f'(x) = 0$

$\Rightarrow x = 1, 2$

Now, $f''(1) = 1 > 0$ and $f''(2) = 0$

$\therefore$ f(x) has a minimum at t = 1

27. Ans: 12

Sol: Given $f(x) = 3x^3 - 7x^2 + 5x + 6$ in [0, 2]

$\Rightarrow f'(x) = 9x^2 - 14x + 5$ & $f''(x) = 18x - 14$

Consider $f'(x) = 0$ for stationary points

$\Rightarrow 9x^2 - 14x + 5 = 0$

$\Rightarrow (x-1)(9x-5) = 0$

$\therefore x = 1, \dfrac{5}{9}$ are stationary points

At $x = 1$, $f''(1) = 18 - 14 = 4 > 0$

$\Rightarrow$ Local minimum exists at x = 1

At $x = \dfrac{5}{9}$, $f''\left(\dfrac{5}{9}\right) = 18\left(\dfrac{5}{9}\right) - 14 = -4 < 0$

$\Rightarrow$ Local maximum exists at $x = \dfrac{5}{9}$

Now, $f(0) = 6$, $f(2) = 12$ and $f\left(\dfrac{5}{9}\right) = 7.13$

$\therefore$ The maximum value of f(x) in $[0, 2] = \max\left\{f(0), f(2), f\left(\frac{5}{9}\right)\right\}$

$= \max\{6, 12, 7.13\} = 12$

28. Ans: (c)

Sol: Given $y = a \log|x| + bx^2 + x$

$\Rightarrow \frac{dy}{dx} = a.\frac{1}{|x|}.\frac{|x|}{x} + 2bx + 1$

$\Rightarrow \frac{dy}{dx} = \frac{a + bx^2 + x}{x}$

Consider $\left[\frac{dy}{dx}\right]_{x=-1} = 0$

$\Rightarrow -a - 2b + 1 = 0$ .... (1)

Consider $\left[\frac{dy}{dx}\right]_{x=2} = 0$

$\Rightarrow \frac{a}{2} + 4b + 1 = 0$ .... (2)

solving (1) & (2), we have

$a = 2, b = \frac{-1}{2}$

29. Ans: 2

Sol: Given $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1, a > 0$

$\Rightarrow f'(x) = 6x^2 - 18ax + 12a^2x$

and $f''(x) = 12x - 18a$

Consider $f'(x) = 0 \Rightarrow x = a$ or $2a$ are stationary points.

Now, $f''(a) = -6a < 0$

and $f''(2a) = 6a > 0$

If $x_1 = a$ then $x_2 = 2a$

Consider $x_2 = x_1^2$

$\Rightarrow 2a = a^2$

$\Rightarrow a = 0$ or 2

Clearly f has a local maximum at x = 2 and a local minimum at x = 4

$\therefore a = 2$ ($\because a > 0$)

30. Ans: 5

Sol: $z = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= a^2 - 2a + 6$

$= (a-1)^2 + 5 \geq 5$

$\therefore$ z is least iff a = 1

least value of $z = [z]_{a=1} = 5$

31. Ans: (b)

Sol: Given $f(x, y) = xy + x - y$

$\Rightarrow f_x = y + 1$ & $f_y = x - 1$

Consider $f_x = y + 1 = 0$ and $f_y = x - 1 = 0$

$\Rightarrow x = 1, y = -1$

$\therefore$ P(1, –1) is a stationary point

Here, $r = f_{xx} = 0, s = f_{xy} = 1, t = f_{yy} = 0$

Now, $rt - s^2 = -1 < 0$

$\therefore$ P(1, –1) is a saddle point

32. Ans: (a)

Sol: Given $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

Consider $f_x = 4x - 4x^3 = 0$

$\Rightarrow x = 0, 1, -1$

Consider $f_y = -4y + 4y^3 = 0$

$\Rightarrow y = 0, 1, -1$

Now, $r = f_{xy} = 4 - 12x^2, s = f_{xy} = 0$

and $t = f_{yy} = -4 + 12y^2$

At (0,1), we have $r > 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has minimum at (0,1)

At (–1, 0), we have $r < 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has a maximum at (–1, 0)

33. Ans: 4

Sol: $\int_0^{2\pi} |x \sin x|\, dx = k\pi$

$$\Rightarrow \int_0^{\pi} x \sin x\, dx + \int_{\pi}^{2\pi} (-x \sin x) dx = k\pi$$

$$\Rightarrow [x(-\cos x) + \sin x]_0^{\pi} - [-x\cos x + \sin x]_{\pi}^{2\pi} = k\pi$$

$$\Rightarrow \pi - [-3\pi] = k\pi$$

$$\therefore k = 4$$

34. Ans: 39

Sol: $\int_4^{10} [x]\, dx$

$$= \int_4^5 [x]\, dx + \int_5^6 [x]\, dx + \ldots\ldots + \int_9^{10} [x]\, dx$$

$$= \int_4^5 4\, dx + \int_5^6 5\, dx + \ldots\ldots + \int_9^{10} 9\, dx$$

$= 4 + 5 + \ldots\ldots + 9$

$= 39$

35. Ans: (a)

Sol: Let $f(x) = \sin^6(x).\cos^4(x)$

Then, $f(\pi - x) = [\sin(\pi - x)]^6.[\cos(\pi - x)]^4$

$\Rightarrow f(\pi - x) = (\sin(x))^6 (-\cos(x))^4$

$\therefore f(\pi - x) = (\sin x)^6 (\cos x)^4 = f(x)$

$$\int_0^{\pi} x.\sin^6(x).\cos^4(x)\, dx = \frac{\pi}{2}\int_0^{\pi} \sin^6(x).\cos^4(x)\, dx$$

$$= \frac{\pi}{2}\left[2.\int_0^{\frac{\pi}{2}} \sin^6(x).\cos^4(x)\, dx\right]$$

$$= \pi\left[\frac{(5.3.1)(3.1)}{10.8.6.4.2}\right].\frac{\pi}{2}$$

$$= \frac{3\pi^2}{512}$$

36. Ans: 0.523

Sol: We have, $\int_a^b f(x)\, dx = \int_a^b f(a + b - x) dx$

Here, $f(x) = \dfrac{1}{1 + \tan^4 x} = \dfrac{\cos^4 x}{\cos^4 x + \sin^4 x}$

Now, $f(a + b - x) = f\left(\dfrac{\pi}{2} - x\right) = \dfrac{\sin^4 x}{\sin^4 x + \cos^4 x}$

Let $I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} f(x)\, dx = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \dfrac{\cos^4 x}{\cos^4 x + \sin^4 x} dx$

again

$I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} f(a + b - x)\, dx = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \dfrac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

adding

$$2I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} dx = \frac{4\pi}{12}$$

$$\therefore I = \frac{\pi}{6}$$

37. Ans: (a)

Sol: $\int_{-\infty}^{0} \sin hx\, dx = |\cos hx|_{-\infty}^{0} = \left|\dfrac{e^x + e^{-x}}{2}\right|_{-\infty}^{0}$

$$= \frac{2}{2} - \left(\frac{e^{-\infty} + e^{\infty}}{2}\right)$$

$$= 1 - 0 - \frac{e^{\infty}}{2} = -\infty$$

38. Ans: (b)

Sol: $\int_{-\infty}^{\infty} x\, e^{-x^2}\, dx = \int_{-\infty}^{\infty} \frac{-2xe^{-x^2}}{2}\, dx + \int_{0}^{\infty} \frac{-2xe^{-x^2}}{-2}\, dx$

$$= \left(\frac{-1}{2}e^{-x^2}\right)_{-\infty}^{0} + \left(\frac{-1}{2}e^{-x^2}\right)_{0}^{\infty}$$

$$= \left(0-\frac{1}{2}\right)+\left(\frac{1}{2}-0\right)$$

$$= 0$$

39. Ans: (a)

Sol: $\int_{-1}^{1}\frac{dx}{x^2} = 2\int_{0}^{1}\frac{dx}{x^2}$ ($\because$ $\frac{1}{x^2}$ is even function)

$= 2\,\underset{x\to 0^+}{\text{Lt}}\int_{0}^{1}\frac{dx}{x^2}$ (since $\frac{1}{x^2}$ is not defined at x = 0)

$= 2\left(\frac{-1}{x}\right)_0^1 = 2\{(-1)-(-\infty)\}$

$= \infty$ (Divergent)

40. Ans: (d)

Sol: Gamma function is $\Gamma n = \int_0^{\alpha} e^{-t}\,.\,t^{n-1}\, dt$

Now $\int_0^{\alpha} t^{-3/2}.\, e^{-t}\, dt = \int_0^{\alpha} e^{-t}\,.\,t^{-3/2}\, dt$

$$= \int_0^{\alpha} e^{-t}\,.\,t^{\left(-3/2+1\right)-1}\, dt$$

$$= \int_0^{\alpha} e^{-t}\,.\,t^{-1/2-1}\, dt$$

$$= \Gamma\left(-1/2\right)$$

$$= -2\,.\,\sqrt{\pi}$$

$\left(\because \Gamma\left(-1/2\right) = -2\sqrt{\pi}\ \&\ \Gamma\, 1/2 = \sqrt{\pi}\right)$

41. Ans: (a)

Sol: Given $I = \int_1^3 \frac{\sqrt{1+x^2}}{(x-1)^2}\, dx$

Let $f(x) = \frac{\sqrt{1+x^2}}{(x-1)^2}$, $f(x)\to\infty$ as $x \to 1$

Let $g(x) = \frac{1}{(x-1)^2}$

$\underset{x\to1}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{x\to1}{\text{Lt}}\frac{\sqrt{1+x}}{(x-1)^2}\times(x-1)^2 = \sqrt{2}$

But $\int_1^3 \frac{1}{(x-1)^2}$ is known to be divergent.

$\therefore$ By comparison test, the given integral also divergent.

42. Ans: (a)

Sol: Given $I = \int_1^2 \frac{x^3+1}{\sqrt{2-x}}\, dx$

Let $f(x) = \frac{x^3+1}{\sqrt{2-x}}$

$f(x)\to\infty$ an $x \to 2$

Let $g(x) = \frac{1}{\sqrt{2-x}}$

$\underset{x\to2}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{x\to2}{\text{Lt}}\left(\frac{x^3+1}{\sqrt{2-x}}\times\sqrt{2-x}\right) = 9\,\text{finite}$

But $\int_1^2 g(x)\, dx$ is known to be convergent

$\therefore$ By comparison test, the given integral also convergent.

43. Ans: (d)

Sol: Given $I = \int_1^{\infty} \frac{e^{-x}}{x^2}\, dx$

Let $f(x) = \frac{e^{-x}}{x^2}$

Choose $g(x) = \frac{1}{x^2}$

$\int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx = -1$ is known to be convergent.

$\therefore$ By comparison test, the given integral also convergent.

44. Ans: 1

Sol: $\int_{x=0}^{1}\int_{x=0}^{2} xy\,dx\,dy = \left(\int_{x=0}^{1} x\,dx\right).\left(\int_{y=0}^{2} y.\,dy\right)$

$$= \left(\frac{x^2}{2}\right)_0^1 . \left(\frac{y^2}{2}\right)_0^2$$

$$= \left(\frac{1}{2} - 0\right).\left(\frac{4}{2} - \frac{0}{2}\right) = 1$$

45. Ans: (c)

Sol: $\int_0^1 \int_0^{y^2} y\,dx\,dy = \int_{y=0}^{1}\left[\int_{x=0}^{y^2} y\,dx\right] dy$

$$= \int_{y=0}^{1} y\,(x)_0^{y^2}\,dy$$

$$= \int_{y=0}^{1} y\,[y^2 - 0]dy$$

$$= \left(\frac{y^4}{4}\right)_0^1$$

$$= \frac{1}{4} - \frac{0}{4}$$

$$= \frac{1}{4}$$

46. Ans: (c)

Sol: $\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r\sin\theta\,dr\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \left(\frac{r^2}{2}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\; 2a^2\; \cos^2\theta\; d\theta$$

$$= -2a^2 \int_1^0 t^2\,dt \qquad \text{Put } \cos\theta = t$$

$$= 2a^2 \int_0^1 t^2\,dt = \frac{2a^2}{3}$$

47. Ans: 0.1143

Sol: $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dz\,dx\,dy$

$$= \int_0^1 \left(\int_{y^2}^1 x(1-x)dx\right) dy$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{y^2}^1 dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy = \frac{4}{35}$$

48. Ans: 0.0536

Sol: $\iint_s \left(x^2y + xy^2\right)dxdy$

$= \int_0^1 \left[\int_{x^2}^{x} \left(x^2\, y + xy^2\right)dy \right] dx$

$= \int_0^1 \left[x^2\left(\frac{y^2}{2}\right)_{x^2}^{x} + x\left(\frac{y^3}{3}\right)_{x^2}^{x} \right]dx$

$= \int_0^1 \left[\frac{x^2}{2}\left(x^2 - x^4\right) + \frac{x}{3}\left(x^3 - x^6\right) \right] dx$

≈ 0.054

49. Ans: 32

Sol: The volume $= \iint_R z\ dx\ dy$

$= \int_0^6 \int_0^2 \left(4 - x^2\right) dx\ dy$

$= \int_0^6 \left[4x - \frac{x^3}{3} \right]_0^2 dy$

$= 32$

50. Ans: (c)

Sol: $\int_0^a \int_{\sqrt{ax}}^{a} \phi(x, y)\ dydx$

By changing the order of integration the above integral becomes $\int_0^a \int_0^{\frac{y^2}{a}} \phi(x, y)\ dydx$

Now,

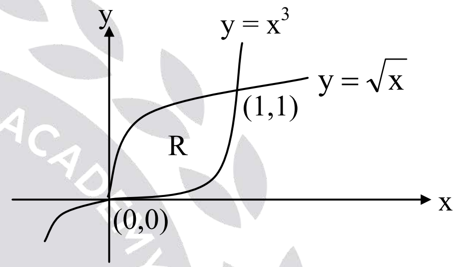
$\int_p^q \int_r^s \phi(x, y)\ dxdy = \int_0^a \int_0^{\frac{y^2}{a}} \phi(x, y)\ dydx$

$\therefore\ q.s = a.\left(\frac{y^2}{a}\right) = y^2$

51. Ans: (a)

Sol: The area of the region R bounded by the curves $y = x^3$ & $y = \sqrt{x}$ is given by

Area $= \int_{x=0}^{1} \left(\sqrt{x} - x^3\right)dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^4}{4}\right)_0^1 = \frac{5}{12}$



52. Ans: (d)

Sol: Length $= \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$

$= \int_0^3 \sqrt{1 + x}\ dx$

$= \frac{2}{3}\left[\left(1 + x\right)^{\frac{3}{2}}\right]_0^3 = \frac{14}{3}$

53. Ans: 25.12

Sol: Volume $= \int_0^4 \pi\, y^2\ dx$

$= \int_0^4 \pi\, x\ dx$

$= 8\pi$ cubic units

54. Ans: 1.88

Sol: Volume = $\int_0^1 \pi\ x^2\ dy$

$$= \pi \int_0^1 y^{\frac{2}{3}}\ dy \approx 1.88$$

55. Ans: (a)

Sol: T = xy + yz + zx

$\Rightarrow \nabla T = \bar{i}(y+z)+\bar{j}(x+z)+\bar{k}(x+y)$

at (1, 1, 1), $\nabla T = 2\bar{i}+2\bar{j}+2\bar{k}$

Given $\bar{a} = 3\bar{i}-4\bar{k}$

$\therefore$ Directional Derivative = $\nabla T. \dfrac{\bar{a}}{|\bar{a}|}$

$$= \left(2\bar{i}+2\bar{j}+2\bar{k}\right).\frac{\left(3\bar{i}-4\bar{k}\right)}{\sqrt{9+16}}$$

$$= \frac{-2}{5}$$

56. Ans: (b)

Sol: $f(x, y, z) = x^2 + y^2 + 2z^2$

$\nabla f = 2x\bar{i}+2y\bar{j}+4z\bar{k}$

at (1,1,2), $\nabla f = 2\bar{i}+2\bar{j}+8\bar{k}$

Given $\bar{a} = \nabla f$

Directional Derivative = $\nabla f. \dfrac{\bar{a}}{|\bar{a}|}$

$$= \nabla f. \frac{\nabla f}{|\nabla f|}$$

$$= |\nabla f| = \sqrt{4+4+64}$$

$$= \sqrt{72}$$

$$= 2\sqrt{18}$$

57. Ans: (a)

Sol: The Directional Derivative is maximum in the direction of $\nabla\phi$

Given $\phi(x, y, z) = x^2y^2z^4$

$\Rightarrow \nabla\phi = (2xy^2z^4)\ \bar{i} + (2x^2yz^4)\ \bar{j} + (4x^2y^2z^3)\ \bar{k}$

At (3, 1, –2), $\nabla\phi = 96\bar{i}+288\bar{j}-288\bar{k}$

$$= 96\left(\bar{i}+3\bar{j}-3\bar{k}\right)$$

58. Ans: (b)

Sol: Div $\bar{f} = \dfrac{\partial f_1}{\partial x}+\dfrac{\partial f_2}{\partial y}+\dfrac{\partial f_3}{\partial z}$

$$= e^x + e^{-x} + 2 \sin hx$$

59. Ans: (a)

Sol: $\nabla \times \bar{V} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 - x^2 + y & x(2y+1) & 0 \end{vmatrix}$

$$= \bar{i}[0-0]-\bar{j}[0-0+\bar{k}[(2y+1)-(2y+1)]$$

$$= \bar{0}$$

60. Ans: (b)

Sol: Given $\bar{V} = \left(x^2+yz\right)\bar{i}+\left(y^2+zx\right)\bar{j}+\left(z^2+xy\right)\bar{k}$

Div $\bar{V}$ = 2x + 2y + 2z ≠ 0

$\Rightarrow \bar{V}$ is not divergence free

Curl $\bar{V} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+yz & y^2+zx & z^2+xy \end{vmatrix}$

$$= \bar{i}[x-x]-\bar{j}[y-y]+\bar{k}[z-z]=\bar{0}$$

$\Rightarrow \bar{V}$ is irrotational

$\therefore$ Option (b) is correct

61. Ans: (c)

Sol: Given $f = x^2 + y^2 + z^2$, $\bar{r} = x\bar{i} + y\bar{j} + 3\bar{k}$

$\Rightarrow f\,\bar{r} = fx\bar{i} + fy\bar{j} + fz\bar{k}$

$$\text{div}\,(f\,\bar{r}) = \frac{\partial}{\partial x}(fx) + \frac{\partial}{\partial y}(fy) + \frac{\partial}{\partial z}(fz)$$

$= [x.(2x) + f] + [y.(2y) + f] + [z.(2y) + f]$

$= 2(x^2 + y^2 + z^2) + 3f = 5f$

62. Ans: (b)

Sol: $\text{L.I} = \int_C \bar{f}.d\bar{r} = \int_A^B (f_1\,dx + f_2\,dy + f_3\,dz)$

$$= \int_{(0,2,1)}^{(4,1,-1)} [(2z)dx + (2y)dy + (2x)dx]$$

$$= \int_{(0,2,1)}^{(4,1,-1)} [(2z\,dx + 2x\,dz) + (2y)dy]$$

$$= \int_{(0,2,1)}^{(4,1,-1)} [2(z\,dx + x\,dx) + (2y)dy]$$

$$= \int_{(0,2,1)}^{(4,1,-1)} [2\,d(xz) + 2y\,dy]$$

$$= \left(2(xz) + 2\frac{y^2}{2}\right)_{(0,2,1)}^{(4,1,-1)}$$

$= [2\,(4)\,(-1) + (1)^2] - [(2)\,(0)\,(1) + (2)^2]$

$= -11$

63. Ans: 202

Sol: Given $\bar{F} = (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$

$$\text{Curl}\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$= \bar{i}[0 - 0] - \bar{j}[3z^2 - 3z^2] + \bar{k}[2x - 2x] = \bar{0}$

$\Rightarrow \bar{F}$ is irrotational

$\Rightarrow$ Work done by $\bar{F}$ is independent of path of curve

$\Rightarrow \bar{F} = \nabla\phi$

where $\phi(x, y, z)$ is scalar potential

$$\Rightarrow (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$\Rightarrow d\phi = (2xy + z^3)\,dx + x^2\,dy + 3xz^2\,dz$

$\Rightarrow \int d\phi = \int (2xy + z^3)dx + x^2\,dy + 3xz^2\,dz$

$\Rightarrow \int d\phi = \int d(x^2y + xz^3)$

$\Rightarrow \phi(x, y, z) = x^2y + xz^3$

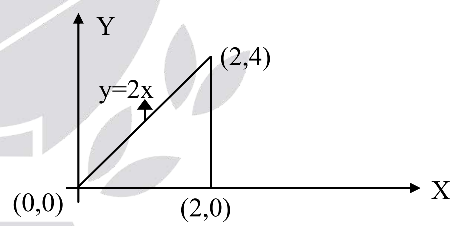
$\therefore$ Workdone $= \int_C \bar{F}.d\bar{r} = \phi(3,1,4) - \phi(1, -2,1)$

$= [9(1)+3(64)] - [1(-2) + 1(1)]$

$= 202$

64. Ans: (d)

Sol:



By Green's Theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

where $M = x + y$, $N = x^2$ and

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 1$$

The given integral= $\int_{x=0}^{2} \int_{y=0}^{2x} (2x - 1)\,dy\,dx$

$$= \int_0^2 \left[2xy - y\right]_0^{2x} dx$$

$$= \int_0^2 \left[4x^2 - 2x\right] dx$$

$$= \frac{20}{3}$$

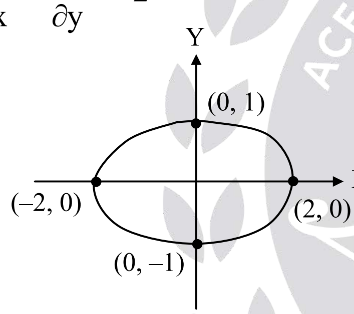
65. Ans: (c)

Sol: By Green's Theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here, $M = 2x - y$ and $N = x + 3y$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$$



The given integral = $\iint_R 2\ dx\ dy$

= 2 Area of the given ellipse

= 2 (π. 2. 1) = 4π

66. Ans: 0

Sol: Given $\overline{A} = \nabla\phi$

$\Rightarrow$ Curl $\overline{A} = \overline{0}$

$\Rightarrow$ $\overline{A}$ is Irrotational

$\therefore$ Line integral of Irrotational vector function along a closed curve is zero

i.e. $\int_C \overline{A}.d\overline{r} = 0$, where $C : \frac{x^2}{4} + \frac{y^2}{9} = 1$ is a closed curve.

