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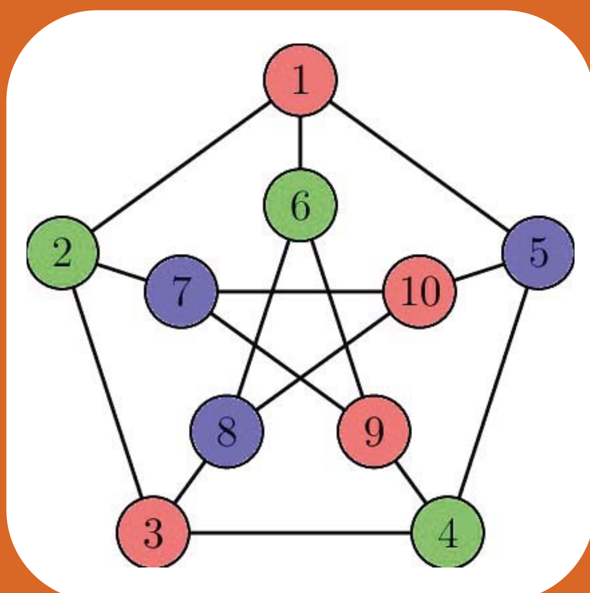


COMPUTER SCIENCE & INFORMATION TECHNOLOGY

DISCRETE MATHEMATICS

Volume-1 : Study Material with Classroom Practice Questions

Discrete Mathematics

(Solutions for Vol-1_Classroom Practice Questions)

1. Logics

Propositional Logic

01. Ans: (a)

Sol: $(\sim (P \vee Q) \vee (\sim P \wedge Q) \vee P)$

$\Leftrightarrow (\sim P \wedge \sim Q) \vee (\sim P \wedge Q) \vee P$ (By Demorgan's law)

$\Leftrightarrow ((\sim P \wedge \sim Q) \vee (\sim P \wedge Q)) \vee P$ (By Associative law)

$\Leftrightarrow (\sim P \wedge (\sim Q \vee Q)) \vee P$ (By Distributive law)

$\Leftrightarrow (\sim P \wedge T) \vee P$ ($\because \sim Q \vee Q$ is a tautology)

$\Leftrightarrow \sim P \vee P$ ($\because \sim P \wedge T \Leftrightarrow \sim P$)

$\Leftrightarrow T$

02. Ans: (c)

Sol: $((P \rightarrow Q) \leftrightarrow (\sim P \vee Q)) \wedge R$

$\Leftrightarrow (T) \wedge R$ $\quad (\because (P \rightarrow Q) \Leftrightarrow (\sim P \vee Q))$

$\Leftrightarrow R$ $\quad (\because T$ is a tautology)

03. Ans: (d)

Sol: In Boolean algebra notation, the given formula can be written as

$\left(\overline{P}.\left(\overline{Q}.R\right)\right)+\left(Q.R\right)+\left(P.R\right)$

$=\left(\overline{P}.\overline{Q}\right).R+\left(Q+P\right).R$

By associative law & distributive law

$=\left(\left(\overline{P}.\overline{Q}\right)+\left(Q+P\right)\right).R$

By distributive law

$=\left(\left(\overline{P+Q}\right)+\left(P+Q\right)\right).R$

By Demorgan's law & commutative law

$= 1 .R$

$= R$

04. Ans: (d)

Sol: (a) L.H.S $\Leftrightarrow A \rightarrow (P \vee C)$

$\Leftrightarrow \sim A \vee (P \vee C)$ $\quad E_{16}$

$\Leftrightarrow (\sim A \vee P) \vee C$ $\quad$ Associative law

$\Leftrightarrow (A \wedge \sim P) \rightarrow C$

E_{16} & Demorgan's law

$=$ R.H.S

(b) L.H.S $= (P \rightarrow C) \wedge (Q \rightarrow C)$

$\Leftrightarrow (\sim P \vee C) \wedge (\sim Q \vee C)$ $\quad$ By E_{16}

$\Leftrightarrow (\sim P \wedge \sim Q) \vee C$ $\quad$ Distributive law

$\Leftrightarrow (P \vee Q) \rightarrow C$ $\quad$ By E_{16}

$=$ R.H.S

(c) $A \rightarrow (B \rightarrow C)$

$\Leftrightarrow \sim A \vee (\sim B \vee C)$ $\quad$ By E_{16}

$\Leftrightarrow (\sim A \vee \sim B) \vee C$ $\quad$ By associative law

$\Leftrightarrow (\sim B \vee \sim A) \vee C$ $\quad$ By commutative law

$\Leftrightarrow \sim B \vee (\sim A \vee C)$ $\quad$ By associative law

$\Leftrightarrow B \rightarrow (A \rightarrow C)$ $\quad$ By E_{16}

(d) When A is false and B is false, we have LHS is true and RHS is false.

$\therefore$ LHS $\neq$ RHS

05. Ans: (d)

Sol: The given compound proposition

$\Leftrightarrow (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$

(Commutative law)

$\Leftrightarrow (p \vee (q \wedge \sim q)) \wedge (\sim p \vee (q \wedge \sim q))$

(Distributive law)

$\Leftrightarrow (p \vee F) \wedge (\sim p \vee F)$

$\Leftrightarrow p \wedge \sim p$

$\Leftrightarrow F$

$\therefore$ The given formula is a contradiction.

06. Ans: (d)

Sol: (A) R.H.S $\Leftrightarrow (a \to c) \vee (b \to c)$

$\Leftrightarrow (\sim a \vee c) \vee (\sim b \vee c)$

Equivalence

$\Leftrightarrow (\sim a \vee \sim b) \vee (c \vee c)$

By associative and commutative laws

$\Leftrightarrow \sim(a \wedge b) \vee c \quad (\because (c \vee c) \Leftrightarrow c)$

$\Leftrightarrow (a \wedge b) \to c$ Equivalence

= L. H. S

$\therefore$ The given formula is a tautology

(B) R.H.S $\Leftrightarrow (p \to q) \wedge (p \to r)$

$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee r)$

Equivalence

$\Leftrightarrow \sim p \vee (q \wedge r)$

Distributive law

$\Leftrightarrow p \to (q \wedge r)$

Equivalence

= L. H. S

(C) R.H.S $\Leftrightarrow (p \to r) \wedge (q \to r)$

$\Leftrightarrow (\sim p \vee r) \wedge (\sim q \vee r)$

Equivalence

$\Leftrightarrow (\sim p \wedge \sim q) \vee r)$

Distributive law

$\Leftrightarrow \sim(p \vee q) \vee r$

Demorgan's law

$\Leftrightarrow (p \vee q) \to r$

Equivalence

= L.H.S

(D) When a is true, b is false and c is true; the given formula has truth value false.

$\therefore$ It is not a tautology

07. Ans: (b)

Sol: **Case 1:** When p is true, the given formula becomes

$(T \to q) \wedge (T \to r) \wedge (q \to \bar{r}) \wedge T$

$\Leftrightarrow q \wedge r \wedge (q \to \bar{r})$

$\Leftrightarrow (q \wedge r) \wedge \sim (q \wedge r)$

$\Leftrightarrow F$

Case 2: When p is false, the given formula becomes

$\{(F \to q) \wedge (F \to r) \wedge (q \to \bar{r}) \wedge F\}$

$\Leftrightarrow F$

$\therefore$ The given statement formula is a contradiction.

08. Ans: (d)

Sol: The truth table of a propositional function in n variables contain 2^n rows. In each row the function can be true or false.

By product rule, number of non equivalent propositional functions (different truth tables) possible $= 2^{(2^n)}$

09. Ans: (c)

Sol: A set of connectives is said to be functionally complete, if equivalent form of every statement formula can be written with those connectives.

(a) $\{\vee, \sim\}$ is functionally complete because the other connectives can be expressed by these two connectives.

$(P \wedge Q) \Leftrightarrow \sim(\sim P \vee \sim Q)$

$(P \rightarrow Q) \Leftrightarrow (\sim P \vee Q)$

$(P \leftrightarrow Q) \Leftrightarrow \sim(\sim(\sim P \vee Q) \vee \sim(\sim Q \vee P))$

(b) $\{\wedge, \sim\}$ is functionally complete

$(P \vee Q) \Leftrightarrow \sim(\sim P \wedge \sim Q)$

$(P \rightarrow Q) \Leftrightarrow \sim(P \wedge \sim Q)$

$(P \leftrightarrow Q) \Leftrightarrow \sim(P \wedge \sim Q) \wedge \sim(Q \wedge \sim P)$

(c) The set $\{\wedge, \vee\}$ is not functionally complete, because, we cannot express 'not' operation using the connectives $\wedge$ and $\vee$

(d) We have, $(p \vee q) \Leftrightarrow (\sim p \rightarrow q)$

$\therefore$ $\{\rightarrow, \sim\}$ is a functionally complete set.

10. Ans: (b)

Sol: Argument I:

This argument is not valid, because it comes under fallacy of assuming the converse.

Argument II:

This argument is valid, by the rule of modus tollens.

Argument III:

Let P : It rains and Q : Erik is sick

In symbolic form the argument is

$P \rightarrow Q$

$\underline{\sim P}$

$\therefore \sim Q$

This argument is not valid, because when P is false and Q is true, the premises are true but conclusion is false.

11. Ans: (a)

Sol: Argument I

1. $\sim p \rightarrow (q \rightarrow \sim w)$	premise
2. $\sim s \rightarrow q$	premise
3. $\sim t$	premise
4. $(\sim p \vee t)$	premise
5. $\sim p$	(3),(4), disjunctive syllogism
6. $q \rightarrow \sim w$	(1), (5), modus ponens
7. $(\sim s \rightarrow \sim w)$	(2), (6), transitivity
8. $(w \rightarrow s)$	(7), contrapositive property

$\therefore$ Argument I is valid.

Argument II

We cannot derive the conclusion from the premises, by applying the rules of inference. Further, when p, q, r, t and w has truth values true, we have all the premises are true but conclusion is false.

$\therefore$ Argument II is not valid

12. Ans: (d)

Sol: (a) The given formula is equivalent to the following argument

(1) $(a \wedge b) \rightarrow c$

$\underline{(2)\ (a \rightarrow b)}$

$\therefore (a \rightarrow c)$

Proof:

(3) a new premise to apply conditional proof

(4) b (2), (3), modus ponens

(5) $(a \wedge b)$ (3), (4), conjunction

(6) c (1), (5), modus ponens

∴ The argument is valid (c.p)

(b) The argument is

(1) $\sim(a \wedge b)$

(2) $(b \vee c)$

(3) $\underline{(c \rightarrow d)}$

$\therefore (a \rightarrow d)$

(4) a new premise to apply C.P

(5) ~ b (1), (4), conjunctive syllogism(c.s)

(6) c (2), (5), D.S

(7) d (3), (6), M.P

∴ The given argument is valid (conditional proof)

(c) The given formula is equivalent to the following argument

(1) a

(2) $\underline{(a \rightarrow b) \vee (c \wedge d)}$

$\therefore (\sim b \rightarrow c)$

Proof:

(3) ~ b new premise to apply c.p

(4) $(a \wedge \sim b)$ (1), (3), conjunction

(5) $\sim(a \rightarrow b)$ (4), E_{17}

(6) $(c \wedge d)$ (2), (5), D.S

(7) c (6), simplification

∴ The argument is valid (c.p)

(d) When a is false, b is true and c is false, the given formula has truth value false.