67. Ans: (b)

Sol: Using Gauss-Divergence Theorem,

$$\int_S \overline{F}.\overline{N}\ ds = \int_V \text{div}\ \overline{F}\ dv$$

$$= \int_V 3\ dv = 3\ V$$

$$= 3 \times \frac{4}{3}\pi r^3$$

$$= 4\pi(4)^3$$

$$= 256\pi$$

68. Ans: 264

Sol: Using Gauss-Divergence Theorem,

$$\iint_S xy\ dy\ dz + yz\ dz dx + zx\ dx\,dy = \iiint_V \text{div}\ \overline{F}\ dv$$

$$= \iiint_V (y + z + x)\ dv$$

$$= \int_{x=0}^{4} \int_{y=0}^{3} \int_{z=0}^{4} (x + y + z)\ dzdydx$$

$$= \int_{x=0}^{4} \int_{y=0}^{3} \left[4x + 4y + 8\right] dy\ dz$$

$$= \int_{x=0}^{4} \left[12x + 18 + 24\right] dx$$

$$= 264$$

69. Ans: 0

Sol: By Stokes' theorem, we have

$$\int_C \overline{F}.d\overline{r} = \iint_S \left(\nabla\times\overline{F}\right).\overline{n}\ ds$$

Here, $\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = \overline{0}$

$\Rightarrow \overline{F}$ is an irrotational

$\therefore \int \overline{F}.d\,\overline{r} = 0$

70. Ans: (d)

Sol: Curl $\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy-y & -yz^2 & -y^2z \end{vmatrix}$

$= \overline{i}[-2yz+2yz]-\overline{j}[0]+\overline{k}[0+1]$

$\Rightarrow$ Curl $\overline{F} = \overline{k}$

Using Stokes' theorem,

$$\int_C \overline{F}.d\overline{r} = \int_S \text{curl } \overline{F}.\overline{N}\ ds = \int_S \overline{k}.\overline{N}\ ds$$

Let R be the protection of s on xy plane

$$\Rightarrow \int_S \overline{k}.\overline{N}\ ds = \iint_R \overline{k}.\overline{N}\ \frac{dxdy}{|\overline{N}.\overline{k}|}$$

$$= \iint_R 1\, dx\, dy$$

= Area of Region

$= \pi r^2 = \pi(1)^2 = \pi$

71. Ans: (d)

Sol: The function f(x) = x.sin x is even function

$\therefore$ The fourier series of f(x) contain only cosine terms.

The coefficient of sin 2x = 0

72. Ans: (b)

Sol: Let $f(x) = \dfrac{(\pi-x)^2}{4}$

The fourier series of f(x) in (0, 2π) is

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} \left\{a_n \cos(nx) + b_n \sin(nx)\right\}$$

Now, $a_o = \dfrac{1}{\pi}\int_0^{2\pi} f(x)\ dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\ dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi-x)^3}{-12}\right]_0^{2\pi}$$

$$= \frac{-1}{12\pi}\left[-\pi^3-\pi^3\right]$$

$$= \frac{2\pi^3}{12\pi} = \frac{\pi^2}{6}$$

$\therefore$ The constant term $= \dfrac{a_0}{2} = \dfrac{\pi^2}{12}$

73. Ans: (b)

Sol: The given function is odd in (–π, π)

$\therefore$ Fourier series of f(x) contains only sine terms.

74. Ans: (b)

Sol: $f(x) = \sum_{n=1}^{\infty} \frac{k}{\pi}\left[\frac{2-2(-1)^n}{n}\right]\sin(nx)$

At $x = \frac{\pi}{2}$

$k = \frac{k}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......\right]$

$\therefore\ 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+........\infty = \frac{\pi}{4}$

75. Ans: (c)

Sol: $f(x) = \pi x - x^2$

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\ dx$

$b_1 = \frac{2}{\pi}\int_0^{\pi}\left[(\pi x - x^2).\sin x\right]dx$

$= \frac{2}{\pi}\left[(\pi x - x^2)(-\cos x) - (\pi - 2x)(-\sin x) + (-2)\cos x\right]_0^{\pi}$

$= \frac{8}{\pi}$

76. Ans: (b)

Sol: $f(x) = (x-1)^2$

The Half range cosine series is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$

$a_n = \frac{2}{\pi}\int_0^{\pi}(x-1)^2\ \cos(n\pi x)\ dx$

$= \frac{2}{\pi}\left[(x-1)^2.\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1).\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$

$= \frac{4}{n^2\pi^2}$

Chapter **3**

Probability & Statistics

C. R. Rao

01.

Sol: A die is rolled two times. Then sample space S is

$$S = \{(1,1)\,(1,2)\ldots\ldots(1,6)$$
$$(2,1)\,(2,2)\ldots\ldots(2,6)$$
$$\vdots$$
$$(6,1)\,(6,2).......(6,6)\}$$

S = 36

(i) Let A = same face appear

$= 6 \rightarrow \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (6,6)\}$

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

(ii) Let B = sum of neither 8 nor 9

$\bar{B}$ = sum is 8 or 9

= {(2,6) (3,5) (4,4) (5,3) (6,2) (3,6) (4,5) (5,4) (6,3)}

= 9

$$P(B) = 1 - P(\bar{B}) = 1 - \frac{9}{36} = \frac{3}{4}$$

(iii) Let C = sum is either 7 or 11

= {(1,6) (2,5) (3,4) (4,3) (5,2) (6,1) (5,6) (6,5)}

= 8

$$P(C) = \frac{8}{36} = \frac{2}{9}$$

(iv) Let D = the second toss results in a value that is higher than first toss

D = {(1,2) (1,3) (1,4) (1,5) (1,6) (2,3) (2,4) (2,5) (2,6) (3,4) (3,5) (3,6) (4,5) (4,6) (5,6)} = 15

$$P(D) = \frac{15}{36} = \frac{5}{12}$$

02.

Sol: As given four fair six sided dice are rolled

So sample space S = 6^4

= 36×36

= 1296

(i) Let A = sum is 22

A = {(5,5,6,6)
(5,6,5,6)
(6,5,5,6)
(6,5,6,5)
(6,6,5,5)
(5,6,6,5)
(6,6,6,6)
(6,6,4,6)
(6,4,6,6)
(4,6,6,6)}

A = 10

$$P(A) = \frac{10}{1296}$$

(ii) Let B = sum is 21

$(6,6,6,3) \rightarrow$ 4 combinations

$(5,5,5,6) \rightarrow$ 4 combinations

$(6,6,5,6) \rightarrow$ 6 combinations

$(5,6,5,6) \rightarrow$ 6 combinations

So total combinations to get sum 21 is 20

$$P(B) = \frac{20}{36 \times 36} = \frac{20}{1296}$$

Calyampudi Radhakrishna Rao, FRS known as C R Rao (born 10 September 1920) is an Indian-born, mathematician and statistician. The American Statistical Association has described him as "a living legend whose work has influenced not just statistics, but has had far reaching implications for fields as varied as economics, genetics, anthropology, geology, national planning, demography, biometry, and medicine.

03. Ans: (c)

Sol: Total cases = $6^3 = 216$

Let x = Number of dice show same face

$P(x \geq 2) = P(x = 2) + P(x = 3)$

$$= \frac{{}^3C_2 \times 6 \times 5}{216} + \frac{6}{216}$$

$$= \frac{96}{216}$$

$$= \frac{4}{9}$$

04. Ans: (c)

Sol: H = event of getting head

T = event of getting tail

X = event of getting 5 or 6

$\overline{X}$ = event of getting number except 5 or 6

$$P(H) = \frac{1}{2}$$

$$P(T) = \frac{1}{2}$$

$$P(X) = \frac{2}{6} = \frac{1}{3}$$

$$P(\overline{X}) = \frac{2}{3}$$

Let Y = event of gets head before gets 5 or 6 on the die

$$P(Y) = P(H) + P(T)P(\overline{X})P(H) + P(T)P(\overline{X})P(T)P(\overline{X})P(H) + \ldots\ldots$$

$$= P(H)(1 + P(T)\,P(\overline{X}) + P(T)P(\overline{X})P(T) + \ldots.)$$

$$P(Y) = \frac{1}{2}\left[1 + \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} \times \frac{2}{3} + \ldots..\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \ldots\ldots\right]$$

$$= \frac{1}{2}\left[\frac{a}{1-r}\right]$$ ($\because$ sum of infinite terms of geometric series)

where $a = 1,\ r = \frac{1}{3}$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{3}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{2}{3}}\right]$$

$$= \frac{3}{4}$$

05.

Sol: Urn contains 12 Red, 16 Blue & 18 Green balls

(i) Let A = 3 red, 2 blue and 2 green balls are withdrawn

$$P(A) = \frac{C_3^{12} \times C_2^{16} \times C_2^{18}}{C_7^{46}}$$

(ii) Let C = atleast 2 red balls are withdrawn

$$P(C) = \frac{\left(C_2^{12} \times C_5^{34}\right) + \left(C_3^{12} \times C_4^{34}\right) + \left(C_4^{12} \times C_3^{34}\right) + \left(C_5^{12} \times C_2^{34}\right) + \left(C_6^{12} \times C_1^{34}\right) + C_7^{12}}{C_7^{46}}$$

(iii) Let D = all withdrawn balls are of the same colour

$$P(D) = \frac{C_7^{12} + C_7^{16} + C_7^{18}}{C_7^{46}}$$

(iv) Let E = either exactly 3 red balls or exactly 3 blue balls are withdrawn

$$P(E) = \frac{\left(C_3^{12} \times C_4^{34}\right) + \left(C_3^{16} \times C_4^{30}\right)}{C_3^{46}}$$

06. Ans: (a)

Sol: Urn contains 5 red and 7 green balls

$P(R_1)$ = Probability of getting red ball in first draw

$= \frac{5}{12}$

$P(R_2)$ = Probability of getting red ball in second draw (extra red ball added) = $\frac{6}{13}$

$P(G_1)$ = Probability of getting green ball in first draw

$= \frac{7}{12}$

$P(R_3)$ = Probability of getting red ball in second draw (extra green ball added)

$= \frac{5}{13}$

The probability of getting a red ball in the next draw

$= [P(R_1) \times P(R_2)] + [P(G_1) \times P(R_3)]$

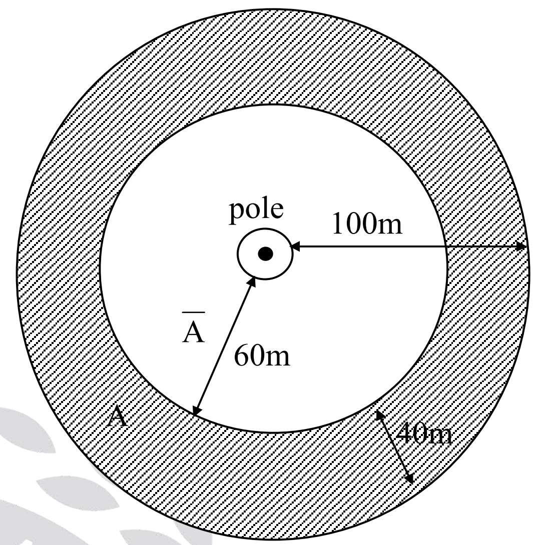
$$= \left[\frac{5}{12}\times\frac{6}{13}\right] + \left[\frac{7}{12}\times\frac{5}{13}\right]$$

$$= \frac{30+35}{12\times 13} = \frac{65}{156}$$

07. Ans: (a)

Sol: Let A = The cow is atleast 60m away from the pole

$\overline{A}$ = The cow is 60m away from the pole



$$P(A) = 1 - P(\overline{A}) = 1 - \left(\frac{(60)^2\pi}{(100)^2\pi}\right)$$

$$= 1 - \frac{36}{100} = \frac{64}{100} = \frac{16}{25}$$

08. Ans: (d)

Sol: p = Probability of birthday on Sunday

$= \frac{1}{7}$

q = Probability of birthday on any day except Sunday

$= \frac{6}{7}$

Then probability in a family of 5 members, exactly 2 members have birthday on Sunday

$= C_2^5\ p^2\ q^3$

$$= C_2^5\left(\frac{1}{7}\right)^2\left(\frac{6}{7}\right)^3$$

$$= \frac{5!}{3!\times 2!}\times\frac{1}{7^2}\times\frac{6^3}{7^3}$$

$$= \frac{5\times4}{2}\times\frac{6^3}{7^5} = \frac{10\times6^3}{7^5}$$

09.

Sol: Two cards are drawn from a pack of 52 cards

(i) Let A = Two cards drawn from pack of 52 cards, both of them from same suit.

In a pack of 52 cards there are 4 suits → clubs, diamonds, hears and spades.

Each suit contains 13 cards

$$\text{So, } P(A) = \frac{C_2^{13}+C_2^{13}+C_2^{13}+C_2^{13}}{C_2^{52}}$$

$$= \frac{4\times C_2^{13}}{C_2^{52}}$$

$$= \frac{4\times13!\times50!\times2!}{11!\times2!\times52!}$$

$$= \frac{4\times13\times12}{52\times51}$$

$$= \frac{4}{17}$$

(ii) Let B = Two cards drawn from a pack of 52 cards both of them from different suits.

[(Club, Diamonds)
(Club, Heart)
(Club, Spade)
(Club, Heart)
(Diamond, Heart)
(Diamond, Spade)
(Heart , Spade)]

$$P(B) = \frac{\left(C_1^{13}\times C_1^{13}\right)+\left(C_1^{13}\times C_1^{13}\right)+\left(C_1^{13}\times C_1^{13}\right)+\left(C_1^{13}\times C_1^{13}\right)+\left(C_1^{13}\times C_1^{13}\right)+\left(C_1^{13}\times C_1^{13}\right)}{C_2^{52}}$$

$$= \frac{6\times13\times13\times50!\times2!}{52!}$$

$$= \frac{6\times13\times13\times2}{52\times51}$$

$$= \frac{13}{17}$$

10.

Sol: A card is selected at random from a pack of 52 cards then

Sample space S = 52

(i) A part of 52 cards contains 13 spade cards and 16 face cards

A = A spade card selected from pack of 52 cards

B = A face card selected from pack of 52 cards

$$P(A) = \frac{13}{52},\ P(B) = \frac{12}{52}$$

In a pack of 52 cards 4 cards are common means 4 spade cards are face cards.

$$P(A\cap B) = \frac{3}{52}$$

$P(A \cup B)$ = probability of selected card is spade or face

$= P(A) + P(B) – P(A\cap B)$

$$= \frac{13}{52}+\frac{12}{52}-\frac{3}{52} = \frac{22}{52}$$

(ii) K = A king card selected from 52 cards

$$P(K) = \frac{4}{52}$$

R = A red card selected from 52 cards

$$P(R) = \frac{26}{52}$$

$$P(K \cap R) = \frac{2}{52}$$

$$P(K \cup R) = P(K) + P(R) - P(K \cap R)$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}$$

$$= \frac{28}{52} = \frac{7}{13}$$

(iii) Q = A queen card selected from a 52 cards

$$P(Q) = \frac{4}{52}$$

$$P(K) = \frac{4}{52}$$

$P(Q \cap K) = 0$

$P(Q \cup K) = P(Q) + P(K) - P(Q \cap K)$

$$= \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52} = \frac{2}{13}$$

11. Ans: (a)

Sol: A = Integers divisible by 6 = 33 (6 to 198)

B = Integers divisible by 8 = 25 (8 to 200)

C = Integers divisible by both 6 and 8

= Integers divisible by 24 (LCM of 6 and 8)

= 8 (24 to 192)

So, probability that the number divisible by 6 or 8

$= P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(C)$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$

$$= \frac{50}{200} = \frac{1}{4}$$

12. Ans: (b)

Sol: S = sample space of ticket number

= { (0,0) (0,1)……..(9,9)}

X = sum of digits on tickets = 9

= { (0,9) (1,8) (2,7) (3,6) (4,5) (5,4) (6,3) (7,2) (8,1) (9,0)}

Y = product of the digits on the tickets = 0

= { (0,0) (0,1) (0,2) (0,3) (0,4) (0,5) (0,6) (0,7) (0,8) (0,9) (1,0) (2,0) (3,0) (4,0) (5,0) (6,0) (7,0) (8,0) (9,0) }

$$P(X) = \frac{10}{100} \quad P(Y) = \frac{19}{100} \quad P(X \cap Y) = \frac{2}{100}$$

$$P(x = 9 / y = 0) = \frac{P(X \cap Y)}{P(Y)} = \frac{2/100}{19/100} = \frac{2}{19}$$

13. Ans: (b)

Sol: S = sample space of a die thrown three times

$= (6)^3 = 6 \times 6 \times 6 = 216$

Let A = sum of the numbers found to be 16

= {(6,6,4)
(6,4,6)
(4,6,6)
(6,5,5)
(5,6,5)
(5,5,6)}

$$P(A) = \frac{6}{216}$$

Let B = 5 appears on the third row

= 36

$$P(B) = \frac{36}{216}$$

$A \cap B = \{(6,5,5)$

$(5,6,5)\}$

$$P(A \cap B) = \frac{2}{216}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{2/216}{6/216}$$

$$= \frac{2}{6} = \frac{1}{3}$$

14. Ans: (d)

Sol: $P(A) = \frac{80}{100} = 0.8$

$$P(B) = \frac{60}{100} = 0.6$$

$$P(\overline{A}) = 1 - P(A) = 0.2$$

$$P(\overline{B}) = 1 - P(B) = 0.4$$

The probability that atleast one of them will solve a problem

= 1 – probability that no one will solve problem

$= 1 - \left[P(\overline{A})\,P(\overline{B})\right]$

$= 1 - [0.2 \times 0.4]$

$= 1 - (0.08)$

$= 0.92$

15. Ans: 0.4

Sol: Given $P(A \cup B) = 0.64$ and

$P(A \cap B) = 0.16$

Now, $P(A \cup B) = 0.64$

$P(A) + P(B) - P(A \cap B) = 0.64$

$P(A) + P(B) - 0.16 = 0.64$

$P(A) + P(B) = 0.80$ .... (1)

$P(A \cap B) = P(A)\,P(B)$

($\because$ A & B are independent events)

$0.16 = P(A)\,P(B)$

$$P(B) = \frac{0.16}{P(A)}$$

$$(1) \Rightarrow P(A) + \frac{0.16}{P(A)} = 0.80$$

$\Rightarrow (P(A))^2 + 0.16 = 0.80\,P(A)$

$\Rightarrow (P(A))^2 - 0.80\,P(A) + 0.16 = 0$

$\Rightarrow [P(A) - (0.4)]^2 = 0$

$\therefore P(A) = 0.4$

16.

Sol: B_1 = sum of the outcomes is 12

B_2 = sum of the outcomes is 7

B_3 = sum of the outcomes is neither 7 nor 12

A = Lunch offered at a five star hotel

$$P(B_1) = \frac{1}{36},\ P(B_2) = \frac{6}{36} = \frac{1}{6},\ P(B_3) = \frac{29}{36}$$

$$P(A\,|\,B_1) = \frac{2}{3},\ P(A\,|\,B_2) = \frac{1}{2},\ P(A\,|\,B_3) = \frac{1}{3}$$

(i) $P(A) = P(B_1)\, P(A|B_1) + P(B_2)\, P(A|B_2) + P(B_3)\, P(A|B_3)$

$$= \left(\frac{1}{36}\times\frac{2}{3}\right)+\left(\frac{6}{36}\times\frac{1}{2}\right)+\left(\frac{29}{36}\times\frac{1}{3}\right)$$

$$= \frac{10}{27}$$

(ii)

$$P(B_1 \mid A) = \frac{P(B_1)P(A \mid B_1)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{36}}{\frac{10}{27}}$$

$$= \frac{1}{20}$$

17.

Sol: Let A = event of cell in a wireless system overloaded.

$\overline{A}$ = event of cell in a wireless system is not overloaded.

B = event of call is blocked

$$P(A) = \frac{1}{3}$$

$$P(\overline{A}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$P(B/A) = 0.3$

$P(B/\overline{A}) = 0.1$

(i) P(B) = Probability that your call is blocked

$$P(B) = P(B/A)\ P(A) + P(B/\overline{A})\, P(\overline{A})$$

$$= \left(0.3\times\frac{1}{3}\right)+\left(0.1\times\frac{2}{3}\right)$$

$$= \frac{0.3+0.2}{3} = \frac{0.5}{3} = \frac{1}{6} = 0.1667$$

(ii) P(A/B) = Probability that the system is overloaded given that your call is blocked.

$$P(A/B) = \frac{P(B/A)\ P(A)}{P(B/A)\ P(A)\ +\ P(B/\overline{A})\ P(\overline{A})}$$

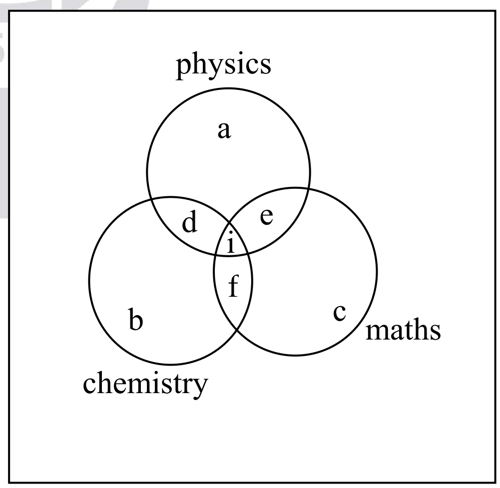
$$= \frac{0.3\times\frac{1}{3}}{\left(0.3\times\frac{1}{3}\right)+\left(0.1\times\frac{2}{3}\right)}$$

$$= \frac{0.3}{0.3+0.2}$$

$$= \frac{0.3}{0.5} = 0.6$$

18.

Sol: First we draw venn diagram:



40 students failed in mathematics

40 = c + e + f + i……..(1)

30 students failed in physics

30 = a + d + e + i…….(2)

25 students failed in chemistry

25 = b + d + f + i…….(3)

20 failed in mathematics and physics

20 = e + i…….(4)

15 failed in physics and chemistry

15 = d = i……..(5)

10 failed in mathematics and chemistry

10 = f + i…….(6)

5 failed in mathematics, physics and chemistry

5 = i……..(7)

By using equation (6) & (7) $\rightarrow$ f = 5

By using equation (5) $\rightarrow$ d = 10

By using equation (4) $\rightarrow$ e = 15

By using f, d, e and i value we find a,b,c value as

a = 30 – (d+e+i)

= 30 – (10+15+5)

a = 0

b = 25 – (d+f+i)

= 25 – (10+5+5)

b = 5

c = 40 – (e+f+i)

= 40 – (15+5+5)

c = 15

(i) Number of students passed in all three subjects is

= 100 – [a+b+c+d+e+f+i]

= 100 – [0+5+15+10+15+5+5]

= 100 – [55]

= 45

The probability that he passed in all three subjects

$$= \frac{45}{100} = 0.45$$

(ii) The number of students failed in atmost one subjects is

= 45 + a + b + c

= 45 + 0 + 5 + 15 = 65

The probability that the a selected students failed in atmost one subject $= \frac{65}{100} = 0.65$

(iii) Number of students failed exactly one subject

= a + b + c

= 0 + 5 + 15

= 20

Probability of a student selected at random failed in exactly one subject $= \frac{20}{100} = 0.2$

(iv) Number of students failed in exactly the subjects

= d + e + f

= 10 + 15 + 5

= 30

Probability of a student selected at random failed exactly two subjects $= \frac{30}{100} = 0.3$

(v) Number of students failed in atleast two subjects

= d + e + f + i

= 10 + 15 + 5 + 5

= 35

Probability of a student selected at random failed in atleast two subjects $= \frac{35}{100} = 0.35$

(vi) The number of students failed in atmost two subjects

$= (d + e + f) + (a + b + c) + 45$

$= 30 + 20 + 45$

$= 95$

Probability of a student selected at random failed in atmost two subjects $= \frac{95}{100} = 0.95$

19. Ans: 0 and 0.4

Sol: Here f (x) is an even function

$$E(X) = \int_{-\infty}^{\infty} x.f(x)\,dx = 0$$

($\because$ x f(x) is an odd function)

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-1}^{0} x^2 (1+x)\,dx + \int_{0}^{1} x^2 (1+x)\,dx = \frac{1}{6}$$

Variance of X = E (X^2) –(E (X))^2 $= \frac{1}{6}$

Standard deviation = $\frac{1}{\sqrt{6}}$

20.