∴ It is not valid

13. Ans: (b)

Sol: From the truth table

$(x * y) \Leftrightarrow (x \wedge \sim y)$

$(p \vee q) \Leftrightarrow \sim(\sim p \wedge \sim q)$

$\Leftrightarrow \sim(\sim p * q)$

14. Ans: (d)

Sol: If $\{(a \rightarrow b) \rightarrow (a \rightarrow c)\}$ has truth value false, then 'a' has truth value true, b has truth value true, and c has truth value false.

For these truth values, only the compound proposition given in option (d) has truth value true.

15. Ans: (c)

Sol: Let us denote the formula by $P \rightarrow Q$,

Where $P = (a \wedge b) \rightarrow c$ and $Q = a \rightarrow (b \vee c)$.

(a) Here Q is false only when a is *true*, b is *false*, and c is *false*. For these truth values P has truth value *true*.

∴ $(P \rightarrow Q)$ has truth value *false*.

∴ The given formula is not a tautology.

(b) When a is *true*, b is *true* and c is *false*; $(P \rightarrow Q)$ has truth value *true*.

∴ The given formula is not a contradiction

(c) The given formula is true in one case and false in other cases

∴ The given formula is a contingency.

16. Ans: (d)

Sol: (a) L. H. S $\Leftrightarrow \sim a \vee (\sim b \vee c)$ Equivalence

$\Leftrightarrow (\sim a \vee \sim b) \vee c$ Associativity

$\Leftrightarrow (a \wedge b) \rightarrow c$ Equivalence

= R. H. S.

(b) L. H. S $\Leftrightarrow (a \wedge b) \rightarrow (\sim c \rightarrow d)$ Equivalence

$\Leftrightarrow (a \wedge b \wedge \sim c) \rightarrow d$ Equivalence

= R. H. S

(c) L. H. S. $= (a \rightarrow (a \vee b)) = T$

R. H. S. $= \sim a \rightarrow (a \rightarrow b)$

$= a \vee (\sim a \vee b)$

$= (a \vee \sim a) \vee b$

$= T \vee b$

$= T$

$\therefore$ L. H. S = R. H. S

(d) When a is false and b is true

We have L. H. S. = F and R. H. S. = T

$\therefore$ Option (d) is not true because

L. H. S. $\neq$ R. H. S.

17. Ans: (a)

Sol: The given formula is equivalent to the following argument

1. $\sim p \rightarrow (q \rightarrow \sim w)$	Premise
2. $\sim s \rightarrow q$	Premise
3. $\sim (w \rightarrow t)$	Premise
4. $\underline{(\sim p \vee t)}$	Premise
$\therefore s$	
5. $(w \wedge \sim t)$	3), Equivalence
6. w	5), Simplification
7. $\sim t$	5), Simplification
8. $\sim p$	4), 7) Disjunctive syllogism
9. $(q \rightarrow \sim w)$	1), 8) Modus ponens
10. $\sim q$	9), 6) Modus tollens
11. s	2), 10) Modus tollens

$\therefore$ The argument is valid

Hence, the given statement is a tautology

18. Ans: (d)

Sol: (a) When p is true, q is false and r is true; the premises are true and conclusion is false. Therefore, the argument is not valid.

(b) When p is false and r is true, the premises are true and conclusion is false.

$\therefore$ The argument is not valid.

(c) When p is true, q is true, r is true and s is false; then the given argument is not valid

(d) The premises are true only when p is true, q is true, r is true, s is false and t is true.

$\therefore$ Whenever the premises are true, the conclusion is also true.

$\therefore$ The argument is valid

19. Ans: (c)

Sol: Argument I is equivalent to the following argument.

$$\{p \rightarrow r, q \rightarrow r, p \vee q\} \Rightarrow r$$

This argument is valid by the rule of dilemma.

$\therefore$ The given argument is also valid by Conditional proof (C.P).

Argument II is equivalent to the following argument

$$\{p \rightarrow q, p \rightarrow r, p\} \Rightarrow (q \wedge r)$$

1) $p \rightarrow q$	premise
2) $p \rightarrow r$	premise
$\underline{3)\ p}$	new premise to apply (C.P)
$\therefore (q \wedge r)$	

4) q (1), (3), modus ponens
5) r (2), (3), modus ponens
6) $q \wedge r$ (4), (5), conjunction

$\therefore$ Argument II is valid (Conditional Proof).

20. Ans: (a)

Sol: The given formula can be written as

$((a\to b) \wedge (c\to d) \wedge (\sim b \vee \sim d)) \to (\sim a \vee \sim c)$

This formula is valid, by the rule of destructive dilemma.

First order Logic

21. Ans: (a)

Sol: To negate a statement formula we have to replace $\forall_x$ with $\exists_x$, $\exists_x$ with $\forall_x$ and negate the scope of the quantifiers.

$\sim\{\exists_x \{P(x) \wedge \sim Q(x)\}\} = \forall x \{P(x) \to Q(x)\}$

(Use the equivalence $\sim (P\to Q) \Leftrightarrow (P \wedge \sim Q)$)

22. Ans: (c)

Sol: $\forall_x (B(x) \to I(x))$

$\Leftrightarrow \sim\sim [\forall_x (B(x) \to I(x))]$

$\Leftrightarrow \sim (\exists_x (B(x) \wedge \sim I(x)))$

23. Ans: (a)

Sol: To negate a statement formula we have to replace $\forall_x$ with $\exists_x$, $\exists_x$ with $\forall_x$ and negate the scope of the quantifiers. Use the equivalence

$\sim (P \to Q) \Leftrightarrow (P \wedge \sim Q)$

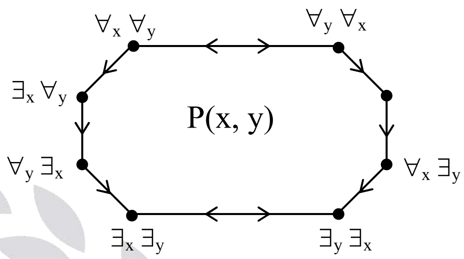
24. Ans: (c)

Sol: $\exists_y \forall_x P(y, x) \to \forall_y \exists_x P(x, y)$

$\Leftrightarrow \exists_x \forall_y P(x, y) \to \forall_y \exists_x P(x, y)$

($\because$ x and y are dummy variables)

Which is valid as per the relationship diagram shown below



The remaining options are not true as per the diagram.

25. Ans: (d)

Sol: The given formula is equivalent to

$$\exists_x \{\forall_y (\sim\alpha) \vee \forall_z(\sim\beta)\}$$

By the rule of negation

$\Leftrightarrow \exists_x \{\exists_y(\alpha)\to \forall_z(\sim\beta)\}$ .... (1)

(By equivalence) $\therefore$ Option (B) is true

$\Leftrightarrow \exists_x \{\exists_z(\beta)\to \forall_y(\sim\alpha)\}$ .... (2)(from (1), by contrapositive equivalence)

$\therefore$ Option (A) is true

$\Rightarrow \exists_x (\forall_z(\sim\beta) \vee \forall_y(\sim\alpha)\}$

(from (2), equivalence) $\therefore$ Option (C) is true

$\therefore$ Options (A), (B) and (C) are equivalent to the given formula.

$\therefore$ Option (D) is not true, because an implication is not equivalent to its converse.

26. Ans: (d)

Sol: S_1 is true

Once we select any integer n, the integer m = 5 – n does exist and

n + m = n + (5 – n) = 5

S_2 is true, because if we choose n=1 the statement nm = m is true for any integer m.

S_3 is false, for example, when m = 0 the statement is false for all n

S_4 is false, here we cannot choose n = – m, because m is fixed.

27. Ans: (b) & (d)

Sol: (a) L.H.S $\Leftrightarrow \exists_x (A(x) \rightarrow B(x))$

$\Leftrightarrow \exists_x (\sim A(x) \vee B(x))$, E_{16}

$\Leftrightarrow \exists_x \sim A(x) \vee \exists_x B(x)$, E_{23}

$\Leftrightarrow \forall_x A(x) \rightarrow \exists_x B(x)$, E_{16}

= R.H.S

(b) L.H.S $\Leftrightarrow \{\forall_x \sim A(x) \vee \forall_x B(x))$

$\Rightarrow \forall_x (\sim A(x) \vee B(x))$

$\Rightarrow \forall_x (A(x) \rightarrow B(x)$ = R.H.S

But converse is not true

$\therefore$ (b) is false

(c) valid equivalence

(d) not valid (converse is not true)

28. Ans: (b)

Sol: (a) The given formula is valid by conditional proof, if the following argument is valid.

(1) $\forall x \{ P(x) \rightarrow Q(x) \}$

(2) $\underline{\forall x\, P(x)}$ new premise to apply C.P

$\therefore \forall x\, Q(x)$

Proof:

(3) $P(a) \rightarrow Q(a)$	(1) , U.S
(4) P(a)	(2), U.S
(5) Q(a)	(3),(4), M.P
(6) $\forall x\, Q(x)$	(5) , U.S

$\therefore$ The given formula is valid (C.P)

(b) The statement need not be true.

Let c and d are two elements in the universe of discourse, such that P(c) is true and P(d) is false and Q(c) is false and Q(d) is false.

Now, the L.H.S of the given statement is true but R.H.S is false.

$\therefore$ The given statement is not valid.

(c) $\forall_x (P(x) \vee Q(x)) \Rightarrow (\forall_x P(x) \vee \exists_x Q(x))$

Indirect proof:

1) $\forall_x (P(x) \vee Q(x))$ Premise

2) $\sim (\forall_x P(x) \vee \exists_x Q(x))$

New premise to apply Indirect proof

3) $\exists_x \sim P(x) \wedge \forall_x \sim Q(x)$

(2), Demorgan's law

4) $\exists_x \sim P(x)$	(3), Simplification
5) $\forall_x \sim Q(x)$	(3), Simplification
6) $\sim P(a)$	(4), E.S
7) $\sim Q(a)$	(5), U.S
8) $(\sim P(a) \wedge \sim Q(a))$	(6), (7),Conjunction
9) $\sim (P(a) \vee Q(a))$	(8), Demorgan's law
10) $(P(a) \vee Q(a))$	(1), U.S
11) F	(9), (10), Conjunction

$\therefore$ valid (Indirect proof)

S_2: The argument is

1) $\forall_x \forall_y (P(x, y) \rightarrow W(x, y))$

2) $\underline{\sim W(a, b)}$

$\therefore \sim P(a, b)$

(d) $\forall_x \{ P(x) \vee Q(x) \}$ follows from

$(\forall_x P(x) \vee \forall_x Q(x))$

$\therefore$ The given statement is valid.

29. Ans: (c)

Sol: Consider

Argument I

1. $\forall x\{p(x) \vee q(x)\}$	premise
2. $\forall x[\{\sim p(x) \wedge q(x)\} \rightarrow r(x)]$	premise
3. $\{p(a) \vee q(a)\}$ (1),	universal specification
4. $\{\sim p(a) \wedge q(a)\} \rightarrow r(a)$	(2), U. S.
5. $\sim r(a)$	new premise to apply C.P
6. $\sim\{\sim p(a) \wedge q(a)\}$	(5), (4), disjunctive syllogism
7. $\{p(a) \vee \sim q(a)\}$	(6), demorgan's law
8. $\{p(a) \vee q(a)\} \wedge \{p(a) \vee \sim q(a)\}$	(3), (7) conjunction
9. $p(a) \vee \{q(a) \wedge \sim q(a)\}$	(8), distributive law
10. $p(a) \vee F$	from (9)
11. $p(a)$	from (10)
12. $\{\sim r(a) \rightarrow p(a)\}$	from (11),
13. $\forall x\{\sim r(x) \rightarrow p(x)\}$	from (12), U.G

$\therefore$ The argument is valid (C.P)

Argument II

1. $\forall x[p(x) \rightarrow \{q(x) \wedge r(x)\}]$	premise
2. $\exists x\{p(x) \wedge s(x)\}$	premise
3. $p(a) \wedge s(a)$	(2), E. S
4. $p(a)$	(3), simplification
5. $s(a)$	(3), simplification
6. $p(a) \rightarrow \{q(a) \wedge r(a)\}$	(1), U.S
7. $q(a) \wedge r(a)$	(4), (6), modus ponens
8. $r(a)$	(7), simplification
9. $r(a) \wedge s(a)$	(5), (8), conjunction
10. $\exists x\{r(x) \wedge s(x)\}$	(9), E. G

$\therefore$ The argument is valid (C.P)

30. Ans: (d)

Sol: The given statement can be represented by S_2.

Further, $S_1 \equiv S_2 \equiv S_3$

$\therefore$ Option (d) is correct

31. Ans: (b)

Sol: **S_1** is equivalent to the following argument

1) $\exists_x P(x)$	premise
2) $\exists_x \{P(x) \rightarrow Q(x)$	premise
$\therefore \exists_x Q(x)$	

Here, we cannot combine (1) and (2), to get the conclusion, because in both the formulae existential quantifiers are used.

S_2 is equivalent to the following argument

1) $\forall_x P(x)$	premise
2) $\forall_x \{P(x) \rightarrow Q(x)$	premise
$\therefore \exists_x Q(x)$	
3) $\forall_x P(x)$	from (1) and (2), by modus ponens,
4) $\exists_x P(x)$	from (3)

$\therefore S_2$ is valid

S_3:

1) $\forall_x P(x)$	premise
2) $\exists_x Q(x)$	premise
$\therefore \exists_x \{(P(x) \wedge Q(x)$	
(3) Q(a)	(2), E.S.
(4) P(a)	(1) U.S.
(5) $P(a) \wedge Q(a)$	(3), (4), conjunction
(6) $\exists_x \{P(x) \wedge Q(x)\}$	(5), U.G.

$\therefore S_3$ is valid

32. Ans: (d)

Sol: I) Let D (x) : x is a doctor

C (x) : x is a college graduate

G (x) : x is a golfer

The given argument can be written as

1) $\forall_x \{D(x) \rightarrow C(x)\}$

2) $\exists_x \{D(x) \wedge \sim G(x)\}$

$\therefore \exists_x \{G(x) \wedge \sim C(x)\}$

3) $\{D(a) \wedge \sim G(a)\}$	2), Existential Specification
4) $\{D(a) \rightarrow C(a)\}$	1), Universal Specification
5) D (a)	3), Simplification
6) $\sim$ G (a)	3), Simplification
7) C (a)	4), 5), Modus ponens Conjunction
8) $\{C(a) \wedge \sim G(a)\}$	7), 6), Conjunction
9) $\exists_x \{G(x) \wedge \sim C(x)\}$	8), Existential Generalization

The argument is not valid

II) Let M (x) x is a mother

N (x) x is a male

P (x) : x is a politician

The given argument is

1) $\forall_x \{M(x) \rightarrow \sim N(x)\}$

2) $\exists_x \{N(x) \wedge P(x)\}$

$\therefore \exists_x \{P(x) \wedge \sim M(x)\}$

3) $N(a) \wedge P(a)$	2), Existential Specification
4) $M(a) \rightarrow \sim N(a)$	1), Universal Specification
5) $N(a)$	3), Simplification
6) $P(a)$	3), Simplification
7) $\sim M(a)$	4), 5), Modus tollens
8) $\{P(a) \wedge \sim M(a)\}$	6), 7), Conjunction
9) $\exists_x\{P(x) \wedge \sim M(x)\}$	8), Existential Generalization

$\therefore$ The argument is valid.

33. Ans: (b)

Sol: S_1 is false. For x = 0. There is no integer y such that '0 is a divisor of y'

S_2 is true. If we choose, x = 1, then the statement is true for any integer y

S_3 is true. If we choose, x = 1, then the statement is true for any integer y

S_4 is false, because there is no integer y which is divisible by all integers.

34. Ans: (b)

Sol: P(x): $x^2 - 7x + 10 = 0$

$\Rightarrow x = 2, 5$

Q(x): $x^2 - 2x - 3 = 0$

$\Rightarrow x = -1, 3$

S_1: For x = 3, Q (x) true and R(x) is false.

$\therefore S_1$ is not true.

S_2: For x = –1, Q(x) is true and R(x) is true.

$\therefore S_2$ is true

S_3: For the integer x = 1, P(x) is false

$\therefore P(x) \rightarrow R(x)$ is true for integer

$\therefore S_3$ is true.

35. Ans: (c)

Sol: (a) When y = 2, the given statement is false

(b) When x = 2 the given statement is false

(c) Solving the equations

$2x + y = 5$ and $x - 3y = -8$

we get x = 1 and y = 3

$\therefore$ The given statement is true

(d) Solving the equations

$3x - y = 7$ and $2x + 4y = 3$

we get $x = \frac{31}{14}$ an $y = -\frac{5}{14}$

$\therefore$ The given statement is false

36. Ans: (b)

Sol: Argument I:

1. $\exists_x A(x)$,
2. $\underline{\forall_x \sim\{A(x) \wedge Q(x)}$

$\therefore \exists_x Q(x)$

3. $A(a)$	from (1), by existential specification
4. $\sim\{A(a) \wedge Q(a)\}$	from (2), by universal specification
5. $(\sim A(a) \vee \sim Q(a))$	(4), demorgan's law
6. $\sim Q(a)$	from (2) and (5) by disjunctive syllogism.

$\therefore$ given argument is not valid.

Argument II:

1. $\{\exists_x \{(P(x) \vee Q(x)) \rightarrow R(x)\}$,
2. $\underline{\forall_x Q(x)\}}$

$\therefore \exists_x R(x)$

3. $\{(P(a) \vee Q(a)) \rightarrow R(a)\}$,	from (1) by existential specification.
4. $Q(a)$	from (2), by universal specification
5. $P(a) \vee Q(a)$	from (4) by addition
6. $R(a)$	from (3) and (5) by modus ponens
7. $\exists_x R(x)$	from (6) by existential genralization

$\therefore$ Argument II is valid.

---

37. Ans: (d)

Sol: The given statement can be written as, $\exists_x \{R(x) \wedge \sim S(x)\}$

Now, $\sim[\exists_x \{R(x) \wedge \sim S(x)\}]$

$\Leftrightarrow \forall_x \{\sim R(x) \vee S(x)\}$

$\Leftrightarrow \forall_x \{R(x) \rightarrow S(x)\}$

$\therefore$ Option (d) is correct.

38. Ans: (c)

Sol: (a) If x is not free then

$\forall x \{W \vee A(x)\} \Leftrightarrow W \vee \forall_x A(x)$ is valid

Refer page 23 of material book.

(b) L.H.S $\Leftrightarrow \exists x \{\sim A(x) \vee W\}$

$\Leftrightarrow \{\exists x \sim A(x)\} \vee \exists x\ W$

$\Leftrightarrow \forall x\, A(x) \rightarrow W$

(c) $\forall x \{A(x) \rightarrow W\}$

$\Leftrightarrow \forall x \{\sim A(x) \vee W\}$ E_{16}

$\Leftrightarrow \{\forall x \sim A(x)\} \vee W$ using (a)

$\Leftrightarrow \{\exists x\, A(x)\} \rightarrow W$ E_{16}

$\neq$ R.H.S

(d) valid refer derivation of option (c).

2. Combinatorics

01. Ans: 2

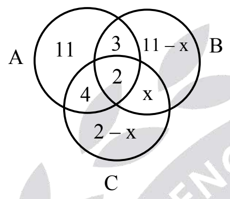
Sol: $\{n(A \cup B \cup C)\}$ = sum of the elements in all the regions of the diagram.

$= 33 - x$

$\Rightarrow n(A \cup B \cup C) = 31 = 33 - x$

$\Rightarrow x = 2$

$\therefore n(\overline{A} \cap B \cap C) = 2$



02. Ans: 600

Sol: By principle of inclusion and exclusion,

Number of integers divisible by 5 or 6 or 8

$= 200 + 166 + 125 - 33 - 25 - 41 + 8$

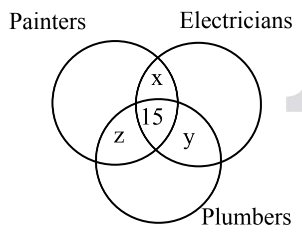
$= 400$

Required number of integers $= 1000 - 400$

$= 600$

03. Ans: (d)

Sol:



By the principle of inclusion and exclusion,

$80 = 45+50+50-(x+15)-(y+15)-(z+15)+15$

$\Rightarrow x + y + z = 35$

Required number of candidates

$= x + y + z + 15 = 50$

04. Ans: 86

Sol: $n(A \cup B \cup C \cup D)$

$= 42 + 36 + 28 + 24 - 6\,(12) + 4\,(8) - 4$

$= 86$

05. Ans: (c)

Sol: If n is even, then number of bit strings of length n which are palindromes $= 2^{\frac{n}{2}}$.

If n is odd, then number of bit strings of length n which are palindromes $= 2^{\frac{n+1}{2}}$

$\therefore$ Required number of bit strings $= 2^{\left\lceil \frac{n}{2} \right\rceil}$.

06. Ans: 45

Sol: For maximum number of points of intersection, we have to draw 10 lines so that no three lines are concurrent. In that case, each point correspond to a pair of distinct straight lines.