Sol: (i) $E(X) = \sum_{i=1}^{4} x_i f(x_i)$

$= x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + x_4 f(x_4)$

$= (1\times0.3) + (2\times0.4) + (3\times0.2) + (4\times0.1)$

$= 2.1$

(ii) var (x) = E(x^2) – $(E(x))^2$

$$E(X^2) = \sum_{i=1}^{4} x_i^2 f(x_i)$$

$= (1\times0.3) + (4\times0.4) + (9\times0.2) + (16\times0.1)$

$= 0.3 + 1.6 + 1.8 + 1.6 = 5.3$

Var(x) = E(x^2) – $(E(x))^2$

$= 5.3 - (2.1)^2$

$= 0.89$

21. Ans: 150

Sol: P(H) = Probability of 1 head occur

$= \frac{1}{2}$

P(T) = Probability of 1 tail occur

$= \frac{1}{2}$

Probability of three heads occur

$$P(3H) = C_3^3 \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^0 = \frac{1}{8}$$

Probability of two heads occur

$$P(2H) = C_2^3 \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^1 = 3\times\frac{1}{4}\times\frac{1}{2} = \frac{3}{8}$$

Probability of one head occur

$$P(1H) = C_1^3 \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = 3\times\frac{1}{8} = \frac{3}{8}$$

Probability of no head occur

$$P(0H) = C_0^3 \left(\frac{1}{2}\right)^0 \times \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Expected value of game

= [500×P(3H)] + [300×P(2H)]

+ [100×P(1H)] + [–500×P(0H)]

$$=\left(500\times\frac{1}{8}\right)+\left(300\times\frac{3}{8}\right)+\left(100\times\frac{3}{8}\right)-\left(500\times\frac{1}{8}\right)$$

$$=\frac{500+900+300-500}{8}$$

$$=\frac{1200}{8}$$

= 150

22. Ans: (d)

Sol: $f(x) = e^{-x}\quad 0 \le x \le \infty$

The probability of $f(x)=\int_0^2 f(x)\,dx$

$$=\int_0^2 e^{-x}\,dx$$

$$=\left.\frac{e^{-x}}{-1}\right|_0^2$$

$= -1\,(e^{-2} - 1)$

$= 1 - e^{-2}$

= 0.865

23.

Sol: (i) We know $\sum_x \sum_y P(x,y)=1$

3k + 6k + 9k + 5k + 8k + 11k + 7k

+ 10k + 13k = 1

72k = 1

$$k=\frac{1}{72}$$

(ii) $P(X\le 1)=\sum_{j=0,k=1}^{j=1,k=3} P(x_j y_k)$

$= P(x_0\, y_1) + P(x_0\, y_2) + P(x_0\, y_3) + P(x_1\, y_1)$

$+ P(x_1\, y_2) + P(x_1\, y_3)$

= 3k + 6k + 9k + 5k + 8k + 11k

= 42k

$$=42\times\frac{1}{72}=\frac{7}{12}$$

(iii) $P(Y\le 2)=\sum_{j=0,k=1}^{j=2,k=2} P(x_j y_k)$

$= P(x_0\, y_1) + P(x_0\, y_2) + P(x_1\, y_1)$

$+ P(x_1\, y_2) + P(x_2\, y_1) + P(x_2\, y_2)$

= 3k + 6k + 5k + 8k + 7k + 10k

= 39k

$$=39\times\frac{1}{72}$$

$$=\frac{13}{24}$$

(iv) $\sum_{j=1,k=1}^{j=2,k=3} P(x_j y_k)$

$= P(x_1\, y_1) + P(x_1\, y_2) + P(x_1\, y_3) + P(x_2\, y_1)$

$+ P(x_2\, y_2) + P(x_2\, y_3)$

= 5k + 8k + 11k + 7k + 10k + 13k

= 54k

$$=54\times\frac{1}{72}=\frac{3}{4}$$

24.

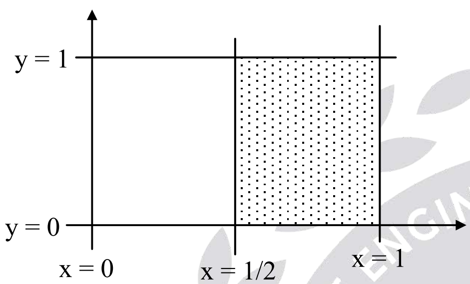
Sol: (i) $P\left(x>\frac{1}{2}\right)=\int_{y=0}^{1}\int_{x=1/2}^{1}(x+y)\,dx\,dy$

$$=\int_{y=0}^{1}\left(\frac{x^2}{2}+yx\right)_{x=1/2}^{1}dy$$

$$= \int_{y=0}^{1}\left[\frac{1}{2}\left(1-\frac{1}{4}\right)+y\left(1-\frac{1}{2}\right)\right]dy$$

$$= \int_{y=0}^{1}\left(\frac{3}{8}+\frac{y}{2}\right)dy = \frac{3}{8}(y)_0^1 + \frac{(y^2)_0^1}{4}$$

$$= \frac{3}{8}+\frac{1}{4} = \frac{5}{8}$$

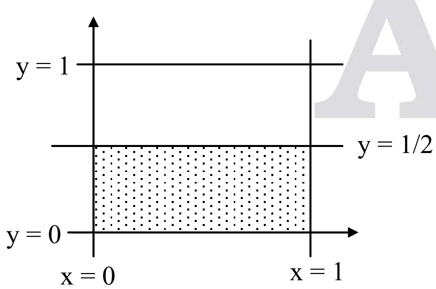


P(x > 1/2) = P(1/2 < x < 1, 0 < y < 1)

(ii) $P\left(y<\frac{1}{2}\right) = \int_{y=0}^{1/2}\int_{x=1/2}^{1}(x+y)\,dx\,\,dy$

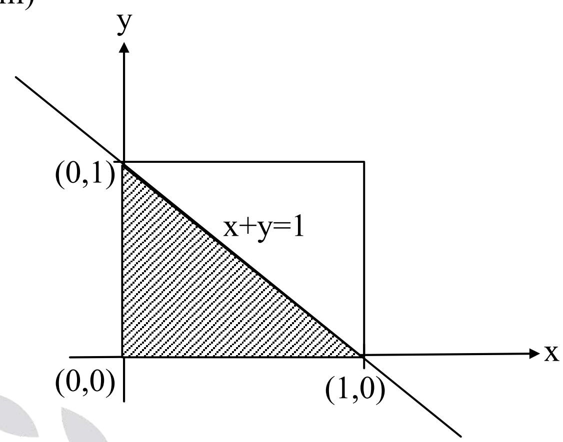
$$= \int_{y=0}^{1/2}\left(\frac{x^2}{2}+yx\right)_{x=0}^{1} dy = \int_{y=0}^{1}\left(\frac{1}{2}+y\right)dy$$

$$= \left[\frac{1}{2}y+\frac{y^2}{y}\right]_{y=0}^{1/2} = \frac{3}{8}$$



P(y < 1/2) = P(0 < x < 1, 0 < y < 1/2)

(iii)



In the given figure, we have indicated the square region $0 \le x \le 1,\ 0 \le y \le 1$, within which the joint density function shown.

The required probability

$$P(X+Y\le 1) = \int_R\int f(x,y)\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{(1-y)}(x+y)\,dx\,dy$$

$$= \int_{y=0}^{1}\left(\frac{x^2}{2}+yx\right)_{x=0}^{(1-y)} dy$$

$$= \int_{y=0}^{1}\frac{(1-y)^2}{2}+y(1-y)\,\,dy$$

$$= \left.\frac{(1-y)^3}{-6}\right|_0^1 + \left.\frac{y^2}{2}\right|_0^1 + \left.\frac{-y^3}{3}\right|_0^1$$

$$= \frac{(0-1)}{-6}+\frac{1}{2}+\frac{-1}{3}$$

$$= \frac{1}{6}+\frac{1}{2}+-\frac{1}{3} = \frac{1}{3}$$

$$P(X+Y\le 1) = \frac{1}{3}$$

25. Ans: (b)

Sol: Let A = Event of getting a total of 7 atleast once in three tosses of a pair of fair dice.

The probability of getting total of 7 is

$$P(B) = \frac{6}{36} = \frac{1}{6}$$

The probability of not getting total of 7 is

$$P(\overline{B}) = 1 - \frac{1}{6} = \frac{5}{6}$$

P(A) = 1 – probability of not getting a seven three times in a row

$$P(A) = 1 - \left(\frac{5}{6}\right)^3$$

$$= 1 - \frac{125}{216}$$

$$P(A) = \frac{91}{216}$$

26. Ans: (b)

Sol: If the person is one step away, then we have two cases:

Case1: 6 forward steps and 5 backward

(or)

Case2: 6 backward steps and 5 forward.

Required Probability

$= C(11,6)(0.4)^6 + C(11,5)(0.6)^6\ (0.4)^5$

$= C(11,5)\ (0.4)^5\ (0.6)^5\ (0.4 + 0.6)$

$= 462 \times (0.24)^5$

27. Ans: (b)

Sol: Probability of getting head is $p = \frac{1}{2}$

Probability of getting tail is $q = \frac{1}{2}$

The probability of getting sixth head at the eleventh toss is given by

$$= \frac{1}{2}\,{}^{10}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 = \frac{{}^{10}C_5}{2^{11}}$$

28.

Sol: In case of poisson's distribution

$$P(X = x) = \frac{\lambda^x\ e^{-\lambda}}{x!}$$

$$\lambda = 240\ \frac{v_{eh}}{hr} \times \frac{30}{3600}(hr)$$

$$\lambda = 2$$

(i) Probability of one vehicle arriving over a 30 second time interval

$$= \frac{(2)^1\ e^{-2}}{1!} = 2e^{-2} = 0.27$$

(ii) Probability of atleast one vehicle arriving over a 30 sec time interval

= 1 – [probability of no vehicle arriving over a 30 sec interval]

$$= 1 - \left[\frac{e^{-\lambda}\ \lambda^0}{0\,!}\right] = 1 - \left(\frac{e^{-2}}{1}\right) = 0.8647$$

(iii) probability of more than two vehicles arriving over a 30 sec time interval

= 1 – (P(0) = P(1))

$$= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}\right]$$

$$= 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

$= 1 - e^{-2}\ (1+2+2)$

$= 1 - 5e^{-2}$

$= 0.3233$

29. Ans: 0.0046

Sol: n = 1000, p = 0.0001

$\lambda = np = 0.1$

Let x = Number of accidents occur in a week.

$p(x \geq 2) = 1 - p(x < 2)$

$= 1 - \{p(x=0) + p(x=1)\}$

$$= 1 - \left\{\frac{\lambda^0\, e^{-\lambda}}{0\,!} + \frac{\lambda^1\, e^{-\lambda}}{1!}\right\}$$

$= 1 - \{e^{-0.1} + 0.1\ e^{-0.1}\}$

$= 1 - 1.1\ e^{-0.1}$

$= 0.0046$ (approx)

30. Ans: 0.122

Sol: We view the number of misprints on one page as the number of successes in a sequence of Bernoulli trials. Here n = 300 since there are 300 misprints, and p = $\frac{1}{500}$, the probability that a misprint appears on a given page. Since p is small, we use the Poisson approximation to the binomial distribution with $\lambda = np = 0.6$.

We have

P(0misprint) = f(0; 0.6)

$$= \frac{(0.6)^0\, e^{-0.6}}{0!} = e^{-0.6} = 0.549$$

P(1 misprint) = f(1; 0.6)

$$= \frac{(0.6)^1\, e^{-0.6}}{1!} = (0.6)\ (0.549)$$

$= 0.329$

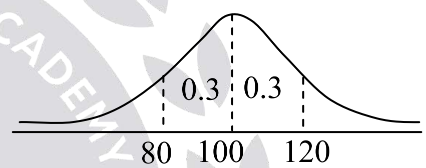
Required probability

$= 1 - (0.549 + 0.329)$

$= 0.122$

31. Ans: 0.2

Sol: The area under normal curve is 1 and the curve is symmetric about mean.



$\therefore P(100 < X < 120) = P(80 < X < 120) = 0.3$

Now, $P(X < 80) = 0.5 - P(80 < X < 120)$

$= 0.5 - 0.3 = 0.2$

32.

Sol: The parameters of normal distribution are $\mu = 68$ and $\sigma = 3$

Let X = weight of student in kgs

Standard normal variable = Z = $\frac{X - \mu}{\sigma}$

(i) When X = 72, we have Z = 1.33

Required probability = P(X > 72)

= Area under the normal curve to the right of Z = 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh greater than 72 kgs = 300 × 0.0918

= 28

(ii) When X = 64, we have Z = –1.33

Required probability = P(X ≤ 64)

= Area under the normal curve to the left of Z = – 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

(By symmetry of normal curve)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh less than 68 kgs = 300 × 0.0918

= 28

(iii) When X = 65, we have Z = –1

When X = 71, we have Z = +1

Required probability = P(65< X < 71)

= Area under the normal curve to the left of Z = – 1 and Z = +1

= 0.6826

(By Property of normal curve)

Expected number of students who weighs between 65 and 71 kgs

= 300 × 0.6826

≈ 205

33. Ans: 0.8051

Sol: The probability of population has Alzheimer's disease is

p = 0.04, q = 0.96, n = 3500

$\mu = np = (3500)(0.04) = 140$

$\sigma^2 = npq = (3500)(0.04)(0.96)$

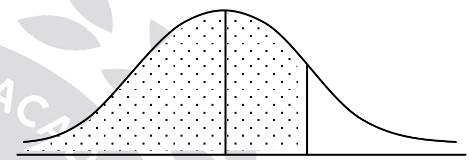
$\sigma^2 = 134.4, \sigma \cong 11.59$

Let X = number of people having Alzheimer's disease

$$P(X < 156) = P\left(\frac{X-\mu}{\sigma} < \frac{150-\mu}{\sigma}\right)$$

$$= P\left(Z < \frac{150-140}{11.59}\right)$$

$= P(Z < 0.86)$



= 0.5 + Area between z = 0 & z = 0.86

= 0.5 + 0.3051

= 0.8051

34.

Sol: (i) $E(x) = \int_0^1 x\, f(x)dx$

$$f(x) = \frac{1}{b-a} = 1$$

$$E(x) = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

(ii) $E(x^2) = \int_0^1 x^2\, f(x)dx$

$$f(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1$$

$$E(x^2) = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

(iii) $E(x^2) = \int_0^1 x^3\ f(x)dx$

$$= \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}$$

(iv) Variance = $E(x^2) - (E(x))^2$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

35. Ans: (d)

Sol: If point chosen is (0,0) then length of position vector (minimum value of P can be O) will be 0 and the maximum value of P be $\sqrt{5}$ when point chosen is (1,2)

Minimum value of P = 0 at (0,0) point

Maximum value of P = $\sqrt{5}$ at (1,2) point

$$E(P^2) = \int_0^{\sqrt{5}} P^2 f(p)dp$$

$f(p) = \frac{1}{\sqrt{5}-0} = \frac{1}{\sqrt{5}}$ as p is random variable

$$E(P^2) = \int_0^{\sqrt{5}} P^2 \frac{1}{\sqrt{5}} dp$$

$$= \frac{1}{\sqrt{5}}\left(\frac{P^3}{3}\right)_0^{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} \times \frac{1}{3}(\sqrt{5})^3 = \frac{1}{3} \times \sqrt{5} \times \sqrt{5}$$

$$E(P^2) = \frac{5}{3}$$

36.

Sol: Given that passenger derives at a bus stop at 10 AM:

While stop arrive time is uniformly distributed between 10AM to 10:30AM

$$f(x) = \frac{1}{b-a} = \frac{1}{30-0} = \frac{1}{30}$$

(i) As we know passenger arrives bus stop at 10:00AM. But as given he want to wait more than 10 minutes means 10:10AM to 10:30AM

$$P(X \geq 10\,\text{min}) = \int_{10}^{30} f(x)\ dx$$

$$= \int_{10}^{30} \frac{1}{30} dx$$

$$= \frac{1}{30}(x)_{10}^{30}$$

$$= \frac{20}{30}$$

$$= \frac{2}{3}$$

(ii) As per given condition passenger will has to wait 10 : 15 AM to 10 : 25 AM.

$$P(15 \leq x \leq 25) = \int_{15}^{25} f(x)\ dx$$

$$= \int_{15}^{25} \frac{1}{30}\ dx$$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

37. Ans: (b)

Sol: In case of exponential distribution

$$f(x)\begin{cases}\theta e^{-\theta x} & x \geq 0 \\ 0 & \text{otherwise}\end{cases}$$

$$\text{Mean} = \frac{1}{\theta}$$

$$\text{Variance} = \frac{1}{\theta^2}$$

$f(x) = 2\ e^{-2x}$

$$\text{mean } = \frac{1}{\theta} = \frac{1}{2}$$

$$\text{variance } = \frac{1}{\theta^2} = \frac{1}{4}$$

38. Ans: 0.3678

Sol: The probability density function is

$P(x) = \theta\ e^{-\theta x} \qquad x \geq 0$

$\quad = 0 \qquad x < 0$

Probability that x exceeds th expected value

$$P\left(X \geq \frac{1}{\theta}\right) = \int_{x=\frac{1}{\theta}}^{\infty} \theta\ e^{-\theta x}\ dx$$

$$= \theta\left[\frac{e^{-\theta x}}{-\theta}\right]_{x=\frac{1}{\theta}}^{\infty}$$

$$= \frac{\theta}{-\theta}\left(0 - e^{-1}\right)$$

$= e^{-1}$

$= 0.3678$

39. Ans: 0.0025

Sol: $f(x) = 2\ e^{-2x}$

The probability that shower more than three minutes

$$= \int_3^{\infty} f(x)\ dx$$

$$= \int_3^{\infty} 2\ e^{-2x}\ dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_3^{\infty}$$

$= -1\ (\ 0 - e^{-6})$

$= e^{-6}$

$= 0.0025$

40.

Sol: Given data: 27, 35, 40, 35, 36, 36, 29

(i) Mean is $\bar{x} = \dfrac{27 + 35 ...... + 29}{7}$

$\bar{x} = 34$

(ii) Arrange the data points in ascending order 27, 29, 35, 35, 36, 36, 40

Median = 35

(iii) Modes = 35, 36

41.

Sol: Given:

Class	Frequency
0-10	4
10-20	5
20-30	7
30-40	10
40-50	12
50-60	8
60-70	4

(i) mean is

$$\bar{x} = \frac{5\times4+15\times5+25\times7+35\times10+45\times12+55\times8+65\times4}{4+5+7+10+12+8+4}$$

$$\bar{x} = 37.2$$

(ii) $\frac{N+1}{2} = \frac{50+1}{2} = 25.5$

Class	Frequency	Cumulative frequency
0-10	4	4
10-20	5	9
20-30	7	16
30-40	10	26 ← Median class
40-50	12	38
50-60	8	46
60-70	4	50

Where $l = 30$, $f = 10$, $m = 16$; $N = 50$ $c = 10$

The median $= \ell + \left\{\frac{\frac{N}{2}-m}{f}\right\}c$

$$= 30 + \left\{\frac{\frac{50}{2}-16}{10}\right\}10$$

$$= 39$$

(iii)

Class	Frequency
0-10	4
10-20	5
20-30	7
30-40	10
40-50	12 ← modal class
50-60	8
60-70	4

$l = 40$, $f = 12$, $f_{-1} = 10$, $f_1 = 8$

$\Delta_1 = f - f_{-1} = 2$, $\Delta_2 = f - f_1 = 4$

$$\text{Mode} = \ell + \left(\frac{\Delta_1}{\Delta_1+\Delta_2}\right)C$$

$$= 40 + \left(\frac{2}{2+4}\right)10$$

$$= 43.33$$

42.

Sol: The regression line of x and y is

$2x - y - 20 = 0$

$2x = y + 20$

$x = \frac{1}{2}y + 10$

The regression coefficient of x and y is

$b_{xy} = \frac{1}{2}$

The regression line of y on x is

$2y - x + 4 = 0$

$2y = x - 4$

$y = \frac{1}{2}x - 2$

The regression coefficient of y on x is

$$b_{yx} = \frac{1}{2}$$

(i) The correlation coefficient is

$$r = \sqrt{b_{yx}\ b_{xy}} = \sqrt{\frac{1}{4}}$$

$$r = \frac{1}{2}$$

(ii) Given $\sigma_y = \frac{1}{4}$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$\frac{1}{2} = \frac{1}{2}\frac{\frac{1}{4}}{\sigma_x}$$

$$\therefore\ \sigma_x = \frac{1}{4}$$

(iii) Both regression lines passing through $(\bar{x}, \bar{y})$, we have

$$2\bar{x} - \bar{y} - 20 = 0$$

$$2\bar{y} - \bar{x} + 4 = 0$$

By solving these two equations, we get

$$\bar{x} = 12 \text{ and } \bar{y} = 4$$

43. Ans: 0.18

Sol: Given: $b_{yx} = 1.6$ and $b_{xy} = 0.4$

$$r = \sqrt{b_{yx}\, b_{xy}}$$

$$r = \sqrt{1.6 \times 0.4}$$

$$r = 0.8$$

Now, $b_{yx} == r\frac{\sigma_y}{\sigma_x}$

$$1.6 = 0.8\frac{\sigma_y}{\sigma_x}$$

$$\frac{\sigma_y}{\sigma_x} = \frac{1.6}{1.8} = \frac{2}{1}$$

$\Rightarrow \sigma_x = 1$ and $\sigma_y = 2$

The angle between two regression lines is

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x\ \sigma_y}{\sigma_x^{\ 2}\ \sigma_y^{\ 2}}\right)$$

$$= \left\{\frac{1-(0.8)^2}{0.8}\right\}\left\{\frac{(1)\,(2)}{(1)^2 + (2)^2}\right\} = 0.18$$

44. Ans: (b)

Sol: Null Hypothesis H_0: The sample has been drawn from a population whit mean $\mu = 280$ days

Alternate Hypothesis H_1: The sample is not drawn from a population with mean $\mu = 280$ i.e. $\mu \neq 280$

Two-tailed test should be used.

Now the test statistic $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

$\mu = 280$, $\bar{x}$ = mean of the sample = 265

$\sigma = 30$, n = size of the sample = 400

$$Z = \frac{265 - 280}{\frac{30}{\sqrt{400}}} = -10$$

$\Rightarrow |Z| = 10$

$Z_\alpha = 1.96$

Since $|Z| = 10 > 1.96$, we reject null hypothesis

The sample is not drawn from population.

45. Ans: (c)

Sol: $H_0 : P = \frac{1}{5}$, i.e., 20% of the product manufactured is of top quality.

$H_1 : P \neq \frac{1}{5}$.

p = proportion of top quality products in the sample

$$= \frac{50}{400} = \frac{1}{8}$$

From the alternative hypothesis H_1, we note that two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$.