$\therefore$ Maximum number of points of intersection = number of ways we can choose two straight lines out of 10 straight lines = C (10, 2) = 45

07. Ans: 1,13,322

Sol: The number of 4-digit number $= \angle 4 = 24$

Each digit occur 6 times in every one of the 4 positions.

The sum of the digits = 17

Hence, sum of these 24 number $= 6.(17)\,(1000 + 100 + 10 + 1) = 113322$

08. Ans: 4096

Sol: In a binary matrix of order 3×4 we have 12 elements. Each element we can choose in 2 ways.

By product rule,

Required number of matrices = 2^{12} = 4096

09. Ans: 188

Sol: An English movie and a telugu movie can be selected in (6)(8) = 48 ways

A telugu movie and a hindi movie can be selected in (8).(10) = 80 ways

A hindi movie and an English movie can be selected in (10)(6) = 60 movies

Required number of ways = 48 + 80 + 60

= 188

10. Ans: 738

Sol: Number of 1-digit integers = 9

Number of 2-digit integers with distinct digits = (9) (9)

= 81

Number of 3-digit integers with distinct digits = (9).(9).(8)

= 648

Required number of integers = 9+81+648

= 738

11. Ans: 2673

Sol: **Case(i):** If the first digit is 6, then each of the remaining digits we can choose in 9 ways.

Case(ii): If the first digit is not 6, then first digit we can choose in 8 ways, digit 6 can appear in 3 ways and each of the remaining digits we can choose in 9 ways.

Required number of integers

= (9)(9)(9) + (8)(3)(9)(9)

= 2673

12. Ans: 2940

Sol: Consider an integer with 5 digits.

Digit 3 can appear in 5 ways

Digit 4 can appear in 4 ways

Digit 5 can appear in 3 ways

Each of the remaining digits we can choose in 7 ways.

By product rule,

Required number of integers

= (5)(4)(3)(7)(7) = 2940

13. Ans: 89

Sol: Each Tennis match eliminates one player and we have to eliminate 89 players.

∴ We have to conduct 89 matches.

14. Ans: 243

Sol: Each element of A can appear in the subsets in 3 ways.

Case 1: The element appears in P but does not appear in Q.

Case 2: The element appears in Q and does not appear in P.

Case 3: The element does not appear in P and does not appear in Q.

By product rule,

Required number of ways = 3^n = 3^5

= 243

15. Ans: 150

Sol: Required number of ways = Number of onto functions possible from persons to rooms

$= 3^5 - C(3, 1)\, 2^5 + C(3, 2)\,.\, 1^5$

$= 243 - 3\,(32) + 3$

$= 150$

16. Ans: P(10, 6) = 151200

Sol: Required number of ways

= Number of ways we can map the 6 persons to 6 of the 10 books

= P(10,6)

= 151200

17. Ans: (a)

Sol: The 3 women can speak as a group in $\angle 3$ ways.

The women group can speak with other 4 men in $\angle 5$ ways.

∴ Required number of ways

$= \angle 5.\ \angle 3$

$= 720$

18. Ans: 2880

Sol: First girls can sit around a circle in $\angle 4$ ways.

Now there are 5 distinct places among the girls, for the 4 boys to sit.

Therefore, the boys can sit in P(5, 4) ways.

By product rule,

Required number of ways = $\angle 4$.P(5, 4)

= 2880

19. Ans: 1152

Sol: Consider 8 positions in a row marked 1, 2, 3,….., 8.

Case 1: Boys can sit in odd numbered positions in $\angle 4$ ways and girls can sit in even numbered positions in $\angle 4$ ways.

Case 2: Boys can sit in even numbered positions in $\angle 4$ ways and girls can sit in odd numbered positions in $\angle 4$ ways.

Required number of ways

$= \angle 4.\angle 4 + \angle 4.\angle 4 = 1152$

20. Ans: 325

Sol: Number of signals we can generate using 1 flag = 5

Number of signals we can generate using two flags = P(5,2) = 5.4 = 20 and so on.

Required number of signals

= 5 + P(5,2) + P(5,3) + P(5,4) + P(5,5)

= 325

21. Ans: 10^6

Sol: Each book we can give in 10 ways.

By product rule, required number of ways $= 10^6$

22. Ans: 243

Sol: Each digit of the integer we can choose in 3 ways.

By product rule,

Required number of integers $= 3^5$

$= 243$

23. Ans: 12600

Sol: Required number of permutations

$$= \frac{\angle 10}{\angle 2.\angle 3.\angle 4} = 12{,}600$$

24. Ans: 210

Sol: Required number of binary sequences

$$= \frac{\angle 10}{\angle 6.\angle 4} = 210$$

25. Ans: 252

Sol: Required number of outcomes

$$= \frac{\angle 10}{\angle 5.\angle 5} = 252$$

26. Ans: 2520

Sol: Required number of ways

$$= \frac{\angle 10}{\angle 3.\angle 2.\angle 5}$$

$$= 2520$$

27. Ans: 2520

Sol: Required number of ways

= number of ordered partitions

$$= \frac{\angle 10}{\angle 3.\angle 2.\angle 5} = 2520$$

28. Ans: 945

Sol: Required number of ways = Number of unordered partitions of a set in to 5 subjects of same size $= \frac{\angle 10}{(\angle 2.\angle 2.\angle 2.\angle 2.\angle 2).\angle 5}$

$$= 945$$

29. Ans: 600

Sol: We can select 4 men in C(5, 4) ways. Those 4 men can be paired with 4 women in P(5, 4) ways.

∴ Number of possible selections

= C(5, 4). P(5, 4)

= 5. (120) = 600

30. Ans: 120

Sol: The 3 zeros can appear in the sequence in C(10,3) ways. The remaining 7 positions of the sequence can be filled with ones in only one way.

Required number of binary sequences

= C(10, 3).1 = 120

31. Ans: 35

Sol: Consider a string of 6 ones in a row. There are 7 positions among the 6 ones for placing the 4 zeros. The 4 zeros can be placed in C(7,4) ways.

Required number of binary sequences

= C(7, 4) = C(7, 3)

= 35

32. Ans: 252

Sol: To meet the given condition, we have to choose 5 distinct decimal digits and then arrange them in descending order. We can choose 5 distinct decimal digits in C(10,5) ways and we can arrange them in descending order in only one way.

Required number of ways = C(10,5).1 = 252

33. Ans: 2n(n–1)

Sol: We have 2n persons.

Number of handshakes possible with 2n persons = C(2n,2)

If each person shakes hands with only his/her spouse, then number of handshakes possible

= n

Required number of handshakes

= C(2n, 2) – n = 2n(n –1)

34. Ans: 1092

Sol: In a chess board, we have 9 horizontal lines and 9 vertical lines. A rectangle can be formed with any two horizontal lines and any two vertical lines.

Number of rectangles possible

$= C(9,2).\ C(9,2) = (36)(36) = 1296$

Number of squares in a chess board

$= 1^2 + 2^2 + 3^2 + \ldots + 8^2 = 204$

Every square is also a rectangle.

Required number of rectangles which are not squares = 1296 – 204 = 1092

35. Ans: 84

Sol: Between H and R, we have 9 letters. We can choose 3 letters in C(9, 3) ways and then arrange them between H and R in alphabetical order in only one way.

Required number of letter strings = C(9,3).1

= 84

36. Ans: 210

Sol: We can choose 6 persons in C(10, 6) ways

We can distinct 6 similar books among the 6 persons in only one ways

∴ Required number of ways

= C(10, 6). 1

= C(10, 4) = 210

37. Ans: 1001

Sol: Required number of ways = V(5,10)

$V(n,k) = C(n–1+k, k)$

⇒ V(5,10) = C(14,10)

= C(14,4)

= 1001

38. Ans: 455

Sol: To meet the given condition, let us put 1 ball in each box, The remaining 12 balls we can distribute in V(4,12) ways.

Required number of ways = V(4,12).1

= C(15,12) = C(15,3) = 455

39. Ans: 3003

Sol: The number of solutions to the inequality is same as the number or solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$$

Where $x_6 \geq 0$

The required number of solutions = V(6,10)

= C(15,10) = C(15,5) = 3003

40. Ans: 10

Sol: Let $x_1 = y_1 + 3$, $x_2 = y_2 – 2$, $x_3 = y_3 + 4$

The given equation becomes

$y_1 + y_2 + y_3 = 3$

Number of solutions to this equation

= V(3, 3)

= C(5, 3) = 10

∴ Required number of solutions = 10

41. Ans: 63

Sol: Let X_1 = units digit, X_2 = tens digit and X_3= hundreds digit

Number of non negative integer solutions to the equation

$X_1 + X_2 + X_3 = 10$ is

V (3, 10) = C (12, 10) = C (12, 2) = 66

We have to exclude the 3 cases where $X_i = 10$ (i = 1, 2, 3)

Required number of integers = 66 – 3 = 63

42. Ans: 1001

Sol: Let $x_1 = y_1 + 2$, $x_2 = y_2 + 3$, $x_3 = y_3 + 4$, $x_4 = y_4 + 5$ and $x_5 = y_5 + 6$

Required number of solutions = Number of non negative integer solutions to the equation

$y_1 + y_2 + y_3 + y_4 + y_5 = 10.$

$= V(5, 10) = C(5 - 1 + 10, 10)$

$= C(14, 10)$

$= C(14, 4) = 1001$

43. Ans: 10

Sol: To meet the given conditions, let us put 2 books on each of the 4 shelves. Now we are left with 2 books to distribute among the 4 shelves. Which ever way we distribute the remaining books, the number of books on any shelf cannot exceed 4.

$\therefore$ Required number of ways = V (4, 2)

$= C(4 - 1 + 2, 2)$

$= C(5, 2) = 10$

44. Ans: (a)

Sol: This is similar to distributing n similar balls in k numbered boxes, so that each box contains atleast one ball.

If we put 1 ball in each of the k boxes, then we are left with (n–k) balls to distribute in k boxes.

Required number of ways = V(k, n – k)

$= C(k - 1 + n - k, n - k)$

$= C(n - 1, k - 1)$

45. Ans: 10800

Sol: The six symbols can be arranged in $\angle 6$ ways. To meet the given condition,

Let us put 2 blanks between every pair of symbols.

The number of ways we can arrange the remaining two blanks = V(5, 2)

$= C(5 - 1 + 2, 2) = 15$

$\therefore$ Required number of ways = $\angle 6 .(15)$

$= (720) .(15) = 10,800$

46. Ans: S_1, S_2, S_5, S_6

Sol: Average number of letters received by an apartment $= A = \frac{410}{50}$

$= 8.2$

Here, $\lceil A \rceil = 9$ and $\lfloor A \rfloor = 8$

By pigeonhole principle, S_1 and S_2 are necessarily true.

S_5 follows from S_1 and S_6 follows from S_2.

S_3 and S_4 need not be true.

47. Ans: 97

Sol: If we have n pigeon holes, then minimum number of pigeons required to ensure that atleast (k+1) pigeons belong to same pigeonhole = kn + 1

For the present example, n=12 and k+1= 9

Required number of persons = kn + 1

$= 8(12) + 1 = 97$

48. Ans: 26

Sol: By Pigeonhole principle,

Required number of balls = kn + 1

$= 5(5) + 1 = 26$

49. Ans: 39

Sol: The favorable colors to draw 9 balls of same color are green, white and yellow.

We have to include all red balls and all green balls in the selection of minimum number of balls. For the favorable colors we can apply pigeonhole principle.

Required number of balls = 6 + 8 + (kn + 1)

Where $k + 1 = 9$

and $n = 3$

$6 + 8 + (8 \times 3 + 1) = 39$

50. Ans: 4

Sol: Suppose $x \geq 6$,

Minimum number of balls required = kn +1=16 where $k + 1 = 6$ and $n = 3$.

$\Rightarrow 5(3) + 1 = 16$

Which is impossible

$\therefore x < 6$

Now, minimum number of balls required

$= x + (kn + 1) = 15$

where $k + 1 = 6$ and $n = 2$

$\Rightarrow x + 5(2) + 1 = 15$

$\Rightarrow x = 4$

51. Ans: 7

Sol: For sum to be 9, the possible 2-element subsets are{0,9}, {1, 8},{2, 7},{3, 6},{4, 5}

If we treat these subsets as pigeon holes, then any subset of S with 6 elements can have at least one of these subsets.

Since we need two such subsets, the required value of k = 7.

52. Ans: 7

Sol: If we divide a number by 10 the possible remainders are 0, 1, 2, …, 9.

Here, we can apply pigeonhole principle. The 6 pigeonholes are

{0}, {5}, {1, 9}, {2, 8}, {3,7}, {4, 6}

In the first two sets both $x + y$ and $x - y$ are divisible by 10. In the remaining sets either $x + y$ or $x - y$ divisible by 10.

$\therefore$ The minimum number of integers we have to choose randomly is 7.

53. Ans: 14

Sol: Every positive integer 'n' can be written as, $n = 2^k m$ where 'm' is odd and $k \geq 0$. Let us call m the odd part.

If we treat the odd numbers 1, 3, 5, …, 25 as pigeonholes then we have 13 pigeonholes.

Every element in S has on odd part and associated with one of the 13 pigeonholes.

The minimum value of k = 14

54. Ans: 6

Sol: For the difference to be 5, the possible combinations are

{1, 6}, {2, 7}, {3, 8}, {4, 9}, {5, 0}

If we treat them as pigeonholes, then we have 5 pigeonholes.

By pigeonhole principle, if we choose any 6 integers in S, then the difference of the two integers is 5.

55. Ans: 40

Sol: The distinct prime factors of 110 are 2, 5 and 11.

Required number of positive integers

$$= \phi(110)$$

$$= 110\left[\frac{(2-1).(5-1).(11-1)}{2.5.11}\right]$$

$$= 40$$

56. Ans: 48

Sol: The distinct prime factors of 180 are 2, 3 and 5.

Required number of +ve integers

$$= \phi(180)$$

$$= 180\left[\frac{(2-1).(3-1).(5-1)}{2.3.5}\right] = 48$$

57. Ans: 288

Sol: The distinct prime factors of 323 are 17 and 19.

Required number of positive integers

$$= \phi(323)$$

$$= 323\left[\frac{(17-1).(19-1)}{17.19}\right] = 288$$

58. Ans: 265

Sol: Required number of 1 – 1 functions

= number of derangements possible with 6 elements

$$= D_6 = \angle 6\left(\frac{1}{\angle 2} - \frac{1}{\angle 3} + \frac{1}{\angle 4} - \frac{1}{\angle 5} + \frac{1}{\angle 6}\right)$$

$$= 265$$

59. Ans: (i) 44 (ii) 76 (iii) 20 (iv) 89 (v) 119 (vi) 0

Sol: (i) Number of ways we can put 5 letters, so that no letter is correctly placed

$$= D_5 = \angle 5\left(\frac{1}{\angle 2} - \frac{1}{\angle 3} + \frac{1}{\angle 4} - \frac{1}{\angle 5}\right)$$

$$= 44$$

(ii) Number of ways in which we can put 5 letters in 5 envelopes = $\angle 5$

Number of ways we can put the letters so that no letter is correctly placed = D_5

Required number of ways = $\angle 5 - D_5$

= 120 – 44

= 76

(iii) Number of ways we can put the 2 letters correctly = C(5,2) = 10

The remaining 3 letters can be wrongly placed in D_3 ways.

Required number of ways = C(5,2) D_3

= (10) 2

= 20

(iv) Number of ways in which no letter is correctly placed = D_5

Number of ways in which exactly one letter is correctly placed = C(5,1) D_4

Required number of ways

= D_5 + C (5,1)D_4

= 44 +5 (9) = 89

(v) There is only one way in which we can put all 5 letters in correct envelopes.

Required number of ways = $\angle 5$ –1= 119

(vi) It is not possible to put only one letter in wrong envelope.

Required number of ways = 0

60. Ans: (i) 1936 (ii) 14400

Sol: (i) The derangements of first 5 letters in first 5 places = D_5

Similarly, the last 5 letters can be deranged in last 5 places in D_5 ways.

The required number of derangements

$= D_5D_5 = (44)(44)$

$= 1936$

(ii) Any permutation of the sequence in which the first 5 letters are not in first 5 places is a derangement. The first 5 letters can be arranged in last 5 places in $\angle 5$ ways. Similarly, the last 5 letters of the given sequence can be arranged in first 5 places in $\angle 5$ ways.

Required number of derangements

$= \angle 5 . \angle 5$

$= 14400$

61. Ans: 4!.D$_4$ = 216

Sol: First time, the books can be distributed in $\angle 4$ ways.

Second time, we can distribute the books in D_4 ways.

Required number of ways = $\angle 4 . D_4 = 216$

62. Ans: (d)

Sol: Given that

a_n = number of ways a path of length n is covered

Case 1: In the first move, if the marker is moved 2 steps ahead, then the remaining length can be covered in a_{n-2} ways.

Case 2: In the first move, if the marker is moved 3 steps ahead, then the remaining length can be covered in a_{n-3} ways.

63. Ans: (c)

Sol: Let a_n = number of n-digit quaternary sequences with even number of zeros

Case 1: If the first digit is not 0, then we can choose first digit in 3 ways and the remaining digits we can choose in a_{n-1} ways. By product rule, number of quaternary sequences in this case is $3a_{n-1}$.

Case 2: If the first digit is 0, then the remaining digits should contain odd number of zeros.

Number quaternary sequences in this case is $(a_{n-1} - 4^{n-1})$

$\therefore$ By sum rule, the recurrence relation is

$\Rightarrow a_n = 3a_{n-1} + (4^{n-1} - a_{n-1})$

$\Rightarrow a_n = 2a_{n-1} + 4^{n-1}$

64. Ans: (d)

Sol: **Case: (1)** If the first square is not red then it can be colored in 2 ways and the remaining squares can be colored in a_{n-1} ways.

Case (2) If the first square is colored in red, then second square can be colored in two ways and remaining squares can be colored in a_{n-2} ways.

By sum rule, the recurrence relation is

$a_n = 2\,a_{n-1} + 2\,a_{n-2}$

$a_n = 2\,(a_{n-1} + a_{n-2})$

65. Ans: (a)

Sol: **Case 1:** If the first digit is 1, then number of bit strings possible with 3 consecutive zeros, is a_{n-1}.

Case 2: If the first bit is 0 and second bit is 1, then the number of bit strings possible with 3 consecutive zeros is a_{n-2}.

Case 3: If the first two bits are zeros and third bit is 1, then number of bit strings with 3 consecutive zeros is a_{n-3}

Case 4: If the first 3 bits are zeros, then each of the remaining n–3 bits we can choose in 2 ways. The number of bit strings with 3 consecutive zeros in this case is 2^{n-3}.

$\therefore$ The recurrence relation for a_n is

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}.$$

66. Ans: (a)

Sol: **Case(i):** If the first bit is 1, then the required number of bit strings is a_{n-1}

Case(ii): If the first bit is 0, then all the remaining bits should be zero

The recurrence relation for a_n is

$$a_n = a_{n-1} + 1$$

67. Ans: (a)

Sol: The recurrence relation is

$$a_n - a_{n-1} = 2n - 2 \;\ldots\ldots\ldots\ldots\; (1)$$

The characteristic equation is $t - 1 = 0$

Complementary function = $C_1 \cdot 1^n$

Here, 1 is a characteristic root with multiplicity 1.

Let particular solution = $(c\,n^2 + d\,n)$

Substituting in (1),

$(cn^2 + d\,n) - \{c\,(n-1)^2 + d(n-1)\} = 2n - 2$

$n = 1 \Rightarrow c + d = 0$

$n = 0 \Rightarrow -c + d = -2$

$\Rightarrow c = 1$ and $d = -1$

$\therefore$ P. S $= n^2 - n$

The solution is

$$a_n = C_1 + n^2 - n \;\ldots\ldots\ldots\ldots\ldots\; (1)$$

Using the initial condition, we get $C_1 = 1$

Substituting C_1 value in equation (1), we get

$\therefore a_n = n^2 - n + 2$

68. Ans: 8617

Sol: $a_n = a_{n-1} + 3(n^2)$

$n = 1 \Rightarrow a_1 = a_0 + 3(1^2)$

$n = 2 \Rightarrow a_2 = a_1 + 3(2^2)$

$= a_0 + 3(1^2 + 2^2)$

$n = 3 \Rightarrow a_3 = a_2 + 3(3^2)$

$= a_0 + 3(1^2 + 2^2 + 3^2)$

$a_n = a_0 + 3(1^2 + 2^2 + \ldots + n^2)$

$= 7 + \frac{1}{2} n\,(n+1)\,(2n+1)$

$a_{20} = 7 + \frac{1}{2}\,(20)\,(21)\,(41) = 8617$

69. Ans: (b)

Sol: The characteristic equation is $t^2 - t - 1 = 0$

$$\Rightarrow t = \frac{1 \pm \sqrt{5}}{2}$$

The solution is

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Using the initial conditions, we get $C_1 = \frac{1}{\sqrt{5}}$

and $C_2 = -\frac{1}{\sqrt{5}}$

70. Ans: (d)

Sol: The recurrence relation can be written as

$$(E^2 - 2E + 1)\,a_n = 2^{n+2}$$

The auxiliary equation is

$$t^2 - 2t + 1 = 0$$

$$t = 1, 1$$

C.F. = $(C_1 + C_2 n)$

$$\text{P.S.} = \frac{2^{n+2}}{(E-1)^2} = 4\left[\frac{2^n}{(E-1)^2}\right]$$

$$= 4\frac{2^n}{(2-1)^2} = 2^{n+2}$$

$\therefore$ The solution is

$$a_n = C_1 + C_2 n + 2^{n+2}$$

71. Ans: 31250

Sol: Let $b_n = a_n^2$

The given recurrence relation becomes

$$b_{n+1} - 5\,b_n = 0$$

The solution is $b_n = 4\,(5^n)$

$$\Rightarrow a_n = 2\left(\sqrt{5}\right)^n$$

$$\Rightarrow a_{12} = 31{,}250$$

72. Ans: (a)

Sol: Replacing n by n+1, the given relation can be written as

$$a_{n+1} = 4a_n + 3(n+1)\,2^{n+1}$$

$\Rightarrow (E - 4)\,a_n = 6\,(n+1)\,2^n$ ……….(1)

The characteristic equation is

$$t - 4 = 0 \Rightarrow t = 4$$

complementary function $= C_1 4^n$

Let particular solution is

$a_n = 2^n(cn + d)$ where c and d are undetermined coefficients.