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{\frac{1}{8}-\frac{1}{5}}{\sqrt{\frac{1}{5}\times\frac{4}{5}\times\frac{1}{400}}}$$

Since the size of the sample is equal to 400.

i.e., $z = \frac{3}{40}\times 50 = -3.75$

Now $|z| = 3.75 > 1.96$.

The difference between p and P is significant at 5% level.

Also H_0 is rejected. Hence H_0 is wrong or the production of the particular day chosen is not a representative sample.

95% confidence limits for P are given by

$$\frac{|p-P|}{\sqrt{\frac{pq}{n}}} \le 1.96$$

Note:

We have taken $\sqrt{\frac{pq}{n}}$ in the denominator, because P is assumed to be unknown, for which we are trying to find the confidence limits and P is nearly equal to p.

i.e. $\left(p - \sqrt{\frac{pq}{n}}\times 1.96\right) \le P \le \left(p + \sqrt{\frac{pq}{n}}\times 1.96\right)$

i.e. $\left(0.125 - \sqrt{\frac{1}{8}\times\frac{7}{8}\times\frac{1}{400}}\times 1.96\right) \le P$

$\le \left(0.125 + \sqrt{\frac{1}{8}\times\frac{7}{8}\times\frac{1}{400}}\times 1.96\right)$

i.e. $0.093 \le P \le 0.157$

Therefore, 95% confidence limits for the percentage of top quality product are 9.3 and 15.7.

46. Ans: (d)

Sol: H_0: p = P, i.e. the hospital is not efficient.

H_1: p < P

One-tailed (left-tailed) test is to be used.

Let LOS be 1%.

Therefore, $z_\alpha = -2.33$.

$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$, where $p = \frac{63}{640} = 0.0984$

P = 0.1726, Q = 0.8274

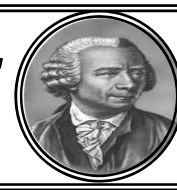
$$z = \frac{0.0984 - 0.1726}{\sqrt{\frac{0.1726\times 0.8274}{640}}} = -4.96$$

$\therefore |z| > |z_\alpha|$

Therefore, difference between p and P is significant. i.e., H_0 is rejected and H_1 is accepted.

That is, the hospital is efficient in bringing down the fatality rate of typhoid patients.

Chapter 4 Differential Equations
(With Laplace Transforms)



Leonhard Euler
(1707 – 1783)

01. Ans: (a)

Sol: Given $y\sqrt{1+x^2}\,dy + x\sqrt{1+y^2}\,dx = 0$

$$\Rightarrow \frac{y}{\sqrt{1+y^2}}dy + \frac{xdx}{\sqrt{1+x^2}} = 0$$

$$\Rightarrow \int \frac{y}{\sqrt{1+y^2}}dy + \int \frac{xdx}{\sqrt{1+x^2}} = c$$

$$\Rightarrow \frac{1}{2}\int \frac{2y}{\sqrt{1+y^2}}dy + \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx = c$$

$$\Rightarrow \frac{1}{2}(2\sqrt{1+y^2}) + \frac{1}{2}(2\sqrt{1+x^2}) = c$$

$$\left(\because \int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}\right)$$

$$\therefore \sqrt{1+y^2} + \sqrt{1+x^2} = c$$

is a required general solution

02. Ans: (a)

Sol: Given $\frac{dy}{dx} = (4x+y+1)^2$ ....(1)

Put t = 4x + y + 1 ……………..(2)

$$\Rightarrow \frac{dt}{dx} = 4 + \frac{dy}{dx} + 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4 \text{ …………..(3)}$$

Using (2) & (3) equation (1) becomes

$$\frac{dt}{dx} - 4 = t^2$$

$$\Rightarrow \frac{dt}{dx} = (4+t^2) \text{ ……….(1)}$$

$$\Rightarrow \int \frac{dt}{t^2+4} = \int dx + c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + c$$

$$\therefore \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c \text{ is a G.S of (1)}$$

03. Ans: (b)

Sol: Given that $\frac{dy}{dx} - x\tan(y-x) = 1$ .... (1)

Put y – x = t .... (2)

$$\Rightarrow \frac{dy}{dx} - 1 = \frac{dt}{dx} \text{ .... (3)}$$

Using (2) and (3), (1) becomes

$$\frac{dt}{dx} + 1 - x\ \tan t = 1$$

$$\Rightarrow \frac{dt}{dx} = x\ \tan\ t$$

$$\Rightarrow \int \cot t\ dt = \int x\ dx + c$$

$$\Rightarrow \log \sin t\ = \frac{x^2}{2} + c$$

$$\therefore\ \log[\sin(y-x)] = \frac{x^2}{2} + c$$

04. Ans: (c)

Sol: $\frac{dy}{dx} = \frac{y^2}{xy - x^2} = \frac{y^2/x^2}{y/x - 1}$ ………..(1)

Put y/x = v (or) y = vx …………(2)

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \text{ ……………..(3)}$$

Leonhard Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He made important discoveries in every branch of mathematic

Using (2) & (3), (1) becomes

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2}{v-1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1} = \frac{v}{v-1}$$

$$\Rightarrow \int dv - \int \frac{1}{v}dv = \log x + \log c$$

$$\Rightarrow v - \log v = \log x + \log c$$

$$\Rightarrow \frac{y}{x} - \log(y/x) = \log x + \log c$$

$$\Rightarrow \frac{y}{x} - \log y + \log x = \log x + \log c$$

$\therefore \frac{y}{x} = \log(cy)$ (or) $y = x\log(cy)$ is a GS of (1)

05. Ans: (d)

Sol: Given $\frac{dy}{dx} = \frac{y}{x} + \cot(y/x)$ …………(1)

Put $\frac{y}{x} = v$ or $y = vx$ …………..(2)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ………………(3)

Using (2) & (3), (1) becomes

$$v + x\frac{dv}{dx} = v + \cot(v)$$

$$\Rightarrow x\frac{dv}{dx} = \cot v$$

$$\Rightarrow \int \tan v = \int \frac{dx}{x} + \log c$$

$\Rightarrow$ log (sec v) = log x + log c

$\Rightarrow$ log (sec v) = log (xc)

$\therefore$ sec (y/x) = xc is a G.S of (1)

06. Ans: (a)

Sol: Given $x^2\frac{dy}{dx} = -(2xy - x + 1)$

$$\Rightarrow \frac{dy}{dx} = \frac{-2y}{x} + \frac{1}{x} - \frac{1}{x^2}$$

$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \left(\frac{1}{x} - \frac{1}{x^2}\right)$ …………..(1)

$$\left(\because \frac{dy}{dx} + P(x).Y = Q(x)\right)$$

If $e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$ then the G.S of (1) is

$$yx^2 = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)x^2 dx + c$$

$$\Rightarrow yx^2 = \int (x-1)dx + c$$

$$\Rightarrow yx^2 = \frac{x^2}{2} - x + c$$

$\therefore y = \frac{1}{2} - \frac{1}{x} + \frac{c}{x^2}$ is a G.S of (1) &

$y = \frac{1}{2} - \frac{1}{x} + \frac{2}{x^2}$ for $c = \frac{1}{2}$ is a particular solution of (1)

07. Ans: (c)

Sol: Given that

(siny–y sinxy)dx+(x cosy – x sinxy) dy = 0

$\Rightarrow$ (sin y) dx + x cos y dy – sin xy (y dx+xdy)= 0

$\Rightarrow$ d (x sin y) – sin(xy) d(xy) = 0

$\Rightarrow \int$ d (x sin y) – $\int$ sin(xy) d(xy) = c

$\therefore$ x sin y + cos(xy) = c is a required general solution

08. Ans: (c)

Sol: Given that $(x^3y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0$

$\Rightarrow (x^2y^2 - 1)\,y\,dx + (x^2y^2 + 1)\,x\,dy = 0$

Let $M = x^3y^2 + x$ and $N = x^2y^3 - y$

$$\text{I.F} = \frac{1}{Mx - Ny} = \frac{-1}{2xy}$$

Multiplying the given differential equation by I.F, we get

$$\left(-\frac{xy^2}{2} + \frac{1}{2x}\right)dx + \left(\frac{-x^2y}{2} - \frac{1}{2y}\right)dy = 0$$

Integrating

$$\left(\frac{-x^2y^2}{4}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\therefore \log\left(\frac{y}{x}\right) + \frac{x^2y^2}{2} = c$$

09. Ans: (a)

Sol: The given equation is

$(5x^3 + 3xy + 2y^2)dx + (x^2 + 2xy)dy = 0$

Let $M = 5x^3 + 3xy + 2y^2$ and $N = x^2 + 2xy$

$$\Rightarrow \frac{\partial M}{\partial y} = 3x + 4y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x + 2y$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{x + 2y}{x^2 + 2xy} = \frac{1}{x}$$

$$\therefore \text{I.F} = e^{\int \frac{1}{x}dx} = x$$

$\Rightarrow k = 1$

Multiplying the given Differential Equation by the integrating factor, we get

$(5x^4 + 3x^2y + 2xy^2)dx + (x^3 + 2x^2y)dy = 0$

which is exact

The solution is

$\int(5x^4 + 3x^2y + 2xy^2)dx = c$

$\therefore x^5 + x^3y + x^2y^2 = c$

10. Ans: (b)

Sol: Given that $\dfrac{dy}{dx} = \dfrac{2x}{(x^2 + y^2 - 2y)}$

$\Rightarrow 2x\,dx = (x^2 + y^2 - 2y)\,dy$

$\Rightarrow 2(x\,dx + y\,dy) = (x^2 + y^2)\,dy$

$$\Rightarrow \left(\frac{2x\,dx + 2y\,dy}{x^2 + y^2}\right) = dy$$

$\Rightarrow d(\log(x^2 + y^2)) = dy$

Integrating both sides, we get

$\log(x^2 + y^2) + c = y$

$\therefore y = \log(x^2 + y^2) + c$ is a general solution

11. Ans: (d)

Sol: Given

$r\sin\theta\,d\theta + (r^3 - 2r^2\cos\theta + \cos\theta)dr = 0$

Let $M = r\sin\theta$ & $N = r^3 - 2r^2\cos\theta + \cos\theta)$

$$\frac{\partial M}{\partial r} = \sin\theta \quad \text{and} \quad \frac{\partial N}{\partial \theta} = +2r^2\sin\theta - \sin\theta$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial \theta} - \frac{\partial M}{\partial r}\right) = 2\left(r - \frac{1}{r}\right)$$

Integrating factor is

$$\text{I.F} = e^{\int 2\left(r - \frac{1}{r}\right)dr} = \frac{e^{r^2}}{r^2}$$

12. Ans: (a)

Sol: Given $x\,dy + y\,dx - x^4y^5dy = 0$ ….(1)

$\Rightarrow (x\,dy + y\,dx) - (x^4y^4)\,.\,y\,dy = 0$

$$\Rightarrow \frac{xdy + ydx}{(xy)^4} - \frac{x^4y^5}{(xy)^4}dy = \frac{0}{(xy)^4}$$

$$\Rightarrow \frac{1}{(xy)^4}d(xy) - ydy = 0$$

$$\Rightarrow \int \frac{1}{(xy)^4}d(xy) - \int ydy = c$$

$$\therefore \frac{1}{-3(xy)^3} - \frac{y^2}{2} = c \text{ is a G.S of (1)}$$

13. Ans: (b)

Sol: Given that $\frac{x\,dy}{(x^2+y^2)} = \left(\frac{y}{x^2+y^2} - 1\right)dx$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = -dx$$

$$\Rightarrow \int d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \int -dx + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = -x + c$$

$\therefore$ $y = x \tan(c - x)$ is general solution

14. Ans: (c)

Sol: Given differential equation is

$$x(y\,dx + x\,dy)\cos\left(\frac{y}{x}\right) = y(x\,dy - y\,dx)\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow x\,d(xy)\cos\left(\frac{y}{x}\right) = y(x\,dy - y\,dx)\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow d(xy) = \left(\frac{y}{x}\right)(x\,dy - y\,dx)\tan\left(\frac{y}{x}\right)$$

Dividing both sides by 'xy', we get

$$\frac{d(xy)}{xy} = \left(\frac{x\,dy - y\,dx}{x^2}\right)\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \int \frac{d(xy)}{xy} = \int \tan\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) + l\text{n}c$$

$$\Rightarrow l\text{n}(xy) = l\text{n}\left[\sec\left(\frac{y}{x}\right)\right] + l\text{n}c$$

$$\Rightarrow l\text{n}(xy) = l\text{n}\left[c\sec\left(\frac{y}{x}\right)\right]$$

$$\Rightarrow xy = c\sec\left(\frac{y}{x}\right)$$

$\therefore$ $xy\cos\left(\frac{y}{x}\right) = c$ is a required solution of (1)

15. Ans: (d)

Sol: Given $\frac{dy}{dx} + \frac{y}{x} = \log x$ with $y(1) = 1$

I.F = $e^{\int \frac{1}{x}dx} = x$

The solution is

$xy = \int \log x.\, x\, dx$

$$\Rightarrow xy = \log x.\left(\frac{x^2}{2}\right) - \frac{x^2}{4} + c$$

$y(1) = 1 \quad \Rightarrow \quad c = \frac{5}{4}$

The solution is

$$y = \frac{x}{2}\log x - \frac{x}{4} + \frac{5}{4x}$$

16. Ans: (a)

Sol: Given $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$ .... (1)

$$\Rightarrow x^2\frac{dy}{dx} + 2xy = 3x^2 + 1$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = \frac{3x^2+1}{x^2}$$

I.F. = $e^{\int \frac{2}{x} dx} = x^2$

The solution is

$$y.x^2 = \int \left(\frac{3x^2+1}{x^2}\right).x^2 \ dx + c$$

$\therefore\ y = x + \frac{1}{x} + \frac{C}{x^2}$ is a G.S of (1)

17. Ans: (c)

Sol: Given that $(x + 2y^3)\left(\frac{dy}{dx}\right) = y$

$\Rightarrow y\ dx - x\ dy = 2y^3\ dy$

$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = 2y\,dy$

$\Rightarrow d\left(\frac{x}{y}\right) = 2y\ dy$

$\Rightarrow \int d\left(\frac{x}{y}\right) = \int 2y\,dy\ + c$

$\Rightarrow \frac{x}{y} = 2\frac{y^2}{2} + c$

$\therefore\ x = cy + y^3$ is a general solution

18. Ans: (d)

Sol: Given that $2xy^1 = (10x^3y^5 + y)$

$\Rightarrow \frac{dy}{dx} - \frac{y}{2x} = 5x^2y^5$

$\Rightarrow \frac{1}{y^5}\frac{dy}{dx} - \frac{y^{-4}}{2x} = 5x^2$ .... (1)

Put $y^{-4} = t$ .... (2)

$\Rightarrow -4\,y^{-5}\frac{dy}{dx} = \frac{dt}{dx}$ .... (3)

Using (2) and (3), (1) becomes

$$-\frac{1}{4}\frac{dt}{dx} - \frac{t}{2x} = 5x^2$$

$\Rightarrow \frac{dt}{dx} + 2\frac{t}{x} = -20x^2$

I.F = $e^{\int \frac{2}{x} dx} = x^2$

The solution is

$t.x^2 = \int -20x^2\,.x^2\ dx + c$

$\Rightarrow \frac{x^2}{y^4} = -20\frac{x^5}{5} + c$

$\therefore\ x^2 + (4x^5 + c)y^4 = 0$ is a general solution

19. Ans: (b)

Sol: Given $\tan y\frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$\sec y \tan y\ \frac{dy}{dx} + \sec y \tan x = \cos^2\ x$ ...... (1)

Put $\sec y = v$ .... (2)

$\Rightarrow \sec y \tan y\ \frac{dy}{dx} = \frac{dv}{dx}$ .... (3)

Using (2) and (3), (1) becomes

$$\frac{dv}{dx} + (\tan x)v = \cos^2\ x$$

I.F = $e^{\int \tan x\ dx} = \sec x$

The solution is

v. sec x = $\int \cos^2 x\ .\sec x\ dx + c$

$\therefore$ sec y = cos x(sin x + c) is a G.S

20. Ans: (a)

Sol: Given f(D)y = 0 ……….(1)

where $f(D) = D^2 + 2D - 5$

The auxiliary equation (A.E) is f(m) = 0

$\Rightarrow m^2 + 2m - 5 = 0$

$\Rightarrow m = -1 \pm \sqrt{6} = a \pm \sqrt{b}$

$\therefore$ The G.S of (1) is $y = y_c$

i.e. $y = C_1 e^{(-1+\sqrt{6})x} + C_2 e^{(-1-\sqrt{6})x}$

21. Ans: (d)

Sol: Given $f(D)\,y = 0$ …………(1)

where $f(D) = D^3 - 4D^2 + 5D - 2$

Consider A.E, $f(m) = 0$

$\Rightarrow m^3 - 4m^2 + 5m - 2 = 0$

$\Rightarrow m = 1, 1, 2$

$\therefore$ The G.S of (1) is

i.e. $y = C_1 e^{2x} + (C_2 + C_3 x)e^x$

22. Ans: (c)

Sol: Given $f(D)y = 0$ ……… (1)

where $f(D) = D^2 - 2D + 5$

Consider $f(m) = 0$

$\Rightarrow m^2 - 2m + 5 = 0$

$\Rightarrow m = 1 \pm 2i$

$\therefore$ The G.S of (1) is

$y = e^x\,[C_1 \cos(2x) + C_2 \sin(x)]$

23. Ans: (c)

Sol: The given equation is $(D^2 - 2D + 5)^2\,y = 0$

The auxiliary equation is $(D^2 - 2D + 5)^2 = 0$

$\Rightarrow D = 1 \pm 2i, \;\; 1 \pm 2i$

The solution is

$y = e^x\,[(C_1 + C_2 x)\cos 2x$

$+ (C_3 + C_4 x)\sin 2x]$

24. Ans: 0.541

Sol: Given $f(D)\,y = 0$ ………….(1),

where $f(D) = D^2 + 4D + 4$

with $y(0) = 1$ ………..(2)

& $y^1(0) = 1$ ………….(3)

Consider $f(m) = 0$

$\Rightarrow m^2 + 4m + 4 = 0$

$\Rightarrow m = -2, -2$

$\therefore$ The G.S of (1) is $y = (c_1 + c_2x)e^{-2x}$

$\Rightarrow y = c_1 e^{-2x} + c_2 xe^{-2x}$ …………(4)

$\Rightarrow y^1 = 2c_1 e^{-2x} + c_2(-2x)e^{-2x} + c_2e^{-2x}$ ….(5)

Using (2), (4) becomes

$1 = c_1 + 0$ (or) $c_1 = 1$ ………… (6)

$1 = -2 + 0 + c_2$ (or) $c_2 = 3$ ……….(7)

Using (6) & (7), (4) becomes

$y = y(x) = e^{-2x} + 3xe^{-2x}$

$\therefore y = y(1) = e^{-2} + 3e^{-2} = 4e^{-2} = 0.541$

25. Ans: –1

Sol: Given $f(D)y = 0$ where $= D^2 + 9$___ (1)

with $y(0) = 0$ _______ (2)

and $y\left(\frac{\pi}{2}\right) = \sqrt{2}$ _______ (3)

Consider A.E, $f(m) = 0$

$\Rightarrow \quad m^2 + 9 = 0$

$\Rightarrow \quad m = 0 \pm 3i = a \pm ib$

$\therefore$ The general solution of (1) is

$y = [C_1\cos(3x) + C_2\sin(3x)].e^{ox}$ .... (4)

Using (2), (4) be comes

$0 = C_1$ _____ (5)

Using (3) and (5), (4) becomes

$\sqrt{2} = 0 + C_2.\sin\left(\frac{3\pi}{2}\right)$

$\Rightarrow \quad C_2 = -\sqrt{2}$ ______ (6)

Using (5) & (6), (4) becomes

$y = y(x) = -\sqrt{2}.\sin(3x)$

$\therefore\ y = y\left(\frac{\pi}{4}\right) = -\sqrt{2}.\sin\left(\frac{3\pi}{4}\right) = -1$

26. Ans: (b)

Sol: If e^{-x} $(C_1 \cos \sqrt{3x} + C_2 \sin \sqrt{3x}) + C_3\ e^{2x}$ is the general solution then the roots of the auxiliary equation are $-1 \pm i\sqrt{3}$, 2.

The corresponding differential equation is

$(D-2)\,[D-(-1+i\sqrt{3}]\,[D-(-1-i\sqrt{3}]y = 0$

$\Rightarrow (D^3-8)y = 0$

$\therefore\ \frac{d^3y}{dx^3} - 8y = 0$

27. Ans: (b)

Sol: The roots of the auxiliary equation are $1,\ \pm 2i$.

The differential equation is

$(D-1)(D+2i)(D-2i)\,y = 0$

$\therefore\ y^{111} - y^{11} + 4y^{1} - 4y = 0$

28. Ans: (a)

Sol: Given $f(D)y = Q(x)$ ______ (1)

where $f(D) = D^2 - 7D + 9$ & $Q(x) = e^{4x}$

CF: Consider $f(m) = 0$

$\Rightarrow\quad m^2 - 7m + 9 = 0$

$\Rightarrow\quad m = \frac{7 \pm \sqrt{13}}{2}$

$\therefore$ The Complementary function is

$$y_c = e^{\frac{7x}{2}}\left[C_1 \cosh\left(\frac{\sqrt{13}}{2}\right)x + C_2 \sinh\left(\frac{\sqrt{13}}{2}\right)x\right]$$

PI: $Q(x) = e^{4x} = ke^{ax+b}$

$\Rightarrow a = 4$

Here, $f(D) = f(a) = f(4) = (4)^2 - 7(4) + 9 = -3$

$\therefore\ y_p = \frac{1}{f(a)}Q(x) = \frac{1}{-3}e^{4x}$

Hence, the general solution of (1) is

$y = y_c + y_p$

$$= e^{\frac{7x}{2}}\left[C_1 \cosh\left(\frac{\sqrt{13}}{2}\right)x + C_2 \sinh\left(\frac{\sqrt{13}}{2}\right)x\right] + \frac{1}{-3}e^{4x}$$

29. Ans: (a)

Sol: Given $f(D)y = Q(x)$

where $f(D) = D^3 - 11D^2 + 11D - 1$

& $Q(x) = 24 = 24.e^{ox+0} = ke^{ax+b}$

$\Rightarrow\quad a = 0$

Here, $f(D) = f(a) = f(0) = -1 \neq 0$

$\therefore\ y_p = \frac{1}{f(a)}Q(x) = \frac{1}{-1}(24)$

30. Ans: (a)

Sol: $P.I = \left(\frac{1}{4D^2 - 4D + 1}\right)e^{\frac{x}{2}}$

$= x.\left(\frac{1}{8D-4}\right)e^{\frac{x}{2}}$

(By Case of failure formula)

$= \frac{x^2}{8}e^{\frac{x}{2}}$ (Replacing D with $\frac{1}{2}$)

31. Ans: (d)

Sol: Particular Integral (P.I) $= \frac{1}{D^2+1}\cosh 3x$

$$= \frac{1}{D^2+1}\left(\frac{e^{3x}+e^{-3x}}{2}\right) = \frac{1}{2}\left[\frac{e^{3x}}{D^2+1} + \frac{e^{-3x}}{D^2-1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{3x}}{10} + \frac{e^{-3x}}{10}\right]$$

32. Ans: (d)

Sol: The given equation is $\frac{d^2y}{dx^2} = e^x$

$\Rightarrow \frac{dy}{dx} = e^x + C_1$

$\Rightarrow y = e^x + C_1x + C_2$ .... (i)

$\because y(0) = 1$

$\Rightarrow C_2 = 0$

$\because y^1(0) = 2$

$\Rightarrow C_1 = 1$

Substituting the values of C_1 & C_2 in (i), we get $y = e^x + x$

33. Ans: (a)

Sol: For the solution $y = C_1 \cos x + C_2 \sin x$ the corresponding roots of the auxiliary equation are $D = \pm i$.

The auxiliary equation is $(D + i)(D - i) = 0$

$\Rightarrow (D^2 + 1) = 0$

The differential equation is $\frac{d^2y}{dx^2} + y = 0$

Comparing above equation with the given equation $\frac{d^2y}{d^2x} + P\frac{dy}{dx} + Q\,y = 0$, we have $P = 0$ and $Q = 1$

Now, the equation $\frac{d^2y}{d^2x} + P\frac{dy}{dx} + (Q-1)y = e^x$ becomes $\frac{d^2y}{d^2x} = e^x$

$\Rightarrow \frac{dy}{dx} = e^x + C_1$

$\therefore y = e^x + C_1 x + C_2$

34. Ans: (c)

Sol: The auxiliary equation is $D^2 + 1 = 0$

$\Rightarrow D = \pm i$

Complementary function (C.F)

$= C_1 \cos x + C_2 \sin x$

$$P.I = \frac{1}{D^2+1}\sin x$$

$$= x.\frac{1}{2D}\sin x = \frac{-x}{2}\cos x$$

The solution is

$y = C_1 \cos x + C_2 \sin x - \frac{x}{2}\cos x$ ...... (i)

$y\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow C_2 = 0$

$y(0) = 1 \quad \Rightarrow C_1 = 1$

substituting the values of C_1 & C_2 in (i),

we get $y = \cos x - \frac{x}{2}\cos x$

35. Ans: (b)

Sol: Given $f(D)y = Q(x)$ ________ (1)

where $f(D) = D^3 + 1 = D.D^2 + 1 = \phi(D^2)$

& $Q(x) = \cos(2x) = k.\cos(ax + b)$

$\Rightarrow a = 2$

Here, $f(D) = \phi(D^2) = \phi(-a^2) = \phi(-4)$

$= 1 - 4D \neq 0$

Now, $y_p = \frac{1}{\phi(-a^2)}Q(x)$

$\Rightarrow y_p = \frac{1}{1-4D}\cos(2x)$

$$\Rightarrow \quad y_p = \frac{1}{1-4D} \times \frac{1+4D}{1+4D}\cos(2x)$$

$$\Rightarrow \quad y_p = \frac{1+4D}{1-16D^2}\cos(2x)$$

$$\Rightarrow \quad y_p = \frac{1+4D}{1}\left[\frac{1}{1-16D^2}\cos(2x)\right]$$

$$\Rightarrow \quad y_p = (1+4D)\left[\frac{1}{1-16(-4)}\cos(2x)\right]$$

$\therefore\ y_p = \frac{1}{65}\cos(2x) - \frac{8}{65}\sin(2x)$ is a general solution of (1)

36. Ans: (c)

Sol: P.I $= \left(\frac{1}{D^4+1}\right)x^5 = (1 + D^4)^{-1}\ x^5$

$= (1 - D^4 + D^8 - \ldots\ldots\ldots)x^5$

$= x^5 - D^4x^4 = x^5 - 120x$

37. Ans: (a)

Sol: Given $f(D)y = Q(t)$……..(1)

where $f(D) = D^2 - 4D + 3$ & $Q(t) = 2t - 3t^2$

Now, $y_P = \frac{1}{f(D)}Q(t)$

$$\Rightarrow \quad y_P = \frac{1}{(D^2-4D+3)}\left(2t-3t^2\right)$$

$$\Rightarrow y_P = \frac{1}{3\left[1+\left(\frac{D^2-4D}{3}\right)\right]}(2t-3t^2)$$

$$\Rightarrow y_P = \frac{1}{3}\left[1+\left(\frac{D^2}{3}-\frac{4D}{3}\right)\right]^{-1}(2t-3t^2)$$

$$\Rightarrow y_P = \frac{1}{3}\left[1-\left(\frac{D^2}{3}-\frac{4D}{3}\right)+\left(\frac{D^2}{3}-\frac{4D}{3}\right)^2\ldots\ldots\right](2t-3t^2)$$

Collect the terms upto D^2 and operate.

$$\Rightarrow y_P = \frac{1}{3}\left[1-\frac{D^2}{3}+\frac{4D}{3}+\frac{16D^2}{9}\right](2t-3t^2)$$

$$= \frac{1}{3}\left[1+\frac{4}{3}D+\frac{13}{9}D^2\right]\left(2t-3t^2\right)$$

$$= \frac{1}{3}\left[2t-3t^2+\frac{4}{3}(2-6t)+\frac{13}{9}(-6)\right]$$

$$= \frac{1}{3}\left[-3t^2+-6t-6\right]$$

$\therefore\ y_P = -t^2 - 2t - 2$

38. Ans: (c)

Sol: Given $f(D)y = Q(x)$……(1)

where $f(D) = D^2 + 4D + 10$

& $Q(x) = e^{-2x}\cos(4x) = e^{cx}\ .\ V(x)$

Now, $y_P = \frac{1}{f(D)}\left[e^{-2x}.\cos(4x)\right]$

$$\Rightarrow y_P = e^{-2x}\left[\frac{1}{f(D-2)}\cos(4x)\right]$$

$$\Rightarrow y_P = e^{-2x}\left[\frac{1}{(D-2)^2+4(D-2)+10}\cos(4x)\right]$$

$$\Rightarrow y_P = e^{-2x}\left[\frac{1}{D^2+6}\cos(4x)\right]$$

$$\Rightarrow y_P = e^{-2x}\left[\frac{1}{-16+6}\cos(4x)\right]$$

$$\therefore Y_P = e^{-2x}\left[\frac{1}{-10}\cos(4x)\right]$$

39. Ans: (a)

Sol: Given $f(D)y = Q(x)$ …………(1)

where $f(D) = D^2 + 4D + 4$

& $Q(x) = x^4\ e^{-2x} = e^{-2x}\ .x^4 = e^x\ .\ V(x)$

Now, $y_P = \frac{1}{f(D)}\left[e^{-2x}\,x^4\right]$

$$\Rightarrow y_P = e^{-2x}\left[\frac{1}{f(D-2)}x^4\right]$$

$$\Rightarrow y_P = e^{-2x}\left[\frac{1}{(D-2)^2+4(D-2)+4}x^4\right]$$

$$\Rightarrow y_P = e^{-2x}\left[\frac{1(x^4)}{D^2}\right]$$

$$\therefore y_p = e^{-2x}.\frac{x^6}{30}$$

40. Ans: (c)

Sol: Given f(D)y = Q(x) ……… (1)

where $f(D) = D^2 + 1$

& Q(x) = x. sin(3x) = x. V(x)

Now, $y_P = \frac{1}{f(D)}[x.\sin(3x)]$

$$\Rightarrow y_P = x.\left[\frac{1}{f(D)}\sin(3x)\right]-\left[\frac{f^1(D)}{(f(D))^2}\sin(3x)\right]$$

$$\Rightarrow y_P = x.\left[\frac{1}{D^2+1}\sin(3x)\right]-\left[\frac{2D}{(1+D^2)^2}\sin(3x)\right]$$

$$\Rightarrow y_P = x.\left[\frac{1}{-9+1}\sin(3x)\right]-\left[\frac{2D}{1}\left(\frac{1}{(1+D^2)^2}\sin(3x)\right)\right]$$

$$\Rightarrow y_P = x.\left[\frac{1}{-8}\sin(3x)\right]-\left[\frac{2D}{1}\left\{\frac{1}{(1-9)^2}\sin(3x)\right\}\right]$$

$$\Rightarrow y_P = \frac{-x}{8}\sin(3x)-\left[\frac{6}{64}\cos(3x)\right]$$

$$\therefore y_P = \frac{-x}{8}\sin(3x)-\frac{3}{32}\cos(3x)$$

41. Ans: (b)

Sol: The given equation is

$$\left(\frac{d^2y}{dx^2}\right)-4\left(\frac{dy}{dx}\right)+4y = \frac{e^{2x}}{x}$$

The auxiliary equation is

$(D-2)^2 = 0 \Rightarrow D = 2, 2$

C.F = $(C_1 + C_2x)e^{2x}$

$= C_1e^{2x} + C_2\,xe^{2x}$

$= C_1\,y_1 + C_2y_2$

where, $C_1 = e^{2x}$ & $C_2 = xe^{2x}$

P.I = $A.y_1 + B.y_2$ .... (i)

where, $A = -\int \frac{Py_2}{W}dx$

where, $W = y_1.y_2' - y_2.y_1' = e^{4x}$

$$A = -\int \frac{e^{2x}}{x}.\frac{x\,e^{2x}}{e^{4x}}\,dx = -x$$

$$B = \int \frac{Py_1}{W}dx = \int \frac{e^{2x}}{x}.\frac{e^{2x}}{e^{4x}}\,dx = \log x$$

Substituting the values of A & B in (i)

P.I = $-xe^{2x} + xe^{2x}\log x$

The solution is

$y = (C_1 + C_2x + x\log x - x)\,e^{2x}$

42. Ans: (b)

Sol: The given equation is

$$y'' + 2y' + y = e^{-x}\log x$$

The auxiliary equation is

$(D+1)^2 = 0 \quad \Rightarrow D = -1, -1$

C.F = $(C_1 + C_2x)e^{-x}$

$= C_1e^{-x} + C_2\,xe^{-x}$

$= C_1\,y_1 + C_2y_2$

where, $C_1 = e^{-x}$ & $C_2 = xe^{-x}$

P.I = $A.y_1 + B.y_2$ .... (i)

where, $A = -\int \frac{Py_2}{W} dx$

where, $W = y_1.y_2' - y_2.y_1' = e^{-2x}$

$\Rightarrow A = -\int \frac{e^{-x} \log x .\, xe^{-x}}{e^{-2x}} dx$

$\therefore A = \frac{x^2}{4}(1 - 2 \log x)$

Now, $B = \int \frac{Py_1}{W} dx = \int \frac{e^{-x} \log x .\, e^{-x}}{e^{-2x}} dx$

$\therefore B = x(\log x - 1)$

43. Ans: (c)

Sol: The given equation is

$$x^2y^{11} + 6xy^1 + 6y = x$$

Let $x = e^t$ and $D_1 = \frac{d}{dt}$

The given equation becomes

$D_1(D_1 - 1) y + 6D_1y + 6y = e^t$

$\Rightarrow (D_1^2 + 5D_1 + 6)y = e^t$

The auxiliary equation is $D_1^2 + 5D_1 + 6 = 0$

$\Rightarrow D_1 = -2, -3$

C.F $= C_1 e^{-2t} + C_2 e^{-3t}$

P.I $= \frac{1}{(D_1^2 + 5D_1 + 6)} e^t = \frac{e^t}{12}$

The solution is

$$y = C_1 e^{-2t} + C_2 e^{-3t} + \frac{e^t}{12}$$

$\therefore y = \frac{C_1}{x^2} + \frac{C_2}{x^3} + \frac{x}{12}$

44. Ans: (c)

Sol: Given $x^2D^2 + xD - y = 0$ ………..(1)

where $D = \frac{d}{dx}$

$\left.\begin{array}{l}\text{Let } x = e^z \text{ (or) } \log x = z \\ \& \ xD = \theta,\ x^2 D^2 = \theta(\theta - 1)\end{array}\right\} \dashrightarrow 2$

where $\theta = \frac{d}{dz}$

Using equation (2), (1) becomes

$[\theta(\theta-1) + \theta - 1]\, y = 0$

$\Rightarrow [\theta^2 - \theta + \theta - 1]\, y = 0$

$\Rightarrow (\theta^2 - 1)\, y = 0$

$\Rightarrow f(\theta)\, y = 0$ ………. (3)

where $f(\theta) = \theta^2 - 1$

Consider A.E, $f(m) = 0$

$\Rightarrow m^2 - 1, -1$

$\Rightarrow m = 1, -1$

$\Rightarrow y_c = c_1 e^z + c_2 e^{-z}$

$\Rightarrow y_c = c_1 x + c_2 \frac{1}{x}$

$\therefore$ The general solution of (1) is

$$y = y_c = c_1 x + c_2 \frac{1}{x}$$

45.

Sol: The given equation is

$$z = ax + by + a^2 + b^2 \text{ .... (i)}$$

$\Rightarrow \quad \frac{\partial z}{\partial x} = p = a$ .... (ii)

and $\quad \frac{\partial z}{\partial y} = q = b$ .... (iii)

substituting the values of a & b from (ii) & (iii) in (i), we get

$$z = px + qy + p^2 + q^2$$

46.

Sol: The given équation is

$$z = xy + y\sqrt{x^2 - a^2 + b^2}$$

$$\Rightarrow\ p = y + y\frac{2x}{2\sqrt{x^2 - a^2 + b^2}} \text{ ...... (i)}$$

and $q = x + \sqrt{x^2 - a^2 + b^2}$

(or) $\sqrt{x^2 - a^2 + b^2} = q - x$ .... (ii)

From (i) & (ii), we get

$$p = y + \frac{xy}{q - x}$$

$\therefore$ $px + qy = pq$ is the required partial differential equation.

47.

Sol: The given equation is

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

$$\Rightarrow\ p = 2f^1\left(\frac{1}{x} + \log y\right).\left(-\frac{1}{x^2}\right) \text{ .... (i)}$$

and $q = 2y + 2f^1\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right)$

$$\Rightarrow q - 2y = 2f^1\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right) \text{ ....(ii)}$$

Dividing (i) by (ii), we get

$$\frac{p}{q - 2y} = -\frac{y}{x^2}$$

$\therefore\ px^2 + qy = 2y^2$

48.

Sol: The given equation can be written as

$z - xy = \phi(x^2 + y^2)$ .... (1)

Differentiating (1) partially with respect to x

$p - y = \phi^1(x^2 + y^2).2x$ .... (2)

Differentiating (2) partially with respect to y

$q - x = \phi^1(x^2 + y^2).2y$ .... (3)

Dividing (2) by (3)

$$\frac{p - y}{q - x} = \frac{x}{y}$$

$\therefore\ qx - py = x^2 - y^2$

49. Ans: (a)

Sol: The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

The general form of 2^{nd} order linear partial differential equation is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}$$

$$+ f\left(x, y, z\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \text{ ....(1)}$$

Equation (1) is said to be

(i) Parabolic if $B^2 - 4AC = 0$

(ii) Elliptic if $B^2 - 4AC < 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Here, A = 1, B = 0 & C = 1

$B^2 - 4AC = -4 < 0$

$\therefore$ The given differntial equation is Elliptic

50.

Sol: The given equation is

$p - q = \log(x + y)$

$\therefore$ The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x + y)}$$

Consider $\dfrac{dx}{1} = \dfrac{dy}{-1}$

$\therefore$ x + y = C

Consider $\dfrac{dx}{1} = \dfrac{dz}{\log(x+y)}$

$\Rightarrow dx = \dfrac{1}{\log C} dz$

$\Rightarrow x = \dfrac{z}{\log C} + C_1$

$\Rightarrow x - \dfrac{z}{\log x + y} = C_1$

$\therefore$ The solution is

$$\phi\left[x + y, x - \frac{z}{\log(x+y)}\right] = 0$$

51.

Sol: The auxiliary equations are

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \text{ .... (i)}$$

Using the multipliers 1, 1, 1 each of the fractions in (i) $= \dfrac{dx+dy+dz}{0}$

$\Rightarrow dx + dy + dz = 0$

$\Rightarrow x + y + z = C_1$ .... (ii)

Using the multipliers x, y, z each of the fractions in (i)

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow x^2 + y^2 + z^2 = C_2$ .... (iii)

$\therefore$ The solution is

$$f(x + y + z, x^2 + y^2 + z^2) = 0$$

52.

Sol: The given equation is

$q = 3p^2$ (Type-I)

Let the solution be

$z = ax + by + c$ .... (1)

$\Rightarrow p = a$ and $q = b$

Substituting in the equation Type-I, we have

$b = 3a^2$ ....(2)

Eliminating b from (1) & (2)

The solution is $z = ax + 3a^2y + c$

53.

Sol: Given $p^2 z^2 + q^2 = 1$ ………..(1)

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$

Let $q = ap$ ……… (2)

$p^2 z^2 + a^2 p^2 = 1$

$\Rightarrow p^2 (z^2 + a^2) = 1$

$\Rightarrow p = \pm \dfrac{1}{\sqrt{a^2 + z^2}}$

$\Rightarrow q = ap = \pm \dfrac{a}{\sqrt{a^2 + z^2}}$

Consider $dz = p\,dx + q\,dy$

$\Rightarrow dz = \dfrac{\pm 1}{\sqrt{a^2+z^2}} dx + \dfrac{\pm a}{\sqrt{a^2+z^2}} dy$

$\Rightarrow \int \sqrt{a^2+z^2}\, dz = \int \pm\, dx \pm \int a\; dy + c$

$\therefore \dfrac{z}{2}\sqrt{a^2+z^2} + \dfrac{a^2}{2}\sinh^{-1}(z/a) = \pm(x + ay) + c$

is a required solution.

54.

Sol: The given equation is

$$p^2 + q^2 = x + y \quad \text{(Type-III)}$$

$$\Rightarrow p^2 - x = y - q^2 = a \quad \text{(say)}$$

$$\Rightarrow p = \sqrt{a+x} \quad \text{and} \quad q = \sqrt{y-a}$$

Consider dz = p dx + q dy

$$\Rightarrow dz = \sqrt{a+x}\ dx + \sqrt{y-a}\ dy$$

Intégrating,

$$z = \left(\frac{2}{3}\right)(a+x)^{3/2} + \left(\frac{2}{3}\right)(y-a)^{3/2} + b$$

55.

Sol: The given equation can be written as

$$z = px + qy + \frac{1}{p-q} \quad \text{(Type-IV)}$$

The solution is

$$z = ax + by + \frac{1}{(a-b)} \quad \text{for } p = a \ \&\ q = b$$

56.

Sol: The given équation is

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \ .... \text{(i)}$$

Let u = X(x).Y(y)

$$\frac{\partial u}{\partial x} = X^1Y \quad \text{and} \quad \frac{\partial u}{\partial y} = XY^1$$

Substituting in equation (i)

$$X^1Y = 4XY^1$$

$$\frac{X'}{X} = \frac{4Y'}{Y} = k$$

$$\frac{X'}{X} = k \quad \text{and} \quad \frac{4Y'}{Y} = k$$

$$\Rightarrow X = C_1 e^{kx} \quad \text{and} \quad Y = C_2 e^{\frac{k}{4}y}$$

Now, the solution is,

$$u = C_1C_2 e^{kx} e^{\frac{k}{4}y}$$

$$u = C_3 e^{kx} e^{\frac{k}{4}y} \ .... \text{(ii)}$$

Given $u(0, y) = 8e^{-3y}$

$$\Rightarrow 8e^{-3y} = u(0,y) = C_3 e^{\frac{k}{4}y}$$

$$\Rightarrow C_3 = 8,\ k = -12$$

$$\therefore u = 8e^{-12x-3y}$$

57.

Sol: The given equation is $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$ ..(1)

Let u = X(x).Y(y)

Then $\dfrac{\partial u}{\partial x} = X^1Y$ and $\dfrac{\partial u}{\partial y} = XY^1$

Substituting in equation (1), we get

$$3X^1Y + 2XY^1 = 0$$

$$\Rightarrow \frac{3X'}{X} = \frac{-2Y'}{Y} = k$$

$$\Rightarrow \frac{3X'}{X} = k \quad \text{and} \quad \frac{-2Y'}{Y} = k$$

$$\Rightarrow X = C_1 e^{\frac{k}{3}x} \quad \text{and} \quad Y = C_2 e^{\frac{-k}{2}y}$$

Now, the solution is,

$$u = C_1 e^{\frac{k}{3}x} C_2 e^{\frac{-k}{2}y}$$

$$u = C_3 e^{\frac{k}{3}x} e^{-\frac{k}{2}y}$$

Given that $u(x,0) = 4e^{-x}$

$\Rightarrow 4e^{-x} = C_3\ e^{\frac{k}{3}x}$

$\Rightarrow C_3 = 4$ and $k = -3$

$\therefore\ u = 4e^{\frac{1}{2}(-2x+3y)}$ is a solution of (1)

58. Ans: (b)

Sol: The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

The general form of 2^{nd} order linear partial differential equation is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, z\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \text{ ....(1)}$$

Equation (1) is said to be

(i) Parabolic if $B^2 - 4AC = 0$

(ii) Elliptic if $B^2 - 4AC < 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Here, $A = C^2$, $B = 0$, $C = 0$

$B^2 - 4AC = 0$

$\therefore$ The equation is parabolic

59. Ans: (b)

Sol: The given equation is $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

Let $u = X(x).T(t)$

$\frac{\partial^2 u}{\partial x^2} = X^{11}T$ and $\frac{\partial u}{\partial t} = XT^1$

Substituting in equation (i)

$X^{11}T = \alpha\ XT^1$

$$\frac{X''}{X} = \frac{\alpha T'}{T} = k$$

$$\frac{X''}{X} = \frac{k}{\alpha} \quad \text{and} \quad \frac{T'}{T} = k$$

$\Rightarrow T = C_1\ e^{kt}$

$$X = C_2\ e^{x\sqrt{\frac{k}{\alpha}}} + C_3\ e^{-x\sqrt{\frac{k}{\alpha}}}$$

The solution is

$$u = C_1 e^{kt}\left[C_2 e^{\left(\sqrt{\frac{k}{\alpha}}\right)x} + C_3 e^{-\left(\sqrt{\frac{k}{\alpha}}\right)x}\right]$$

60. Ans: (d)

Sol: The equation given in option (d) represents one dimensional wave equation.