Substituting in the given recurrence relation, we have

$2^n\,(c\,n + d) - 4\;2^{n-1}\,\{c(n-1)+d\} = 3n2^n$

$\Rightarrow (c\,n + d) - 2\{c(n-1)+d\} = 3n$

Equating coefficients of n and constants on both sides, we get

$c = -3$ and $d = -6$

$\therefore$ Particular solution $= 2^n\,(-3n - 6)$

Hence the solution is

$a_n = C_1 4^n - (3n + 6)\,2^n$ …………(2)

$x = 0 \Rightarrow 4 = C_1 - 6 \Rightarrow C_1 = 10$

$a_n = 10(4^n) - (3n + 6)\,2^n$

73. Ans: (a)

Sol: Required generating function

$= f(x) = 0 + x + 3x^2 + 9x^3 + 27x^4 + \ldots$

$= x(1 + 3x + 3^2 x^2 + 3^3 x^3 + \ldots \infty)$

$$= x\sum_{n=0}^{\infty} 3^n x^n = x.(1-3x)^{-1}$$

74. Ans: (d)

Sol: Required generating function

$f(x) = 0 + 0\,x + 1\,x^2 - 2x^3 + 3x^4 - 4x^5 + \ldots$

$= x^2\,(1 - 2x + 3x^2 - 4x^3 + \ldots \infty)$

$= x^2(1 + x)^{-2}$ (Binomial theorem)

75. Ans: (c)

Sol: The generating function is

$f(x) = 1 + 0.x + 1.x^2 + 0.x^3 + 1.x^4 + \ldots\ldots \infty$

$= 1 + (x^2) + (x^2)^2 + \ldots\ldots \infty$

$= (1 - x^2)^{-1}$

76. Ans: (a)

Sol: $(x^4 + 2x^5 + 3x^6 + 4x^7 + \ldots\infty)^5$

$= x^{20}\,(1 + 2x + 3x^2 + 4x^3 + \ldots\infty)^5$

$= x^{20}.\,[(1 - x)^{-2}]^5$

$= x^{20}\,[1 - x]^{-10}$

$$= x^{20}\sum_{n=0}^{\infty} C(n+9, n)x^n$$

Coefficient of $x^{27} = C(16, 7)$

$= C(16, 9)$

77. Ans: (b)

Sol: Required number of ways =

Number of non negative integer solutions to the equation

$x_1 + x_2 + x_3 = 15$ where $1 \le x_1, x_2, x_3 \le 7$

= coefficient of x^{15} in the expansion of f(x)

where, $f(x) = (x + x^2 + ... + x^7)^3$

$= x^3 (1 + x + x^2 + ... + x^6)^3$

$= x^3\left(\frac{1-x^7}{1-x}\right)^3$

$= x^3 (1 - 3x^7 + 3x^{14} - x^{21}) (1 - x)^{-3}$

$= x^3 - 3x^{10} + 3x^{17} - x^{24})$

$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

coefficient of $x^{15} = \frac{(13)(14)}{2} - 3\left(\frac{(6).(7)}{2}\right)$

$= 91 - 63 = 28$

3. Graph Theory

01. Ans: (a)

Sol: For any simple graph,

$$\delta(G)|V| \le 2|E| \le \Delta(G)|V|$$

$\Rightarrow \delta(G)\ (10) \le 2(16)$

$\Rightarrow \quad \delta(G) \le 3.2$

$\Rightarrow \quad \delta(G) \le 3$

02. Ans: 19

Sol: By sum of degrees of regions theorem, if degree of each vertex is k, then

$k|V| = 2|E|$

$\Rightarrow 4|V| = 2(38)$

$\Rightarrow \ |V| = 19$

03. Ans: (c)

Sol: If degree of each vertex is k,

$k|V| = 2|E|$

$\Rightarrow k|V| = 2\,(12)$

$\Rightarrow |V| = \frac{24}{k}$ (k = 1,2,3,4)

$\Rightarrow |V|$ = 24 or 12 or 8 or 6

$\therefore$ only option (c) is possible.

04. Ans: (e)

Sol: (a) {2, 3, 3, 4, 4, 5}

Here, sum of degrees

= 21, an odd number.

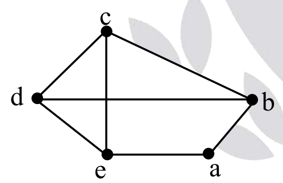
$\therefore$ The given sequence cannot represent a simple non directed graph

(b) {2, 3, 4, 4, 5}
In a simple graph with 5 vertices, degree of every vertex should be ≤ 4.
$\therefore$ The given sequence cannot represent a simple non directed graph.

(c) {1, 3, 3, 4, 5, 6, 6}
Here we have two vertices with degree 6. These two vertices are adjacent to all the other vertices. Therefore, a vertex with degree 1 is not possible.
Hence, the given sequence cannot represent a simple non directed graph.

(d) {0, 1, 2,….., n –1}
Here, we have n vertices, with one vertex having degree n-1. This vertex is adjacent to all the other vertices. Therefore, a vertex with degree 1 is not possible.
Hence, the given sequence, cannot represent a simple non directed graph.

(e) A graph with the degree sequence {2, 3, 3, 3, 3} is shown below.



05. Ans: (S_2) {6, 5, 5, 4, 3, 3, 2, 2, 2}

06. Ans: 8

Sol: By sum of degrees theorem,

$(5 + 2 + 2 + 2 + 1) = 2\,|E|$

$\Rightarrow |E| = 7$

$\therefore$ Number of edges in G = 7

$|E(G)| + |E(\overline{G})| = |E(K_6)|$

$\Rightarrow \quad 7 + |E(\overline{G})| = C(6, 2)$

$\Rightarrow \quad |E(\overline{G})| = 8$

07. Ans: 12

Sol: G is a tree

By sum of degrees theorem,

$n.1 + 2\,(2) + 4.\,(3) + 3.\,(4) = 2\,|E|$

$\therefore\ n + 28 = 2\,(|v| - 1)$

$= 2\,(n + 2 + 4 + 3 - 1)$

$\Rightarrow n + 28 = 2n + 16$

$\Rightarrow n = 12$

08. Ans: 8

Sol: G has 8 vertices with odd degree.

For any vertex $v \in G$,

Degree of v in G + degree of v in $\overline{G} = 8$

If degree of v in G is odd, then degree of v in $\overline{G}$ is also odd. If degree of v in G is even, then degree of v in $\overline{G}$ is also even.

$\therefore$Number of vertices with odd degree in G = 8

09. Ans: 27

Sol: By sum of degrees theorem, if degree of each vertex is atmost K,

then $\quad K|V| \geq 2\,|E|$

$\Rightarrow 5\,(11) \geq 2\,|E|$

$\Rightarrow \quad |E| \leq 27.5$

$\Rightarrow \quad |E| \leq 27$

10. Ans: (d)

Sol: (a) Let G be any graph of the required type.
Let p be the number of vertices of degree 3.
Thus, (12 – p) vertices are of degree 4.
Hence, according to sum of degrees theorem,

$$3p - 4(12 - p) = 56.$$

Thus, p = –8 (Which is impossible)

$\therefore$ Such a graph does not exist.

(b) Maximum number of edges possible in a simple graph with 10 vertices
C(10, 2) = 45

(c) Maximum number of edges possible in a bipartite graph with 9 vertices = $\lfloor \frac{9^2}{4} \rfloor$
= 20

$\therefore$ Such a graph does not exist.

(d) A connected graph with n vertices and n–1 edges is a tree. A tree is a simple graph.

11. Ans: (b)

Sol: G is a simple graph with 5 vertices.
For any vertex v in G,
deg(v) in G + deg(v) in $\overline{G}$ = 4
$\therefore$ The degree sequence $\overline{G}$ is
{4 – 3, 4 – 2, 4 – 2, 4 – 1, 4 – 0}
= {1, 2, 2, 3, 4}
= {4, 3, 2, 2, 1}

12. Ans: 455

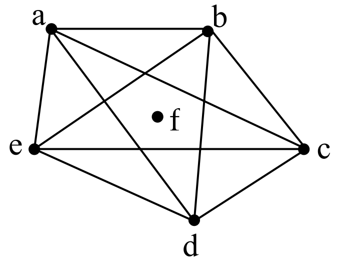
Sol: Maximum number of edges possible with 6 vertices is C(6, 2) = 15. Out of these edges, we can choose 12 edges in C(15, 2) ways.
$\therefore$ Number of simple graphs possible
$= C(15, 12) = C(15, 3) = \frac{15.14.13}{1.2.3} = 455$

13. Ans: 20

Sol: Maximum number of edges possible $= \left\lfloor \frac{n^2}{4} \right\rfloor$, where n = 10 = 25
We have, $|E(G)| + |E(\overline{G})| = |E(K_{10})|$
$\Rightarrow 25 + |E(\overline{G})| = C(10, 2)$
$\Rightarrow |E(\overline{G})| = 20$

14. Ans: 7

Sol: G is a star graph
$\therefore \chi(G) = 2$
The graph $\overline{G}$ is shown below



Here, the vertices a, b, c, d, e form a complete graph
$\therefore \chi(\overline{G}) = 5$
Now, $\chi(G) + \chi(\overline{G}) = 7$

15. Ans: 3

Sol: Applying welch-powel's algorithm we can see that 3 - colouring is possible

Vertex	d	a	b	c	e	f	g
Color	C_1	C_2	C_3	C_3	C_2	C_2	C_3

$\therefore \chi(G) \le 3$ .... (1)
since, G has cycles of odd length, $\chi(G) \ge 3$ .... (2)
From (1) and (2), we have $\chi(G) = 3$.

16. Ans: 2

Sol: In the given graph, all the cycles are of even length.
$\therefore$ G is a bipartite graph and every bipartite graph is 2-colorable
$\therefore$ Chromatic number of G = 2.