61. Ans: (d)

Sol: Given $\frac{\partial^2 u}{\partial x^2} = 25\frac{\partial^2 u}{\partial t^2}$

(or) $\frac{\partial^2 u}{\partial t^2} = \frac{1}{25}\frac{\partial^2 u}{\partial x^2}$ .... (1)

with $u(0) = 3x$ .... (2)

and $\frac{\partial u(0)}{\partial t} = 3$ .... (3)

If the given one dimensional wave equation is of the form $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial u^2}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$ and $c > 0$, satisfying the conditions $u(x, 0) = f(x)$ and $\frac{\partial u(x,0)}{\partial t} = g(x)$, where $f(x)$ & $g(x)$ are given functions representing the initial displacement and initial velocity, respectively then its general solution is given by

$$u(x, t) = \frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds$$

Comparing the given problem with above general problem, we have

$c = \frac{1}{5}$, f(x) = 3x, g(x) = 3

Now,

$$u(1, 1) = \frac{1}{2}\left[f\left(1-\frac{1}{5}\right)+f\left(1+\frac{1}{5}\right)\right]+\frac{1}{2\left(\frac{1}{5}\right)}\int_{1-\frac{1}{5}}^{1+\frac{1}{5}} 3\ ds$$

$$\Rightarrow u(1, 1) = \frac{1}{2}\left[3\left(\frac{4}{5}\right)+3\left(\frac{6}{5}\right)\right]+\frac{5}{2}(3)\ (s)_{\frac{4}{5}}^{\frac{6}{5}}$$

$$\Rightarrow u(1, 1) = \frac{1}{2}\left[\frac{3}{5}\times(4+6)\right]+\frac{15}{2}\left[\frac{6}{5}-\frac{4}{5}\right]$$

$$\Rightarrow u(1, 1) = 3+\frac{15}{2}\left(\frac{2}{5}\right)$$

$\therefore$ u(1, 1) = 6

62. Ans: (a)

Sol: Given $u_{tt} = u_{xx}$ ……..(1) $(\because u_{tt} = c^2 u_{xx})$

with B.C's

$$\left.\begin{aligned} u(0,t)&=0 \\ u(\pi,t)&=0 \end{aligned}\right\} \text{...... (2)} \qquad \left[\begin{aligned} \because u(0,t)&=0 \\ u(\ell,t)&=0 \end{aligned}\right]$$

and I.C's

$$\left.\begin{aligned} u(x,0) &= 2\sin(x) \\ \frac{\partial}{\partial t}u(x,0)&=0 \end{aligned}\right\} \text{.......(3)} \qquad \left[\begin{aligned} \because u(x,0)&=f(x) \\ \frac{\partial u}{\partial t}(x,0)&=0 \end{aligned}\right]$$

Now, the solution of the wave equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right).\cos\left(\frac{n\pi ct}{\ell}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx).\cos(nt) \text{ ……(4)}$$

$(\because c = 1, l = \pi)$

$$\Rightarrow u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{for } t = 0$$

$$\Rightarrow 2\sin(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$(\because u(x, 0) = 2 \sin x)$

$\Rightarrow 2\sin(x) = a_1 .\sin x + a_2 \sin(2x) + \ldots\ldots\ldots$

$\Rightarrow a_1 = 2, a_2 = 0. a_3 = 0$ ……….(5)

Using (5), (4) becomes

$u(x, t) = a_1.\sin(x) \cos(t) + a_2 .\sin(2x) .\cos(2t)$

$\therefore$ u(x, t) = 2.sin(x) cos(t) + 0 + ….

$\Rightarrow$ g(t).sin(x) = 2 sin(x).cos(t)

$\Rightarrow$ g(t) = 2.cos(t)

$\therefore$ g(π/3) = 2.cos(π/3) = 1

63. Ans: (b)

Sol: Given $u_{tt} = 2^2 u_{xx} \to (1)$ $(\because u_{tt} = c^2 u_{xx})$

with B.C's

$$\left.\begin{aligned} u(0,t)&=0 \\ u(\pi,t)&=0 \end{aligned}\right\} \to (2) \qquad \left[\begin{aligned} \because u(0,t)&=0 \\ u(\ell,t)&=0 \end{aligned}\right]$$

and I.C's

$$\left.\begin{aligned} u(x,0)&=0 \\ \frac{\partial}{\partial t}u(x,0) &= 2\sin(x) \end{aligned}\right\} \text{........(3)} \qquad \left[\begin{aligned} \because u(x,0)&=0 \\ \frac{\partial u}{\partial t}(x,0)&=g(x) \end{aligned}\right]$$

The solution of (1) is given by

$$u(x,y) = \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{\ell}\right).\sin\left(\frac{n\pi ct}{\ell}\right) \to (4)$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty} b_n \sin(nx).\sin(n2t)$$

$(\because l=\pi, C=2)$

$$\Rightarrow \frac{\partial}{\partial t}u(x,t)=\sum_{n=1}^{\infty} b_n \sin(nx).\cos(2nt).2n$$

$$\Rightarrow \frac{\partial}{\partial t}u(x,0)=\sum_{n=1}^{\infty} b_n \sin(nx).2n \quad \text{for } t=0$$

$\Rightarrow \sin(x) = b_1\sin(x).\ 2 + b_2.\ 2(2).\ \sin(2x)$ ……

$\Rightarrow 2b_1 = 1$

$\Rightarrow b_1 = \dfrac{1}{2}$

$\therefore\ u(x, t) = b_1.\sin(x)\ .\ \sin(2t) + 0 + 0$…….

$$=\frac{1}{2}\sin(x).\sin(2t):$$

Hence, $u\left(\dfrac{\pi}{3},\dfrac{\pi}{6}\right)=\dfrac{1}{2}\sin\left(\dfrac{\pi}{3}\right).\sin\left(\dfrac{2\pi}{6}\right)$

$$=\frac{1}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}=\frac{3}{8}$$

64. Ans: (c)

Sol: Given $u_t = \left(\sqrt{2}\right)^2 u_{xx}$ ………………..(1)

$(\because u_t = c^2\ u_{xx})$

with B.C's

$$\left.\begin{aligned} u(0,t)&=0\\ u(\pi,t)&=0\end{aligned}\right\} \ldots\ldots (2) \quad \left[\begin{aligned}\because u(0,t)&=0\\ u(\ell,t)&=0\end{aligned}\right]$$

and I.C

$u(x, 0) = \sin(x)+2\sin(4x)$…..(3)

$(\because u(x, 0) = f(x))$

The solution of (1) is

$$u(x,t)=\sum_{n=1}^{\infty} a_n.\sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n\pi c}{\ell}\right)^2 t} \ldots\ldots(4)$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty} a_n.\sin(nx)\ e^{-2n^2 t} \quad \text{for } \ell=\pi$$

$$\Rightarrow u(x,0)=\sum_{n=1}^{\infty} a_n.\sin(nx) \qquad (\text{for } t=0)$$

$\Rightarrow \sin(x) + 2\sin(4x)$

$= a_1\sin x+a_2\sin(2x)+a_3\sin(3x)+a_4\sin(4x)+$…..

$\Rightarrow a_1 = 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = 2,\ a_5 = 0,$ ….

$\therefore$ The solution of (1) from (4), using (2) & (3)

$u(x, t) = a_1\sin(x).\ e^{-2t} + a_4\sin(4x)\ e^{-2(4)^2 t}$

Hence,

$u(\pi/2, \log 5) = 1.\ \sin(\dfrac{\pi}{2}).\ e^{-2\log 5}$

$+ 2.\ \sin(4.\ \dfrac{\pi}{2}).\ e^{-32}\log 5$

$= 5^{-2} = 0.04$

65. Ans: (d)

Sol: Given $u_t = c^2\ u_{xx}$……………(1)

with B.C's

$$\left.\begin{aligned} u(0,t)&=0\\ u(\pi,t)&=0\end{aligned}\right\} \ldots\ldots\ldots\ldots(2)$$

& I. C $u(x, 0) = \sin x = f(x)$ …..(3)

Now, the solution of (1) is given by

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{\ell} e^{\left(\frac{-n^2\pi^2c^2}{\ell^2}\right)t} \ldots\ldots(4)$$

$$\text{where } a_n=\frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx \ldots\ldots(5)$$

put t = 0 in (4), we get

$$u(x,0)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$\Rightarrow \sin(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)$

$= a_1\sin(x) + a_2 \sin(2x) +$

$\Rightarrow \sin(x) = a_1\sin x + ...\qquad (\ell = \pi,\ n = 1)$

$\therefore a_1 = 1,\ a_2 = 0\ldots.\ a_3 = 0\ \ldots.$

From (4), the solution of (1) with (2) & (3) is

$u(x, t) = a_1 \sin(x).\ e^{-c^2t}$

$\therefore u(x, t) = \sin(x).\ e^{-c^2t}$

66. Ans: (a)

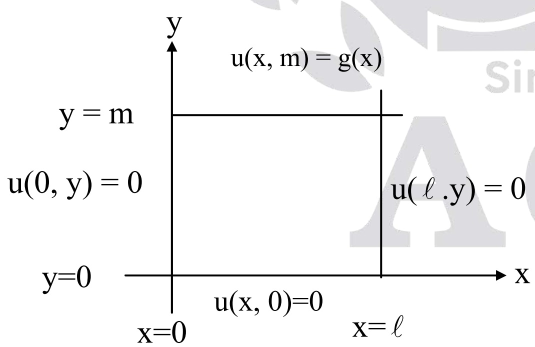
Sol: Given $u_{xx} + u_{yy} = 0$ …..(1)

with B.C's

$$\left.\begin{aligned} u(0,y) &= 0 \\ u(\ell,y) &= 0 \end{aligned}\right\}\quad 0 \le y \le m$$

$$\left.\begin{aligned} u(x,0) &= 0 \\ u(x,a) &= \sin\frac{n\pi x}{\ell} \end{aligned}\right\}\quad 0 \le x \le \ell$$

The solution of (1) is given by



$$u(x, y) = \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{\ell}\right).\sinh\left(\frac{n\pi y}{\ell}\right)$$

where

$$b_n = \frac{2}{\ell \sinh\left(\frac{n\pi m}{\ell}\right)} \int_0^{\ell} g(x).\sin\left(\frac{n\pi x}{\ell}\right) dx$$

Now,

$$b_n = \frac{2}{\ell.\sinh\left(\frac{n\pi m}{\ell}\right)} \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right).\sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell.\sinh\frac{n\pi m}{\ell}} \int_0^{\ell} \frac{1+\cos\frac{2n\pi x}{\ell}}{2}\, dx$$

$$= \frac{2}{\ell.\sinh\left(\frac{n\pi m}{\ell}\right)} \left[\frac{x}{2} + \frac{1}{2}\frac{\sin\left(\frac{2n\pi x}{\ell}\right)}{\frac{2n\pi}{\ell}}\right]_0^{\ell}$$

$$= \frac{1}{\ell.\sin h\left(\frac{n\pi m}{\ell}\right)} \left[(\ell + 0) - (0 + 0)\right]$$

$$b_n = \frac{1}{\sinh\left(\frac{n\pi m}{\ell}\right)} = \frac{1}{\sinh\frac{n\pi a}{\ell}} \qquad \text{for } m = a$$

$\therefore$ The solution of (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{1}{\sinh\left(\frac{n\pi a}{\ell}\right)}.\sin\left(\frac{n\pi x}{\ell}\right).\sinh\frac{n\pi y}{\ell}$$

67. Ans: (a)

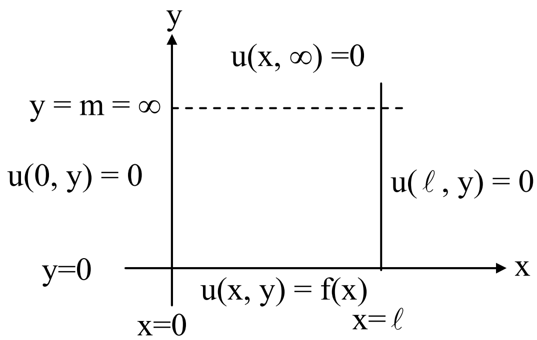
Sol: Given $u_{xx} + u_{yy} = 0 \rightarrow (1)$

$u(0, y) = 0 \qquad \rightarrow (2) \qquad \forall\ y > 0$

$u\ (l, y) = 0 \qquad \rightarrow (3) \qquad \forall\ y > 0$

$u(x, 0) = f(x) = u_0 \rightarrow (4) \qquad 0 < x\ l$

$u(x, \infty) = 0 \qquad \rightarrow (5) \qquad 0 < x < l$



The G.S of (1) satisfying above all boundary conditions is

$$u(x,y)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right).e^{-\left(\frac{n\pi y}{\ell}\right)} \rightarrow (6)$$

where $b_n = \frac{2}{\ell}\int_0^{\ell} f(x).\sin\left(\frac{n\pi x}{\ell}\right)dx$

Now, $b_n = \frac{2}{\ell}\int_0^{\ell} u_0.\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$\Rightarrow b_n = \frac{2u_0}{\ell}\left[\frac{-\cos\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}\right]_0^{\ell}$$

$$\Rightarrow b_n = \frac{2u_0}{n\pi}\left[1-\cos(n\pi)\right]$$

$$\Rightarrow b_n = \frac{2u_0}{n\pi}\left[1-(-1)^n\right] \rightarrow (7)$$

Using (7) (i.e. the value of b_n in (6), the required solution is), the equation (6) becomes

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2u_0}{n\pi}\left[1-(-1)^n\right].\sin\left(\frac{n\pi x}{\ell}.e^{\left(\frac{-2\pi y}{\ell}\right)}\right)$$

(or)

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2u_0}{(2n-1)\pi}(2).\sin\left[\frac{(2n-1)\pi x}{\ell}\right].e^{-\left[\frac{(2n-1)\pi y}{\ell}\right]}$$

68. Ans: (c)

Sol: Given $f(t) = \begin{cases} 2, & \text{when } 0 \le t < 1 \\ 2t, & \text{when } t \ge 1 \end{cases}$

$$L\{f(t)\} = \int_0^{\infty} e^{-st}.f(t)\,dt$$

$$= \int_0^1 e^{-st}.2\,dt + \int_1^{\infty} e^{-st}.2t\,dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 + 2\left[t.\left[\frac{e^{-st}}{-s}\right] - 1\left[\frac{e^{-st}}{s^2}\right]\right]_1^{\infty}$$

$$= 2\left[\frac{e^{-s}}{-s}+\frac{1}{s}\right]+2\left[\frac{e^{-s}}{s}+\frac{e^{-s}}{s^2}\right]$$

$$= 2\left[\frac{e^{-s}}{s^2}+\frac{1}{s}\right]=\frac{2}{s}\left(1+\frac{e^{-s}}{s}\right)$$

69. Ans: (d)

Sol: $L\,(1+t\,e^{-t})^2$

$= L\,(1+2t\,e^{-t}+t^2\,e^{-2t})$

$$=\frac{1}{s}+\frac{2}{(s+1)^2}+\frac{2}{(s+2)^3}$$

(By first shifting property)

70. Ans: (b)

Sol: $L(\cos t) = \frac{s}{s^2+1}$

By first shifting property

$$L(e^{-t}\cos t) = \frac{(s+1)}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

By multiplication by t^n property

$$L(t\,e^{-t}\cos t) = (-1)\,\frac{d}{ds}\left(\frac{s+1}{s^2+2s+2}\right)$$

$$= \frac{s^2+2s}{\left(s^2+2s+2\right)^2}$$

71. Ans: (a)

Sol: $L(1-e^t) = \frac{1}{s}-\frac{1}{s-1}$

By division property

$$L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s}-\frac{1}{s-1}\right) ds$$

$$= \left[\log s - \log(s-1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty$$

$$= 0 - \log\left(\frac{s}{s-1}\right) = \log\left(\frac{s-1}{s}\right)$$

72. Ans: (c)

Sol: $L(\sin t) = \frac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\{f^1(t)\} = s.\,L\{f(t)\} - f(0)$$

$$\Rightarrow L\{f^1(t)\} = s.L\left\{\frac{\sin t}{t}\right\} - f(0)$$

$$L\{f^1(t)\} = s\cot^{-1} s - f(0) = \cot^{-1} s - 1$$

73. Ans: (b)

Sol: $L(\cos t) = \frac{s}{s^2+1}$

By fist shifting property

$$L(e^{-t}.\cos t) = \frac{(s+1)}{(s+1)^2+1}$$

By integral property

$$\therefore\ L\left[\int_0^t e^{-t}\cos dt\right] = \frac{1}{s}\left(\frac{s+1}{s^2+2s+2}\right)$$

74. Ans: (d)

Sol: $f(t) = \begin{cases} t, & 0 < t \le 1 \\ 0, & 1 < t < 2 \end{cases}$

$\because$ f(t) is periodic function with period 2

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 t.e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2s}}\left[t.\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_0^1$$

$$= \frac{1}{1-e^{-2s}}\left[\left(\frac{e^{-s}}{-s}\right) - \left(\frac{e^{-s}}{s^2}\right) + \frac{1}{s^2}\right]$$

$$= \frac{1-e^{-s}-s\,e^{-s}}{s^2\left(1-e^{-2s}\right)}$$

75. Ans: (c)

Sol: $L\{e^t\} = \dfrac{1}{s-1}$

$e^t\, u(t-3) = [e^{t-3}.\, u(t-3)]e^3$

By second shifting property

$L[e^t.u(t-3)] = e^3.\, L[e^{t-3}.\, u(t-3)]$

$$= e^3.\left(\frac{e^{-3s}}{s-1}\right) = \frac{e^{3-3s}}{s-1}$$

76. Ans: (a)

Sol: $L\,(\sin t) = \dfrac{1}{s^2+1}$

$$L\,(t\sin t) = \int_0^\infty e^{-st}\,(t\sin t)\,dt$$

$$\Rightarrow (-1).\,\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \int_0^\infty e^{-st}(t\sin t)\,dt$$

$$\Rightarrow \frac{2s}{(s^2+1)^2} = \int_0^\infty e^{-st}\,(t\sin t)\,dt$$

Put s = 3

$$\Rightarrow \frac{2(3)}{(3^2+1)^2} = \int_0^\infty e^{-st}\ t\sin t\ dt$$

$$\therefore \int_0^\infty e^{-st}\ t.\sin t\ dt = \frac{3}{50}$$

77. Ans: (a)

Sol: $f(t) = L^{-1}\left[\dfrac{1}{s^2(s+1)}\right]$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$\Rightarrow 1 = A\,(s+1) + B(s+1) + cs^2$

C = 1, B = 1, A = –1

$$\therefore\ f(t) = L^{-1}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right]$$

$$= -1 + t + e^{-t}$$

78. Ans: (c)

Sol: $L^{-1}\left(\dfrac{1}{s^2}\right) = t$

By first shifting property

$$L^{-1}\left[\frac{1}{(s-2)^2}\right] = e^{2t}.t$$

By second shifting property

$$L^{-1}\left[\frac{e^{-4s}}{(s-2)^2}\right] = e^{2(t-4)}.u(t-4)$$

79. Ans: (d)

Sol: $L^{-1}\left[\dfrac{1}{s(s-1)}\right] = L^{-1}\left[\dfrac{1}{s-1} - \dfrac{1}{s}\right] = e^t - 1$

80. Ans: (b)

Sol: $L^{-1}\left[\dfrac{s}{(s^2+4)^2}\right] = L^{-1}\left[\left(\dfrac{s}{s^2+4}\right)\left(\dfrac{1}{s^2+4}\right)\right]$

$$L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t \quad \text{and}$$

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{\sin 2t}{2}$$

By convolution theorem,

$$L^{-1}\left[\left(\frac{s}{s^2+4}\right).\left(\frac{1}{s^2+4}\right)\right] = \int_0^t \cos\ 2x.\frac{\sin 2(t-x)}{2}\ dx$$

$$= \frac{1}{4}\left\{t \sin(2t) + \left[\frac{\cos(2t-4x)}{4}\right]_0^t\right\}$$

$$= \frac{t \sin 2t}{4}$$

81. Ans: (a)

Sol: $L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$3s + 1 = A(s^2 + 1) + (Bs + C)(s - 1)$

$s = 1 \Rightarrow 4 = 2A$

$\Rightarrow A = 2$

$A + B = 0 \quad \Rightarrow B = -2$

$3 = -B + C \Rightarrow C = 1$

$$\therefore L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = L^{-1}\left[\frac{2}{s-1} - 2\left(\frac{s}{s^2+1} + \frac{1}{s^2+1}\right)\right]$$

$$= 2e^t - 2\cos t + \sin t$$

82. Ans: (b)

Sol: $L^{-1}\left[\frac{1}{(s-1)(s-2)^2}\right]$

$$\frac{1}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$1 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$

$s = 1 \quad \Rightarrow A = 1$

$A + B = 0 \Rightarrow B = -1$

$s = 2 \Rightarrow C = 1$

$$L^{-1}\left[\frac{1}{(s-1)(s-2)^2}\right] = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{(s-2)^2}\right]$$

$$= e^t - e^{2t} + t\,e^{2t}$$

83. Ans: (d)

Sol: Given $y^1(t) + 5\,y(t) = u(t)$……………..(1)

with $y(0) = 1$ ……………………(2)

Applying L. T on both sides of (1), we get

$L\{y^1(t) + 5\,y(t)\} = L\{u(t)\}$

$\Rightarrow L\{y^1(t)\} + 5\,L\{y(t)\} = \frac{1}{s}$

$\Rightarrow [s.\bar{y}(s) - y(0)] + 5.\bar{y}(s) = \frac{1}{s}$

$\Rightarrow (s+5)\bar{y}(s) - 1 = \frac{1}{s}$

$\Rightarrow (s+5)\bar{y}(s) = 1 + \frac{1}{s} = \frac{s+1}{5}$

$\Rightarrow \bar{y}(s) = \frac{s+1}{s.(s+5)}$

$\Rightarrow \bar{y}(s) = \frac{1}{5}.\frac{1}{s} + \frac{4}{5}\frac{1}{s+5}$

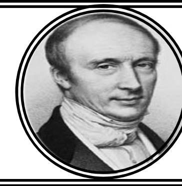
Applying inverse Laplace transform on both sides of above, we get

$$L^{-1}\{\bar{y}(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{5}L^{-1}\left\{\frac{1}{s+5}\right\}$$

$\therefore y(t) = \frac{1}{5} + \frac{4}{5}e^{-5t}$ is a solution of (1)

Chapter **5**

Complex Variables



Augustin-louis Cauchy
(1789 –1857)

01. Ans: (a)

Sol: Let $u + iv = f(z) = z^2 = (x+iy)^2$

Then $u + iv = f(z) = (x^2 - y^2) + i(2xy)$

$\Rightarrow u = x^2 - y^2$ and $v = 2xy$

$\Rightarrow u_x = 2x$ | $v_x = 2y$

$\Rightarrow u_y = -2y$ | $v_y = 2x$

Here $u_x = v_y$ and $v_x = -u_y$ at every point and also u, v, u_x, u_y, v_x, v_y are continuous at every point.

$\therefore$ f(z) is analytic at every point.

02. Ans: (a)

Sol: Let $u + iv = f(z) = z\,\text{Im}(z) = (x + iy)\,y$

Then $u + iv = f(z) = xy + iy^2$

$\Rightarrow u = xy$ and $v = y^2$

$\Rightarrow u_x = y$ | $v_x = 0$

$u_y = x$ | $v_y = 2y$

Here, $u_x = v_y$ and $v_x = -u_y$ only at one point origin. i.e., C.R equations $u_x = v_y$ and $v_x = -u_y$ are satisfied only at origin. Further u, v, v_x, v_y, u_x, u_y are also continuous at origin.

$\therefore$ $f(z) = z\,\text{Im}(z)$ is differentiable only at origin (0,0).

03. Ans: (d)

Sol: sin(z), cos(z) and polynomial $az^2 + bz+c$ are analytic everywhere.

$\therefore$ sin(z), cos(z) and az^2+bz+c are an entire functions.

Here, $\dfrac{1}{z-1}$ is analytic at every point except at z = 1 because the function $\dfrac{1}{z-1}$ is not defined at z = 1.

$\Rightarrow \dfrac{1}{z-1}$ is not analytic at z = 1

$\therefore \dfrac{1}{z-1}$ is not an entire function

04. Ans: (a)

Sol: Given that $z = \sin hu.\cos v + i\cosh u.\sin v$

$\Rightarrow z = \sinh u.\cosh(iv) + \cosh u.(i \sin v)$

$(\because \cosh(ix) = \cos x \;\&\; i \sin x = \sin h(ix)$

$\Rightarrow z = \sinh u.\cosh(iv) + \cosh u.\sinh(iv)$

$\Rightarrow z = \sinh(u+iv)$

$(\because \sinh(A+B) = \sinh A \cosh B + \cosh A.\sinh B)$

$\Rightarrow z = \sinh(w)$ $(\because w = u + iv)$

$\Rightarrow w = \sinh^{-1}(z)$

$\Rightarrow w = f(z) = \sinh^{-1}(z)$

$\Rightarrow w^1 = f^1(z) = \dfrac{1}{\sqrt{1+z^2}}$

Here, $f^1(z)$ is defined for all values of z except at $\sqrt{1+z^2} = 0$ (or) $1+z^2 = 0$ (or) $z = i, -i$

$\Rightarrow f^1(z)$ does not exist at $z = i, -i$

$\Rightarrow f(z)$ is not differentiable at $z = i, -i$

$\therefore f(z)$ is not analytic at $z = i, -i$

Augustin-Louis Cauchy was a French mathematician. "More concepts and theorems have been named for Cauchy than for any other mathematician". Cauchy was a prolific writer; he wrote approximately eight hundred research articles and almost single handedly founded complex analysis.

05. Ans: (d)

Sol: Given that $v = x^3 - 3xy^2$

$\Rightarrow v_x = 3x^2 - 3y^2$ and $v_y = -6xy$

Consider $du = u_x\, dx + u_y\, dy$

$\Rightarrow du = (v_y)\, dx + (-v_x)\, dy$

$(\because u_x = v_y \;\&\; v_x = -u_y)$

$\Rightarrow du = (-6xy)\, dx + (-3x^2 + 3y^2)\, dy$

which is an exact differential form

$\Rightarrow \int du = \int (-6xy)\, dx + \int (3y^2)\, dy + k$

$\therefore u(x, y) = -3x^2y + y^3 + k$

06. Ans: (c)

Sol: Given $u(r, \theta) = e^{-\theta} \cos(\log r)$

$\Rightarrow u_r = -e^{-\theta} \sin(\log r).\dfrac{1}{r}$ and

$u_\theta = -e^{-\theta} \cos(\log r)$

Consider $dv = \left(\dfrac{\partial v}{\partial r}\right) dr + \left(\dfrac{\partial v}{\partial \theta}\right) d\theta$

$\Rightarrow dv = (v_r)\, dr + (v_\theta)\, d\theta$

$= \left(\dfrac{-1}{r}\, u_\theta\right) dr + (r\, u_r)\, d\theta$

$\Rightarrow dv = \dfrac{1}{r} e^{-\theta} \cos(\log r)\, dr + \left(-e^{-\theta}.\sin(\log r)\right) d\theta$

$\Rightarrow \int dv = \int e^{-\theta}.\dfrac{1}{r}.\cos(\log r)\, dr + \int 0\, d\theta + c$

$\therefore v(r, \theta) = e^{-\theta} \sin(\log r) + c$

07. Ans: (c)

Sol: Given that $\text{Re}\{f^l(z)\} = 2x + 2$, $f(0) = 2$ and $f(1) = 1 + 2i$

Let $f^l(z) = u + iv$, then $u = 2x + 2$

Consider $f^{11}(z) = u_x + i\, v_x = u_x - i\, u_y$

$= 2 - i\, 0$

$\Rightarrow f^l(z) = 2z + c$

$\Rightarrow f(z) = z^2 + cz + k$

$\because f(0) = 2$

$\Rightarrow k = 2$

$\because f(i) = 1 + 2i$

$\Rightarrow (i)^2 + c(i) + k = 1 + 2i$

$\Rightarrow c = 2$

$\therefore f(z) = z^2 + 2z + 2$

$\Rightarrow f^l(z) = 2z + 2$

$\Rightarrow f^l(z) = 2(x + i\, y) + 2 = 2(x + 1) + i(2y)$

$\therefore$ Imaginary part of $f^l(z) = 2y$

08. Ans: (c)

Sol: Given that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\Rightarrow u_x = 3x^2 - 3y^2 + 6x$ and $u_y = -6xy - 6y$

Consider $f^l(z) = u_x - iu_y$

$\Rightarrow f^l(z) = (3x^2 - 3y^2 + 6x) - i(-6xy - 6y)$

$\Rightarrow f^l(z) = (3z^2 - 0 + 6z) - i\,(0 - 0)$

(Replacing 'x' by 'z' and 'y' by '0')

$\Rightarrow \int f^l(z)\, dz = \int (3z^2 + 6z)\, dz + c$

$\therefore f(z) = 3\dfrac{z^3}{3} + 2\dfrac{z^2}{2} + c$

$= z^3 + 3z^2 + c$ is a required analytic function where $c = c_1 + ic_2$ is a integral constant & $c = 1 + ic_2$ because given real part 'u' is containing constant '1'.

09. Ans: (c)

Sol: Given $u = (x - 1)^3 - 3xy^2 + 3y^2$

$\Rightarrow u_x = 3(x - 1)^2 - 3y^2$ and $u_y = -6xy + 6y$

Consider $f^l(z) = u_x - i\, u_y$

$\Rightarrow f^l(z) = 3(x - 1)^2 - 3y^2 - i\,(-6xy + 6y)$

$\Rightarrow f^l(z) = 3(z - 1)^2 - 0 - i(-0 + 0)$

(Replacing 'x' by 'z' & 'y' by '0')

$\Rightarrow \int f^{1}(z)\,dz = \int 3(z-1)^2\,dz + c,\ c = c_1 + ic_2$

$\therefore$ $f(z) = (z-1)^3 + ic_2$ because the given real part does not contain any constant.

10. Ans: (a)

Sol: Given that $v = e^x[y\cos y + x\sin y]$

$\Rightarrow v_x = e^x[0 + \sin y] + e^x[y\cos y + x\sin y]$

and $v_y = e^x[-y\sin y + \cos y + x\cos y]$

Consider $f^{1}(z) = u_x - iu_y$

$\Rightarrow f^{1}(z) = v_y + i\,v_x$ ($\because$ $u_x = v_y$ & $v_x = -u_y$)

$\Rightarrow f^{1}(z) = e^x[-y\sin y + \cos y + x\cos y]$
$\quad + i\,e^x[\sin y + y\cos y + x\sin y]$

$\Rightarrow \int f^{1}(z) = ze^z - e^z + e^z + c$

$\therefore$ $f(z) = z\,e^z + c,\ c = c_1 + ic_2$ is a required analytic function.

11. Ans: (b)

Sol: Let $f(z) = e^z + \sin z$ and $z_0 = \pi$

Then Taylor's series expansion of f(z) about a point $z = z_0$ (or) in power of $(z - z_0)$ is given by $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

Here, the coefficient of $(z - z_0)^n$ in the Taylor's series expansion of f(z) about $z = z_0$ is given by $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

$$\therefore a_2 = \frac{f''(z_0)}{2!} = \frac{f''(\pi)}{2!}$$

$$= \frac{(e^z - \sin z)_{z=\pi}}{2} = \frac{e^{\pi}}{2}$$

12. Ans: (a)

Sol: Given $f(z) = \dfrac{3}{3z - z^2}$ and $z_0 = 1$

The given function is analytic at $z = 1$

$\therefore$ Taylor's series expansion of f(z) is possible at $z = 1$

Now, $f^{1}(z) = \dfrac{-3(3-2z)}{(3z-z^2)^2} = \dfrac{6z-9}{(3z-z^2)^2}$

$$\Rightarrow f^{11}(z) = \frac{(3z-z^2)(6) - (6z-9)2(3z-z^2)(3-2z)}{(3z-z^2)^4}$$

$\Rightarrow f^{1}(1) = \dfrac{-3}{4},\ f^{11}(1) = \dfrac{18}{8}$ and $f(1) = \dfrac{3}{2}$

The Taylor's series of f(z) about $z = z_0$ is given by

$$f(z) = f(z_0) + (z - z_0)\,f^{1}(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \ldots.$$

$$\Rightarrow f(z) = f(1) + (z-1)\,f^{1}(1) + \frac{(z-1)^2}{2!}f''(1) + \ldots.$$

$$\therefore \frac{3}{(3z-z^2)} = \frac{3}{2} + \left(\frac{-3}{4}\right)(z-1) + \frac{18}{8}\cdot\frac{1}{2!}(z-1)^2 + \ldots$$

13. Ans: 1

Sol: Let $f(z) = \log\left(\dfrac{z}{1-z}\right)$ and $|z| > 1$

(or) $\left|\dfrac{1}{z}\right| < 1$

$$\text{Then } f(z) = \log\left[\frac{z}{z\left(1-\frac{1}{z}\right)}\right] = \log\left(\frac{1}{1-\frac{1}{z}}\right)$$

$$\Rightarrow f(z) = \log\left(1-\frac{1}{z}\right)^{-1} = -\log\left(1-\frac{1}{z}\right), \quad \left|\frac{1}{z}\right|<1$$

$$\Rightarrow f(z) = -\left[-\left\{\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^2+\frac{1}{3}\left(\frac{1}{z}\right)^3+........\right\}\right], \quad \left|\frac{1}{z}\right|<1$$

$$\Rightarrow \log(1-z) = \frac{1}{z}+\frac{1}{2}.\frac{1}{z^2}+\frac{1}{3}.\frac{1}{z^3}+......, \quad \left|\frac{1}{z}\right|<1$$

$\therefore$ The coefficient of $\frac{1}{z}$ is 1

14. Ans: (b)

Sol: Given $f(z) = \dfrac{1}{z^2-3z+2}$ in $|z|>2$

(or) $|z| > 2 > 1$

$$\Rightarrow f(z) = \frac{1}{(z-1)(z-2)}$$

$$\Rightarrow f(z) = \frac{1}{(z-2)}-\frac{1}{z-1} \quad \text{in } |z|>2 \;\&\; |z|>1$$

$$\Rightarrow f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)} \quad \text{in } \left|\frac{2}{z}\right|<1$$

and $\left|\frac{1}{z}\right|<1$

$$\Rightarrow f(z) = \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$\Rightarrow f(z) = \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+.....\right]$$

$$-\frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+.....\right]$$

$$\Rightarrow f(z) = \frac{1}{z}\left[\sum_{n=0}^{\infty}2^n\left(\frac{1}{z}\right)^n-\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right]$$

$$\text{(or) } f(z) = \sum_{n=0}^{\infty}2^n\left(\frac{1}{z}\right)^{n+1}-\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

15. Ans: –1

Sol: Given $f(z) = \dfrac{1}{z-1}-\dfrac{1}{z-2}$ and $|z|>2$

Now, $|z|>2$

$\Rightarrow \quad |z| > 2 > 1$

$\Rightarrow |z| > 2$ and $|z| > 1$

$\therefore \left|\frac{2}{z}\right|<1$ and $\left|\frac{1}{z}\right|<1$

Consider $f(z) = \dfrac{1}{z-1}-\dfrac{1}{z-2}$

$$\Rightarrow f(z) = \frac{1}{z\left(1-\frac{1}{z}\right)}-\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$\Rightarrow f(z) = \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}-\frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$$=\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+....\right]-\frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+......\right]$$

$$\Rightarrow f(z) = (1-1)\,\frac{1}{z}+(1-2)\,\frac{1}{z^2}$$

$$+(1-2^2)\frac{1}{z^3}+.......$$

$\therefore$ The coefficient of $\frac{1}{z^2} = -1$

16. Ans: (a)

Sol: Given that $f(z) = \dfrac{1}{4z-z^2}$ in $0 < |z| < 4$

$$\Rightarrow f(z) = \frac{1}{z(4-z)} \text{ in } |z| < 4 \text{ or } \left|\frac{z}{4}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{4z\left(1-\frac{z}{4}\right)} = \frac{1}{4z}\left[1-\frac{z}{4}\right]^{-1} \text{ in } \left|\frac{z}{4}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{4z}\left[1+\left(\frac{z}{4}\right)+\left(\frac{z}{4}\right)^2+\ldots..\right] \text{ in } \left|\frac{z}{4}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n \text{ in } \left|\frac{z}{4}\right| < 1$$

$$\therefore f(z) = \sum_{n=0}^{\infty}\frac{1}{4^{n+1}}z^{n-1} \text{ in } \left|\frac{z}{4}\right| < 1$$

17. Ans: (a)

Sol: Given $f(z) = \dfrac{1}{(z-1)(z+3)}$ in $0 < |z+1| < 2$

Let $z + 1 = t$ then $z = t - 1$ and $0 < |t| < 2$

Now, $f(z) = \dfrac{1}{(z+1)(z+3)} = \dfrac{1}{t(t+2)}$

Instead of expanding $\dfrac{1}{(z+1)(z+3)}$ in powers of $z + 1$ it is enough to expand $\dfrac{1}{t(t+2)}$ in powers of t in $0 < |t| < 2$

$$f(z) = \frac{1}{t(t+2)} \text{ in } 0 < |t| < 2 \text{ or } \left|\frac{t}{2}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{t}.\frac{1}{2\left(1+\frac{t}{2}\right)} = \frac{1}{t}.\frac{1}{2}.\left[1+\frac{t}{2}\right]^{-1}, \left|\frac{t}{2}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{2t}\left[1-\left(\frac{t}{2}\right)+\left(\frac{t}{2}\right)^2-\ldots..\right], \left|\frac{t}{2}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{2t}-\frac{1}{4}+\frac{1}{8}t-\ldots\ldots\ldots.$$

$$\therefore f(z) = \frac{1}{2(z+1)}-\frac{1}{4}+\frac{(z+1)}{8}-\frac{(z+1)^2}{16}+\ldots.$$

18. Ans: (c)

Sol: The given function $f(z) = z^2$ is analytic at every point.

$\therefore$ The value of the given integral is independent of the path joining $z = 0$ and $z = 3 + i$

Now, $I = \displaystyle\int_{z=0}^{3+i} z^2\, dz$

$$\Rightarrow I = \left(\frac{z^3}{3}\right)_0^{3+i} = \frac{(3+i)^3}{3}-\frac{0}{3}$$

$$= \frac{(27-27i-9-i)}{3}$$

$$\therefore I = 6+\left(\frac{26}{3}\right)i$$

19. Ans: 0

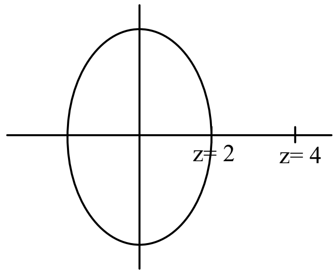
Sol: Let $f(z) = \dfrac{4z^2+z+5}{z-4}$

Then the singular point of $f(z)$ is given by $z - 4 = 0$ (or) $z = 4$

Given that C: $9x^2 + 4y^2 = 36$

$$\Rightarrow \frac{9x^2}{36}+\frac{4y^2}{36} = 1$$

$$\Rightarrow \frac{x^2}{2^2}+\frac{y^2}{3^2} = 1$$



Here the singular point of the function f(z) lies outside the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

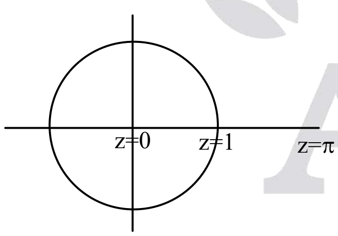
$\therefore$ The given function f(z) has no singular inside and on the curve 'C'

Hence by Cauch's Integral Theorem, we have $\oint_C f(z)\,dz = 0$.

20. Ans: (a)

Sol: Let f(z) = $\dfrac{\sin^2(z)}{\left(\dfrac{z-\pi}{6}\right)^3} = \dfrac{6^3.\sin^2 z}{(z-\pi)^3}$

Then the singular point of f(z) is given by $(z-\pi)^3 = 0$ (or) $z = \pi$.



Here the singular z = π lies outside the given circle C: |z| = 1.

$\therefore$ By Cauchy's Integral Theorem, we have

$\oint_C f(z)\,dz = 0$

21. Ans: (c)

Sol: Let f(z) = $\dfrac{\cos(\pi z)}{z-1} = \dfrac{\phi(z)}{z-z_0}$

Then the singular point of f(z) is given by z –1 = 0 (or) z = 1

Here, the singular point z = 1 lies inside the given circle C: |z –1| = 2.

$\therefore$ By Caychy's Integral Formula, we have

$$\oint_C f(z)\,dz = 2\pi i\,[\cos(\pi z)]_{z=1} = 2\pi i(-1)$$

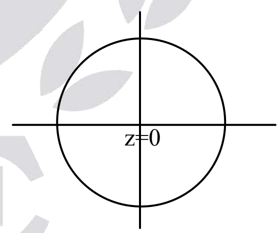
$$= -2\pi i$$

22. Ans: (c)

Sol: Let f(z) = $\dfrac{1}{z^2 e^z} = \dfrac{e^{-z}}{z^2} = \dfrac{e^{-z}}{(z-0)^2}$

Then the singular point of the function f(z) is given by $z^2e^z = 0$ (or) z = 0 ($\because$ $e^z \neq 0\ \forall z$)

Here, the singular point z = 0 of the function f(z) lies inside the circle C: |z| = 1.



Let f(z) = $\dfrac{\phi(z)}{[z-z_0]} = \dfrac{e^{-z}}{[z-0]^{1+1}}$

Then by Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \frac{2\pi i}{1!}\left(\frac{d}{dz}e^{-z}\right)_{z=0}$$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i\,(-e^{-z})_{z=0}$$

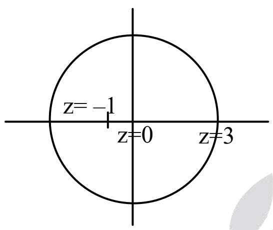
$$\therefore \oint_C f(z)\,dz = -2\pi i$$

23. Ans: (d)

Sol: Let $f(z) = \dfrac{e^{2z}}{(z+1)^4}$

Then the singular point of f(z) is given by $(z+1)^4 = 0$

$\Rightarrow z = -1$



Here, the singular point z = –1 lies inside the given circle C: |z| = 3.

Let $f(z) = \dfrac{\phi(z)}{[z-z_0]^{n+1}} = \dfrac{e^{2z}}{[z-(-1)]^{3+1}}$

Then by Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \frac{2\pi i}{3!}\left(\frac{d^3}{dz^3}e^{2z}\right)_{z=-1}$$

$$\Rightarrow \oint_C f(z)\,dz = \frac{2\pi i}{3!}\left(8e^{2z}\right)_{z=-1}$$

$$\therefore \oint_C f(z)\,dz = \left(\frac{8}{3}\right)\pi i\,e^{-2}$$

24. Ans: 0

Sol: Let $f(z) = \dfrac{z^2+z}{(z-1)^{10}}$

Then the singular point of f(z) is z = 1 and the singular z = 1 lies inside the circle |z| = 2.

Now, $f(z) = \dfrac{\phi(z)}{[z-z_0]^{n+1}} = \dfrac{z^2+z}{[z-1]^{9+1}}$

$\therefore$ By Cauchy's Integral formula, we have

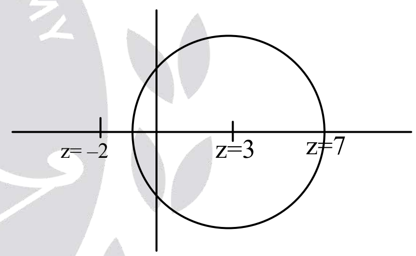
$$\oint_C f(z)\,dz = \frac{2\pi i}{9!}\left[\frac{d^9}{dz^9}\left(z^2+z\right)\right]_{z=1}$$

$$= \frac{2\pi i}{9}(0) = 0$$

25. Ans: (d)

Sol: Let $f(z) = \dfrac{e^z}{(z+2)(z-3)^2}$

Then the singular points of f(z) are z = –2 & z = 3 Of these two singular points z = –2 and z = 3 only z = 3 lies inside the circle |z –3| = 4.



Let $f(z) = \dfrac{\phi(z)}{[z-z_0]^{n+1}} = \dfrac{\left(\dfrac{e^z}{z+2}\right)}{[z-3]^{1+1}}$

Then by Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\left(\frac{e^z}{z+2}\right)\right]_{z=3}$$

$$= 2\pi i\left[\frac{(z+2)e^z - e^z(1)}{(z+2)^2}\right]_{z=3}$$

$$= 2\pi i\left[\frac{(3+2)e^3 - e^3}{(3+2)^2}\right]$$

$$= \frac{8\pi i\,e^3}{25}$$

26. Ans: –1.047

Sol: Let $f(z) = \dfrac{1}{z^2+9} = \dfrac{1}{(z+3i)(z-3i)}$

Then the singular points of f(z) are given by $z^2 + 9 = 0$ (or) $z = 3i, -3i$

But only one singular point $z = -3i$ lies inside the given circle C: $|z + 3i| = 2$

Consider $f(z) = \dfrac{\phi(z)}{z-z_0} = \dfrac{\frac{1}{z-3i}}{[z-(-3i)]}$

$\therefore$ By Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = 2\pi i\left[\frac{1}{z-3i}\right]_{z=-3i}$$

$$= 2\pi i\left(\frac{1}{-3i-3i}\right)$$

$$= \frac{-\pi}{3}$$

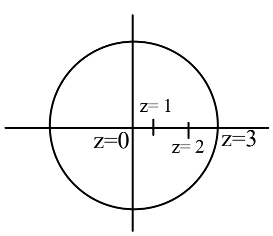
$$= -1.04719$$

27. Ans: (d)

Sol: Let $f(z) = \dfrac{\cos(\pi z^2)}{(z-2)(z-1)}$

Then the singular points of f(z) are given by $(z-2)(z-1) = 0$

$\Rightarrow z = 1$ and $z = 2$



Here, the two singular points $z = 1$ and $z = 2$ lie inside the circle C: $|z| = 3$

Now, $f(z) = \cos(\pi z^2).\left[\dfrac{1}{(z-2)(z-1)}\right]$

$$= \cos(\pi z^2)\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$\Rightarrow f(z) = \frac{\cos(\pi z^2)}{z-2} - \frac{\cos(\pi z^2)}{z-1}$$

$\therefore$ By Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \oint_C \frac{\cos(\pi z^2)}{z-2}dz - \oint_C \frac{\cos(\pi z^2)}{z-1}dz$$

$$= 2\pi i\,[\cos(\pi z^2)]_{z=2} - 2\pi i[\cos(\pi z^2)]_{z=1}$$

$$= 2\pi i\,(1) - 2\pi i\,(-1)$$

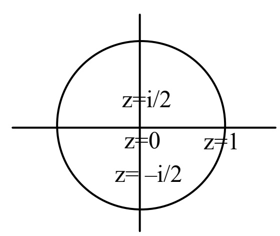
$$= 4\pi i$$

28. Ans: 0

Sol: Let $f(z) = \dfrac{\cosh(z)}{4z^2+1} = \dfrac{\cosh(z)}{4\left[z^2+\frac{1}{4}\right]}$

$$= \frac{\left(\frac{\cosh(z)}{4}\right)}{\left(z-\frac{i}{2}\right)\left(z+\frac{i}{2}\right)}$$

Then the singular points of f(z) are $z = \dfrac{i}{2}, \dfrac{-i}{2}$.