17. Ans: 5

Sol: $\overline{G}$ is a disconnected graph with two components, one component is the complete graph K_5 and the other component is the trivial graph with only an isolated vertex

$\therefore$ Chromatic number of $\overline{G} = 5$

18. Ans: (b)

Sol: $\alpha = n - 2\lfloor n/2 \rfloor + 2$

$\beta = n - 2\lceil n/2 \rceil + 4$

$\alpha + \beta = 2n - 2\{\lfloor n/2 \rfloor + \lceil n/2 \rceil\} + 6$

$= 2n - 2n + 6 = 6$

19. Ans: (c)

Sol: Chromatic number of $K_n = n$

If we delete an edge in K_{10}, then for the two vertices connecting that edge we can assign same color.

$\therefore$ Chromatic number = 9

20. Ans: (a)

Sol: Matching number of $K_{m,n}$ = minimum of {m, n}

Here, $G = K_{1,\,n-1}$

$\Rightarrow$ Matching number of G = 1

21. Ans: 13

Sol: A disconnected graph with 10 vertices and maximum number of edges has two components K_9 and an isolated vertex.

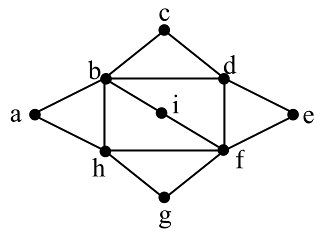
Matching number of $K_9 = \left\lfloor \frac{9}{2} \right\rfloor = 4$

$\therefore$ Matching number of G = 4

Chromatic number of G = 9

22. Ans: 4

Sol:



The graph has 9 vertices. The maximum number of vertices we can match is 8. A matching in which we can match 8 vertices is { a – b, c – d, e – f, g – h}

$\therefore$ Matching number of the graph = 4

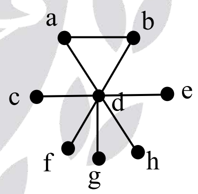
23. Ans: 2

Sol: The given graph is $K_{2,4}$

$\therefore$ Matching number = 2

24. Ans: 2

Sol: The given graph is

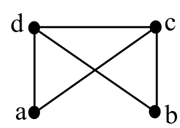


If we delete the edge {a,b} then the graph is a star graph. If we match a with b, then in the remaining vertices we can match only two vertices.

$\therefore$ Matching number = 2

25. Ans: 3

Sol: Let us label the vertices of the graph as shown below

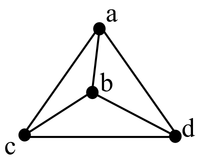


There are 3 maximal matchings as given below

{a–d, b–c}, {a–c, b–d} and { c–d}

26. Ans: 3

Sol: The given graph is



The maximal matchings are

{a–b, c–d}, {a–c, b–d}, {a–d, b–c}

27. Ans: 10

Sol: The graph has 3 maximal matchings, 6 matchings with one edge, and a matching with no edges.

∴ Number of matchings = 10

28. Ans: 8

Sol: By sum of degrees theorem,

(5 + 2 + 2 + 2 + 2 + 1) = 2 |E|

⇒ |E| = 7

∴ Number of edges in G = 7

$|E(G)| + |E(\overline{G})| = |E(K_6)|$

$\Rightarrow \quad 7 + |E(\overline{G})| = C(6, 2)$

$\Rightarrow \quad |E(\overline{G})| = 8$

29. Ans: (a)

Sol: If n is even, then a bipartite graph with maximum number of edges is $k_{n/2,n/2}$

$\therefore$ Matching number of $G = \frac{n}{2}$

If n is odd, then a bipartite graph with maximum number of edges = $k_{m,n}$

Where $m = \frac{n-1}{2}$ and $n = \frac{n+1}{2}$

∴Matching number of G

$$= \begin{cases} \frac{n}{2}, \text{ if } n \text{ is even} \\ \frac{n-1}{2}, \text{ if } n \text{ is odd} \end{cases}$$

$\therefore$ Matching number of $G = \left\lfloor \frac{n}{2} \right\rfloor$

30. Ans: (a)

Sol: If G has n vertices and k components, then

$$(n-k) \le |E| \le \frac{(n-k)(n-k+1)}{2}$$

$$\Rightarrow 7 \le |E| \le 28$$

31. Ans: (d)

Sol: If |E| < (n – 1), then G is disconnected

If $|E| > \frac{(n-1)(n-2)}{2}$, then G is connected.

then G may or may not be connected.

32. Ans: (d)

Sol: The given graph is a complete graph K_6, with 6 vertices of odd degree.

∴ G is not traversable

33. Ans: 3

Sol: d is the cut vertex of G

⇒ vertex connectivity of G = 1

G has no cut edge.

⇒ λ(G) ≥ 2 .... (1)

By deleting the edges d – e and d – f, we can disconnect G.

∴ Edge connectivity = λ(G) = 2

34. Ans: 105

Sol: If G is a simple graph with n vertices and k components then $|E| \le \frac{(n-k)(n-k+1)}{2}$

Here n = 20 and k = 5

$\therefore$ Maximum number of edges possible $= \frac{(20-5).(20-6)}{2} = 105$

35. Ans: (a)

Sol: If G is any graph having p vertices and $\delta(G) \ge \frac{p-1}{2}$, then G is connected.

(theorem)

36. Ans: (b)

Sol: If a component has n vertices, then maximum number of edges possible in that component

= C(n, 2)

$\therefore$ The maximum number of edges possible in G = C(5, 2) + C(6, 2) + C(7, 2) + C(8, 2)

= 10 + 15 + 21 + 28

= 74

37. Ans: 9

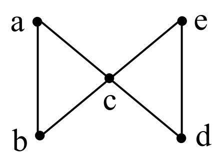
Sol: In a tree, each edge is a cut set.

Number of edges in a tree with 10 vertices = 9

$\therefore$ Number of cut sets possible on a tree with 10 vertices = 9

38. Ans: 1, 2

Sol: The graph can be labeled as



c is a cut vertex of the graph G.

$\therefore$ vertex connectivity of G = K(G) = 1

G has no cut edge.

$\Rightarrow$ Edge connectivity = λ (G) $\ge$ 2 ….. (1)

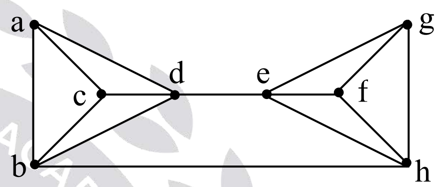
We have, λ (G) $\le \delta$ (G) = 2 …… (2)

From (1) and (2), we have

λ (G) = 2

39. Ans: 2, 2

Sol: The graph G can be labeled as



G has no cut edge and no cut vertex. By deleting the edges {d, e} and {b, h} we can disconnect G.

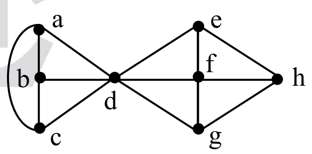
$\therefore \lambda$ (G) = 2

By deleting the vertices b and d, we can disconnect G.

$\therefore$ K (G) = 2

40. Ans: 1 & 3

Sol: The graph G can be labeled as



The vertex d is a cut vertex of G.

$\therefore$ K (G) = 1

We have $\lambda(G) \le \delta$ (G) = 3 ...... (1)

G has no cut edge and by deleting any two edges of G we cannot disconnect G.

$\therefore \lambda$ (G) = 3

41. Ans: S_1, S_3 & S_4

Sol: **S_1:** This statement is true.

Proof:

Suppose G is not connected G has atleast 2 connected components.

Let G_1 and G_2 are two components of G.

Let u and v are any two vertices in G

We can prove that there exists a path between u and v in G.

Case1: u and v are in different component of G.

Now u and v are not adjacent in G.

$\therefore$ u and v are adjacent in $\overline{G}$

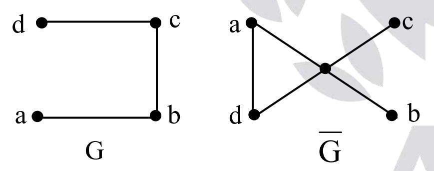
Case2: u and v are in same component G_1 of G. Take any vertex $w \in G_2$.

Now u and v are adjacent to w in G.

$\therefore$ There exists a path between u and v in G. Hence, $\overline{G}$ is connected.

S_2: The statement is false.

we can give a counter example.



Here, G is connected and $\overline{G}$ is also connected.

S_3: Suppose G is not connected

Let G_1 and G_2 are two connected components of G.

Let $v \in G_1$

$$\Rightarrow \deg(v) \geq \frac{n-1}{2} \qquad \left(\because \delta(G) = \frac{n-1}{2}\right)$$

Now $\quad |V(G_1)| \geq \left(\frac{n-1}{2} + 1\right)$

Similarly, $|V(G_2)| \geq \frac{n+1}{2}$

Now, $|V(G)| = |V(G_1)| + |V(G_2)|$

$\Rightarrow |V(G)| \geq n+1$

Which is a contradiction

$\therefore$ G is connected.

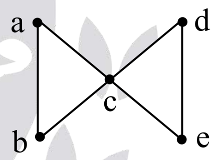
S_4: If G is connected, then the statement is true. If G is not connected, then the two vertices of odd degree should lie in the same component,

By sum of degrees of vertices theorem.

$\therefore$ There exists a path between the 2 vertices.

42. Ans: S_1, S_2 & S_4

Sol: The graph G can be labeled as



The number of vertices with odd degree is 0.

$\therefore$ S_1 and S_2 are true

C is a cut vertex of G.

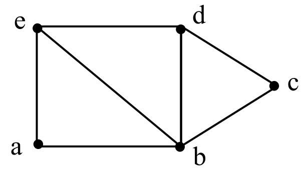
$\therefore$ Hamiltonian cycle does not exists.

By deleting the edges {a, c} and {c, e}, there exists a Hamiltonian path

a – b – c – d – e

43. Ans: S_1, S_3 & S_4

Sol: The graph G can be labeled as



The number of vertices with odd degree = 2

$\therefore$ Euler path exists but Euler circuit does not exist.

There exists a cycle passing through all the vertices of G.

a – b – c – d – e – a is the Hamiltonian cycle of G.

The Hamiltonian path is a – b – c – d – e

44. Ans: S_1 & S_2

Sol: The number of vertices with odd degree = 0

$\therefore$ S_1 and S_2 are true.

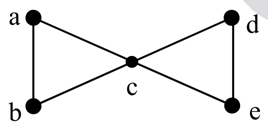
To construct Hamiltonian cycle, we have to delete two edges at each of the vertices a and f. Then, we are left with 4 edges and 6 vertices.

$\therefore$ G has neither Hamiltonian cycle nor Hamiltonian path.

45. Ans: (b)

Sol: S_1 is false. We can prove it by giving a counter example.

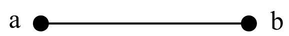
Consider the graph G shown below



'e' is a cut vertex of G. But, G has no cut edge

S_2 is false. We can prove it by giving a counter example.

For the graph K_2 shown below,



The edge {a, b} is a cut edge. But K_2 has no cut vertex.

46. Ans: 33

Sol: If G has K components, then

$$|E| = |V| - K$$

$$\Rightarrow \quad 26 = |V| - 7$$

$$\Rightarrow |V| = 33$$

47. Ans: (b)

Sol: A 2-regular graph G has a perfect matching iff every component of G is an even cycle.