Here, the two singular points $z = \dfrac{i}{2}$ and $z = \dfrac{-i}{2}$ lie inside the circle $|z| = 1$.

Now, $f(z) = \dfrac{\cos(z)}{4}\left[\dfrac{1}{\left(z-\frac{i}{2}\right)\left[z-\left(-\frac{i}{2}\right)\right]}\right]$

$\Rightarrow f(z) = \left[\dfrac{\cos(z)}{4}\left[\dfrac{1}{\left(z-\frac{i}{2}\right)\left[\left(\frac{i}{2}+\frac{i}{2}\right)\right]}\right]\right.$

$\left. + \dfrac{1}{\left(z+\frac{i}{2}\right)\left(\frac{-i}{2}-\frac{i}{2}\right)}\right]$

$\Rightarrow f(z) = \dfrac{\left(\frac{\cosh(z)}{4i}\right)}{\left[z-\frac{i}{2}\right]} + \dfrac{\left(\frac{\cosh(z)}{-4i}\right)}{\left[z-\left(\frac{-i}{2}\right)\right]}$

By Caychy's Integral Formula, we have

$\oint_C f(z)\,dz$

$= \dfrac{1}{4i}\oint_C \dfrac{(\cosh(z))}{\left[z-\frac{i}{2}\right]}dz + \left(\dfrac{1}{-4i}\right)\int_C \dfrac{\cosh(z)}{\left[z-\left(\frac{-i}{2}\right)\right]}\,dz$

$= \left(\dfrac{1}{4i}\right)\left[2\pi i\,(\cos z)_{z=\frac{i}{2}}\right] + \left(\dfrac{-1}{4i}\right)\left[2\pi i(\cos hz)_{z=\frac{-i}{2}}\right]$

$= \dfrac{\pi}{2}\left[\cosh\left(\dfrac{i}{2}\right)\right] + \left(\dfrac{-\pi}{2}\right)\left[\cosh\left(-\dfrac{i}{2}\right)\right]$

$= \dfrac{\pi}{2}\cosh\left(\dfrac{i}{2}\right) - \dfrac{\pi}{2}\cosh\left(\dfrac{-i}{2}\right)$

$= 0$ $\quad (\because \cosh(-z) = \cosh(z))$

29. Ans: (b)

Sol: Given $F(a) = \displaystyle\int_C \dfrac{5z^2-4z+3}{z-a}\,dz$

where 'C' is $16x^2 + 9y^2 = 144$ (or)

$\dfrac{x^2}{3^2} + \dfrac{y^2}{4^2} = 1$

Let a= i for finding the value of $F^1(i)$.

Then the singular point z = a = i of the function $\dfrac{5z^2-4z+3}{z-a}$ lies inside the ellipse

$\therefore$ By Cauchy's integral formula, we have

$F(a) = \displaystyle\int_C \dfrac{5z^2-4z+3}{z-a}dz$

$= 2\pi i\,(5z^2 - 4z + 3)_{z=a}$

$\Rightarrow F(a) = 2\pi i\,(5a^2 - 4a + 3)$

$\Rightarrow F^1(a) = 2\pi i\,(10a - 4)$

$\therefore F^1(i) = 2\pi i\,(10i - 4) = -4\pi\,(5 + 2i)$

30.

Sol: Given $f(z) = \dfrac{z-1}{(z+1)(z-3)}$

$\Rightarrow$ the singular points are z = –1 & z = –3

$\Rightarrow$ z = –1 & z = –3 are first order poles.

If the algebraic function f(z) has a first order pole at a singular point $z = z_0$ then the residue of f(z) is given by

$\text{Res}(f(z): z = z_0) = \underset{z\to z_0}{\text{Lt}}\left[(z-z_0)\,.\,f(z)\right]$

$R_1 = \text{Res}(f(z): z = -1)$

$= \underset{z\to -1}{\text{Lt}}\left[[z-(-1)].\dfrac{z-1}{(z+1)(z-3)}\right]$

$\therefore R_1 = \underset{z\to -1}{\text{Lt}}\left[\dfrac{z-1}{z-3}\right] = 1/2$

$R_2 = \text{Res}(f(z): z = 3)$

$$= \underset{z\to 3}{\text{Lt}}\left[(z-3).\frac{z-1}{(z-1)(z-3)}\right]$$

$$= \underset{z\to 3}{\text{Lt}}\left(\frac{z-1}{z+1}\right)=\frac{1}{2}$$

Hence, the sum of the residues of f(z) at its singular points is $R_1 + R_2 = \frac{1}{2}+\frac{1}{2} = 1$.

31.

Sol: Given $f(z)=\frac{\sin(z)}{(z-3\pi/2)}=\frac{\phi(z)}{[z-z_0]}$

$\Rightarrow$ Singular point is $z = 3\pi/2$

$\Rightarrow z=\frac{3\pi}{2}$ is a 1st order pole

If the function $f(z)=\frac{\phi(z)}{(z-z_0)}$ has a 1st order pole at $z = z_0$ then its residue is given by

$\text{Res}(f(z) : z = z_0) = \phi(z_0)$

$\therefore R_1 = \text{Res}(f(z) : z = z_0) = \phi(3\pi/2)$

$= \sin(3\pi/2) = -1$

32. Ans: 0

Sol: The singular points of $f(z) = \frac{\sin z}{z.\cos(z)}$ are given by $z.\cos(z) = 0$

$\Rightarrow z = 0$ and $z = (2n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow z = 0$ and $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, ...

$\Rightarrow z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ are the given singular points of f(z).

Here, $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ are simple poles of

$f(z) = \frac{\sin z}{z\cos(z)} = \frac{\phi(z)}{\psi(z)}$,

where $\psi^1(z) = \cos(z) - z\sin z$

$R_1 = \text{Res}(f(z): z = \frac{\pi}{2})$

$$= \frac{\phi\left(\frac{\pi}{2}\right)}{\psi'\left(\frac{\pi}{2}\right)} = \frac{1}{0-\frac{\pi}{2}} = \frac{-2}{\pi}$$

$R_2 = \text{Res}(f(z): z = -\frac{\pi}{2})$

$$= \frac{\phi\left(-\frac{\pi}{2}\right)}{\psi'\left(-\frac{\pi}{2}\right)} = \frac{-1}{0-\frac{\pi}{2}} = \frac{2}{\pi}$$

Hence, the sum of the residues of the function f(z) at given singular points $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ is $R_1 + R_2 = \left(\frac{-2}{\pi}\right) + \left(\frac{2}{\pi}\right) = 0$

33.

Sol: Given $f(z)=\frac{\sin(z)}{z\cos(z)}=\frac{\tan(z)}{z}$

$\Rightarrow$ Singular point of f(z) is $z = 0$

Now, $f(z)=\frac{\tan(z)}{z}=\frac{1}{z}\left[z+\frac{z^3}{3}+\frac{2z^5}{15}+......\right]$

$\Rightarrow f(z)=1+\frac{z^2}{3}+\frac{2}{15}z^4+......$

$\Rightarrow f(z)=1+\frac{1}{3}(z-0)^2+\frac{2}{15}(z-0)^4+......$

The above power series of f(z) is a Lanrent series about z = 0 & z = 0 is a removable singular point.

$\therefore$ Res (f(z): z = 0) = the coefficient of $\frac{1}{(z-0)}$ in Laurent series of f(z) = 0

34. Ans: 0.33

Sol: Given $f(z) = \frac{1}{z^3} - \frac{1}{z^5}\left[\sin^2(z)\right]$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{z^5}\left[\frac{1-\cos(2z)}{2}\right]$$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{z^5}\left[\frac{1}{2} - \frac{1}{2}\cos(2z)\right]$$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{2z^5} + \frac{1}{2z}\left[1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - - \frac{(2z)^6}{6!} + ...\right]$$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{2z^5} + \left[\frac{1}{2z^5} - \frac{1}{z^3} + \frac{2^4}{2.4!}.\frac{1}{z} - \frac{2^6}{2.6!}z + ...\right]$$

$\therefore$ Res(f(z) : z = 0) = The coefficient of $\frac{1}{z}$ in above series = $\frac{2^4}{2.4!} = \frac{1}{3} = 0.333.....$

35. Ans: 1

Sol: The given singular point z = 0 is a simple pole (or) 1st order pole of

$$f(z) = \frac{1+e^z}{z\cos(z)+\sin(z)}$$

Now R_1 = Res (f(z) : z = 0) = $\underset{z\to 0}{\text{Lt}}(z-0)\,f(z)$

$$\Rightarrow R_1 = \underset{z\to 0}{\text{Lt}}(z-0).\frac{1+e^z}{z\cos(z)+\sin(z)}$$

$\left(\frac{0}{0}\text{ form}\right)$

$$\Rightarrow R_1 = \underset{z\to 0}{\text{Lt}}\ \frac{z(0+e^z)+(1+e^z)}{-z\sin(z)+\cos(z)+\cos(z)}$$

$$\therefore R_1 = \frac{0+1+1}{0+1+1} = 1$$

36. Ans: (c)

Sol: Let $f(z) = z^2\ e^{\frac{1}{z}}$

Then

$$f(z) = z^2\left[1 + \frac{\left(\frac{1}{z}\right)}{1} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3\,!} + \frac{\left(\frac{1}{z}\right)^4}{4!} + ...\right]$$

$$\Rightarrow f(z) = z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + ...$$

$$\Rightarrow f(z) = (z-0)^2 + (z-0) + \frac{1}{2\,!} + \frac{1}{3!}\frac{1}{(z-0)} + \frac{1}{4!}\frac{1}{(z-0)^2} + ...$$

$\Rightarrow$ f(z) has a singular point at z = 0.

Here, the singular point z = 0 lies inside the circle |z| = 1.

R_1 = Res(f(z) : z = 0) = The coefficient of $\frac{1}{(z-0)}$ in Laurent series

$$\Rightarrow R_1 = \text{Res}\ (f(z) : z = 0) = \frac{1}{3\,!} = \frac{1}{6}.$$

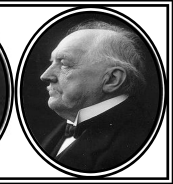
$\therefore$ By Cauchy's Residue Theorem, we have

$$\oint_C f(z)\,dz = 2\pi i(R_1) = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

Chapter **6**

Numerical Methods

Carl David Tolme Runge (1856 – 1927)



Martin Wilhelm Kutta (1867-1944)

01. Ans: (c)

Sol: $f(x) = x^3 - 4x - 9 = 0$

$f(2) = -9 < 0$, $f(3) = 6 > 0$

Let $x_1 = \dfrac{2+3}{2} = 2.5$ is first approximation to the root

$\therefore\ f(x_1) = f(2.5) = -3.375 < 0$

Now, Root lies in [2.5, 3]

Let $x_2 = \dfrac{2.5+3}{2} = 2.75$ is second approximation root.

02. Ans: 0.67

Sol: $f(x) = x^3 + x - 1 = 0$

Let $x_0 = 0.5$, $x_1 = 1$

$f(x_0) = f(0.5) = -0.375$

$f(x_1) = f(1) = 1$

$\therefore\ x_2 = \dfrac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$

is first approximation root

$$= \frac{1(0.5) - (-0.375)(1)}{1 - (-0.375)}$$

$$= \frac{0.5 + 0.375}{1.375} = \frac{0.875}{1.375}$$

$$= 0.6363$$

$f(x_2) = f(0.6363)$

$$= (0.6363)^3 + (0.6363) - 1$$

$$= 0.2576 + 0.6363 - 1$$

$$= -0.1061 < 0$$

Root lies in (0.6363, 1)

$x_3 = \dfrac{f(x_1)x_2 - f(x_2).x_1}{f(x_1) - f(x_2)}$

$$= \frac{1(0.6363) - (-0.1061)1}{1 + 0.1061}$$

$$= 0.6711$$

03. Ans: (b)

Sol: $f(x) = xe^x - x = 0$

$f(0) = -2 < 0$, $f(1) = 2.7183 - 2 > 0$

Let $x_0 = 0$, $x_1 = 1$

$x_2 = \dfrac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$

$$= \frac{0.7183(0) - (-2).1}{0.7183 - (-2)}$$

$$= \frac{2}{2.7183}$$

$$= 0.7357$$

$f(x_2) = f(0.7357)$

$$= 0.7357\ .e^{0.7357} - 2 = -0.4644$$

Take $x_0 = 0.7357$ & $x_1 = 1$

$\therefore\ x_3 = \dfrac{f(x_2).x_1 - f(x_1).x_2}{f(x_2) - f(x_1)}$

$$= \frac{0.9929}{1.1827} = 0.8395$$

***C. Runge** and **M. W. Kutta** (German mathematicians) developed an important family of implicit and explicit iterative methods, which are used in temporal discretization for the approximation of solutions of ordinary differential equations. In numerical analysis, these techniques are known as Runge–Kutta methods.*

04. Ans: (b)

Sol: $f(x) = x^4 - x - 10 = 0$; $f^1(x) = 4x^3 - 1$

$f(1) = -10 < 0$, $f(2) = 4 > 0$

Let $x_0 = 2$ is initial approximation

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$= 2 - \frac{4}{31} = 1.871$$

05. Ans: (c)

Sol: $f(x) = 3x - \cos x - 1$

$f(x_0) = f(0) = -2$

$f^1(x_0) = f^1(0) = 3$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = -\frac{(-2)}{3} = \frac{2}{3}$$

06. Ans: (a)

Sol: Let $x = \sqrt{N}$

$f(x) = x^2 - N = 0$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \text{......... (i)}$$

07. Ans: (b)

Sol: Taking N = 18 & $x_0 = 4$ in equation (i) of previous examples(06), we get

$$x_1 = \frac{4^2 + 18}{8} = 4.25$$

08. Ans: (a)

Sol: $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$$

Let $x_{n+1} = x_n = x$

$$x = \frac{1}{2}\left(x + \frac{3}{x}\right)$$

$x^2 = 3$

09. Ans: (b)

Sol: $$\text{Trap.rule} = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.01}{2}[(0.2474 + 0.2860) + 2(0.2571 + 0.2667 + 0.2764)]$$

$$= \frac{0.01}{2}[0.5334 + 1.6004]$$

$= 0.005[2.1338]$

$= 0.0106$

10. Ans: (a)

Sol:

x	–1	0	1
$f(x) = 5x^3 - 3x^2 + 2x + 1$	–9	1	5

$$\int_{-1}^{1} f(x)\,dx = \frac{h}{3}[(y_0 + y_2) + 2(0) + 4(y_1)]$$

$$= \frac{1}{3}[(-4) + 4(1)] = 0$$

11. Ans: (a)

Sol: Error = Exact value of the integral – The value of the integral by the simpson's rule

= 0 – 0 = 0

12. Ans: (b)

Sol: The area

$$= \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{0.5}{3}\left[(2 + 2.1) + 2(2.7 + 3) + 4(2.4 + 2.8 + 2.6)\right]$$

$= 7.783$

13. Ans: (c)

Sol:

x	0	1	2	3	4	5	6
f(x) = $\frac{1}{1+x^2}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[\left(1 + \frac{1}{37}\right) + 2\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26}\right)\right]$$

$= 1.4107$

14. Ans: (a)

Sol: The volume of cylinder = $\pi\int_0^1 y^2\ dy$

$$= \pi\frac{h}{2}\left[(y_0^2 + y_4^2) + 2y_2^2 + 4(y_1^2 + y_3^2)\right]$$

$$= \pi\frac{0.25}{3}\left[(1+1) + 2(9) + 4(4+1)\right]$$

$$= \pi\frac{0.25}{3}[40]$$

$$= \frac{10\pi}{3}$$

15. Ans: (a)

Sol: Error = $\text{Max}\left|\frac{b-a}{12} \times h^2 \times f''(x)\right|$

$$= \frac{1}{12} \times \frac{1}{100} \times 6(2.718)$$

$= 0.0136$

Here,

$f(x) = e^{x^2}$

Max $|f^{11}(x)|_{[0,1]} = 6e$

$\therefore\ h = \frac{b-a}{n}$

$$= \frac{1}{10}$$

16. Ans: (c)

Sol: $\left|\frac{b-a}{180}\times h^4 \times \max f^{iv}(x)\right| \le 10^{-5}$

Let $h = \frac{b-a}{n} = \frac{1}{n}$

$f(x) = \frac{1}{x}$

$\text{Max } |f^{iv}(x)|_{\text{at } x=1} = 24$

$\left(\frac{1}{180}\times\frac{1}{n^4}\times 24\right) \le 10^{-5}$

$\Rightarrow n \ge 10.738$

$\therefore\ n \ge 10.738$

17. Ans: x = 0.9, y = 1 & z = 1

Sol: Let

$10x + y + z = 12$

$2x + 10y + z = 13$

$2x+2y + 10z = 14$ and

$x_0 = 0, y_0 = 0, z_0 = 0$

Then first iteration will be

$x_1 = \frac{1}{10}(12 - y_0 - z_0)$

$= 1.2$

$y_1 = \frac{1}{10}(13 - 2x_1 + 10y_0)$

$= \frac{1}{10}(13 - 2(1.2) - 0) = 1.06$

$z_1 = \frac{1}{10}(14 - 2x_1 - 2y_1)$

$= \frac{1}{10}(14 - 2(1.2) - 2.(1.06)) = 0.95$

Second iteration will be

$x_2 = \frac{1}{10}(12 - y_1 - z_1)$

$= 0.90$

$y_2 = \frac{1}{10}(13 - 2x_2 + 10y_1)$

$= 1.00$

$z_2 = \frac{1}{10}(14 - 2x_2 - 2y_2)$

$= 1.00$

The required solution after second iteration is $x = 0.9$, $y = 1$ & $z = 1$

18. Ans: 0.6

Sol: $y^1 = f(x, y) = 4 - 2xy$

$x_0 = , y_0 = 0.2, h = 0.1$

By Taylor's theorem,

$y(x) = y(x_0 + h)$

$= y(x_0) + h\, y^1(x_0) + \frac{h^2}{2\,!} y^{11}(x_0)$

$= 0.2 + 0.1(4) + \frac{(0.1)^2}{2!}(-0.4)$

$= 0.598 = 0.6$

19. Ans: 0.6

Sol: $f(x, y) = 4 - 2xy$

$x_0 = 0, y_0 = 0.2, f_1 = 0.1$

By Euler's formula

$y_1 = y_0 + h\, f(x_0, y_0) = 0.2 + 0.1(4 - 0)$

$= 0.6$

20. Ans: 0.04

Sol: By Euler's formula,

$y_1 = y_0 + h\, f(x_0, y_0)$

$y_1 = 0 + (0.2)\,(0 + 0) = 0$

$y_2 = y_1 + h\, f(x_1, y_1)$

$y_2 = 0 + 0.2(0.2 + 0)$

$y_2 = 0.04$

21. Ans: 0.095

Sol: $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$k_1 = hf(x_0, y_0) = 0.1\,(1 - 0) = 0.1$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.1\,(1 - 0.1) = 0.09$

$y_1 = 0 + \frac{1}{2}(0.1 + 0.09)$

$= 0.095$

22. Ans: 1.1961

Sol: $f(x, y) = x + \sin y$

$x_0 = 0,\ y_0 = 1,\ h = 0.2$

$k_1 = h(f_0, y_0)$

$= 0.2(0 + \sin 1)$

$= 0.2(0.8414) = 0.1682$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.2(0.2 + \sin(1.1682))$

$= 0.2(0.2 + 0.9200)$

$= 0.2(1.1200)$

$= 0.2240$

$y_1 = 1 + \frac{1}{2}(0.1682 + 0.2240) = 1.1961$

23. Ans: 1.1165

Sol: $f(x, y) = x + y^2$,

$x_0 = 0,\ y_0 = 1,\ f_1 = 0.1$

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1\left[\left(x_0 + \frac{h}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right]$

$= 0.1168$

$k_3 = hf\left(x_0 + h, y_0 + \frac{k_2}{2}\right)$

$= 0.1[0.05 + 1.1185]$

$= 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1347$

$y_1 = y_0 = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + 0.1164$

$y_1 = 1.1164$

24. Ans: 2.6 – 1.3x, 2.3

Sol: The various summations are given as follows:

	x_i	y_i	x_i^2	x_iy_i
	–2	6	4	–12
	–1	3	1	–3
	0	2	0	0
	1	2	1	2
Σ	–2	13	06	–13

Thus, $\Sigma y_i = na + b\,\Sigma x_i$

$\Sigma x_i y_i = a\,\Sigma x_i + b\,\Sigma x_i^2$

These are called normal equations. Solving for a and b, we get

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \bar{y} - b\bar{x}$$

$$b = \frac{4\times(-13)-(-2)\times 13}{4\times 6-6}$$

$$= -1.3$$

$$a = \frac{13}{4} - 1.3\times\frac{(-2)}{4} = 2.6$$

Therefore, the linear equation is

$y = 2.6 - 1.3x$

The least squares error $= \sum_{i=1}^{4}\{y_i - (a+bx_i)\}^2$

$= (6-5.2)^2 + (3-3.9)^2 + (2-2.6)^2 + (2-1.3)^2$

$= 2.3$

25. Ans: i. $8x^2 - 19x + 12$ ii. 6 iii. 13

Sol: $f(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$

$f(x) = 8x^2 - 19x + 12$

$f(2) = 6$

$f^1(2) = 13$

$f(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$

$= 1 + (x-1)\,13 + (x-1)(x-3)\,8$

$= 8x^2 - 19x + 12$

$p(2) = 6$

$p^1(2) = 13$

26. Ans: $8x^2 - 19x + 12$, 6, 13

Sol:

x	P(x)	Δp	$\Delta^2 p$
1	1	$\frac{27-1}{3-1} = 13$	$\frac{37-13}{4-1} = 8$
3	27	$\frac{64-27}{4-3} = 37$	
4	64		

By Newton's divided difference formula

$P(x) = P(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$

$= 1 + (x-1)13 + (x-1)(x-3).8$

$= 8x^2 - 19x + 12$

$P^1(x) = 16x - 19$

$P(2) = 6$

$P^1(2) = 13$

27. Ans: $x^2 + 2x + 3$, 4.25, 3

Sol: Since the given observations are at equal interval of width unity.

Construct the following difference table.

x	f(x)	Δ f(x)	Δ^2f(x)	Δ^3f(x)
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

Therefore f(x)

$f(x) = f(0) + C(x,1)\ \Delta f(0) + C(x, 2)\ f(0)$

$= 3 + (x \times 3) + \left(\dfrac{x(x-1)}{2!} \times 2\right)$

$f(x) = x^2 + 2x + 3$

$f^1(x) = 2x + 2$

$f(0.5) = 4.25$

$f^1(0.5) = 3$

28. Ans: $x^3 + 6x^2 + 11x + 6$, 990, 299

Sol: Let us apply Newton's forward formula

Let $u = \dfrac{x-a}{h} = \dfrac{x-1}{2}$

To calculate forward differences

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Now by Newton's forward interpolation formula, we have

$$f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$y(x) = 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$= x^3 + 6x^2 + 11x + 6$

$y^1(x) = 3x^2 + 12x + 11$

$y(8) = 990$

$y^1(8) = 299$