$\therefore$ S_2 and S_4 are true.

S_1 need not be true. For example the complete graph K_2 has a perfect matching but K_2 has no cycle.

S_3 need not be true. For example G can have two components where each component is K_2.

48. Ans: 21

Sol: In a simple graph with n vertices and K components,

$$|E| > \frac{(n-k)(n-k+1)}{2}$$

$\therefore$ Required minimum number of edges

$$= \frac{(n-k)(n-k+1)}{2} = 21$$

Where n = 10 and k = 4

49. Ans: (d)

Sol: G has exactly two vertices of odd degree. Therefore, Euler path exists in G but Euler circuit does not exist.

In Hamiltonian cycle, degree of each vertex is 2. So, we have to delete 2 edges at vertex 'd' and one edge at each of the vertices 'a' and 'g'. Then we are left with 8 vertices and 6 edges. Therefore, neither Hamilton cycle exists nor Hamiltonian path exists.

50. Ans: (b)

Sol: G has cycles of odd length

$\therefore$ Chromatic number of

$G = \chi(G) \geq 3$ ……(1)

For the vertices c and h we can use same color C_1

The remaining vertices from a cycle of length 6.

A cycle of even length require only two colors for its vertex coloring.

For vertices a, d and f we can apply same color C_2

For the vertices {b, e, g} we can use same color C_3

$\therefore \chi(G) = 3$

A perfect matching of the graph is

a–b, c – d, e –f, g – h

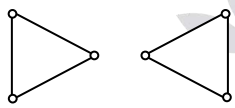
$\therefore$ Matching number = 4

Hence, chromatic number of G

+ Matching number of G = 3 + 4 = 7

51. Ans: (c)

Sol: S_1 need not be true. Consider the graph



Here, we have 6 vertices with degree 2, but the graph is not connected.

S_2 need not be true. For the graph given above, Euler circuit does not exist, because it is not a connected graph.

A simple graph G with n vertices is necessarily connected if $\delta(G) \geq \frac{n-1}{2}$.

$\therefore S_3$ is true.

52. Ans: (a)

Sol: Vertex connectivity of $G = k(G) \leq \delta(G)$

$\Rightarrow \delta(G) \geq 3$

By sum of degrees theorem

$3|V| \leq 2|E|$

$\Rightarrow |E| \geq 15$

$\therefore$ Minimum number of edges necessary = 15

4. Set Theory

01. Ans: (c)

Sol: (a) $A \cup (A \cap B) = A$ (Absorption law)

$\therefore$ Option (a) is false

(b) $A \cap (A \cup B) = A$ (Absorption law)

$\therefore$ Option (b) is false

(c) $(A \cup B) \cap (A \cup \overline{B})$

$= A \cup (B \cap \overline{B})$ Distribution law

$= A$

$\therefore$ Option (c) is true

02. Ans: (b)

Sol: If $S = \{\phi\}$ then

(a) $P(S) \cap S = \{\phi\}$

$\therefore$ option (a) is false

(b) $P(S) \cap P(S) = \{\phi, \{\phi\}\}$

$\therefore$ Option (b) is true

(c) If S = {a, b} then

$P(S) \cap S = \phi$

$\therefore$ option (c) is false

(d) false

Refer option (b)

03. Ans: (a)

Sol: Given that

$A \subseteq B \subseteq S$

The venn-diagram is shown here

Here, each element of S can appear in 3 ways

i.e., $x \in A$ or $x \in (B - A)$ or $x \in (S - B)$

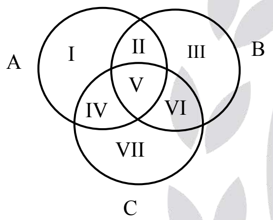
In all 3 cases, $A \subseteq B \subseteq S$.

By product rule, the 6 elements of S can appear in 3^6 ways.

$\therefore$ Required number of ordered pairs = 3^6 = 729

04. Ans: (a, c & d)

Sol:



I, II,…….., VII are regions

(a) $(A - B) - C = \{I, IV\} - \{IV, V, VI, VIII\}$
$= \{I\}$

$(A - C) - B = \{I, II, IV, V\} - \{IV, VII\}$
$= \{I, II, V\}$

$\therefore$ Option (a) is false

(b) $A-(B\cup C) = (A-B)\cap (A-C)$ is true by Demorgan's law

(c) $A-(B-C) = \{I, II, IV, V\} - \{II, III\}$
$= \{I, IV, V\}$

$A-(C-B) = \{I, II, IV, V\} - \{IV, VII\}$
$= \{I, II, V\}$

$\therefore$ Option (c) need not be true

(d) Similarly show that option (d) is not true

05. Ans: (c)

Sol: Let $x \in X$

Case 1: If x is even number then it can appear in two ways i.e., either $x \in A - B$ or $x \in B - A$

Case 2: If x is odd number then it can appear in two ways i.e., $x \in A \wedge B$ or $x \in \left(\overline{A \vee B}\right)$

$\therefore$ By product rule, required number of subsets = 2^{100}

06. Ans: (c)

Sol: If $A \Delta B = (A\cap B)^C$

Then $\left(\overline{A \cup B}\right) = \phi$

But $(A\cup B) \cup \left(\overline{A \cup B}\right) = U$

$\Rightarrow (A\cup ssB) = U$

Where U is universal set

07. Ans: (d)

Sol: (a) Let $A\oplus B = A$

$\Rightarrow A\oplus B = A\oplus\phi$

$\Rightarrow \quad B = \phi$

(b) $(A\oplus B) \oplus B$
$= A \oplus (B \oplus B)$
$= A \oplus \phi$
$= A$

(c) $A \oplus C = B \oplus C$
$\Rightarrow A = B$ (cancellation law)

(d) LHS = $A \oplus B = (A \vee B) - (A \wedge B)$

RHS = $(A \vee B) \vee (A \wedge B)$
$= (A \vee B)$

$\therefore$ L.H.S $\neq$ R.H.S

08. Ans: (c)

Sol: (A) Let A = {1}, B = {2}, C = {3}

Now $(A\cap B) = (B\cap A) = \phi$

But $A \neq B$

$\therefore$ (A) is not true

(B) Let A = {1}, B = {2}, C = {1, 2}

Now $A\cup C = B\cup C = C$

But $A \neq B$

$\therefore$ (B) is not true

(C) Let $x \in A$.

Consider the two cases

Case1: $x \in C$

$\Rightarrow x \notin (A \Delta C)$ $(\because x \in (A \Delta C))$

$\Rightarrow x \notin (B \Delta C)$ $(\because A \Delta C = B \Delta C)$

$\Rightarrow x \in B$ ……….(1)

Case2: $x \notin C$

$\Rightarrow x \in (A \Delta C)$

$\Rightarrow x \in (B \Delta C)$

$\Rightarrow x \in B$ $(\because x \notin C)$ ………(2)

$\therefore A \subseteq B$ (Form (1) and (2))

Similarly we can show that $B \subseteq A$.

$\therefore$ A = B

Hence, (C) is true

(D) Let A = {1, 2}

B = {2, 3}

C = {1, 3}

Here, A – C = {2} = B – C

But, $B \neq C$

$\therefore$ (D) is not true

09. Ans: 126

Sol: Required number of multi sets = V(6, 4)

= C(6 – 1 + 4, 4)

= C(9, 4) = 126

10. Ans: (a)

Sol: Suppose R is irreflexive and transitive but not anti symmetric.

Let $a\,^R\,b$ and $b\,^R\,a$

$(\because$ R is not anti symmetric)

$\Rightarrow a\,^R\,a$ $(\because$ R is transitive)

$\Rightarrow$ R is not irreflexive.

Which is a contradiction.

$\therefore$ R is anti symmetric

11. Ans: (d)

Sol: Let A = set of all +ve integers

(A) We have a – a = 0 = an even integers

$\Rightarrow a \not{R} a \quad \forall a \in A$

$\Rightarrow$ R is not reflexive

(B) Let $a^R b$

$\Rightarrow$ (a – b) is an odd positive integer

$\Rightarrow$ (b – a) is an odd negative integer

$\Rightarrow b \not{R} a$

$\Rightarrow$ R is not symmetric

(C) Let $(a^R b)$ and $(b^R c)$

$\Rightarrow$ (a–b) and (b – c) are odd +ve integers

Now, a – c = (a – b) + (b – c)

= An even +ve integer

$\Rightarrow a \not{R} c$

$\Rightarrow$ R is not transitive

(D) Let $(a^R b)$

$\Rightarrow$ (a – b) is an odd +ve integer

$\Rightarrow$ (b – a) is an even +ve integer

$\Rightarrow b \not{R} a$

$\Rightarrow$ R is asymmetric

$\Rightarrow$ R is antisymmetric

12. Ans: (c)

Sol: R_1 is not a function because, the element d of A has no image in B

R_2 is not a surjection, because the element 3 in B is not mapped by any element of A.

R_3 is a surjection, because each element of B is mapped by atleast one element of A.

R_4 is not a function, because the element a in A is mapped with two elements in B.

13. Ans: (a)

Sol: R = {(1, 1), (2, 2), (2, 3), (2, 4), (3, 2), (3,3), (3, 4), (4, 2), (4, 3), (4, 4)}

R is an equivalence relation

[1] = {1}

[2] = {2, 3, 4}

[3] = {2, 3, 4}

[4] = {2, 3, 4}

14. Ans: (d)

Sol: (A) We have $\phi \in P(S)$

and $(\phi \cap \phi) = \phi$

$\Rightarrow \phi$ is not related to ϕ

$\Rightarrow$ R is not reflexive

(B) R is not reflexive

(C) We have $\{1\} \in P(S)$

and $\{1\} \cap \{1\} = \{1\} \neq \phi$

$\Rightarrow \{1\}$ is related to 1

$\Rightarrow$ R is not irreflexive

(D) Let $A\ ^R\ B$

$\Rightarrow A \cap B = \phi$

$\Rightarrow B \cap A = \phi$

$\Rightarrow B\ ^R\ A$

$\Rightarrow$ R is symmetric on P(A)

Let A = {1, 2}, B = {2, 3}, C = {3, 4}

Here $A\ ^R\ B$ and $B\ ^R\ C$, but $A^R\ B$

$\Rightarrow$ R is not transitive.

15. Ans: (b)

Sol: Suppose R is symmetric

$\Rightarrow (a, b) \in R$ and $(b, a) \in R$

$\Rightarrow (a, a) \in R$ which is impossible because R is irreflexive.

$\therefore$ R is not symmetric

Suppose, R is not asymmetric

$\Rightarrow (a, b) \in R$ and $(b, a) \in R$

$\Rightarrow (a, a) \in R$ (By transitivity)

which is impossible because R is irreflexive.

$\therefore$ R is asymmetric

$\Rightarrow$ R is anti-symmetric

16. Ans: (d)

Sol: (a) we have $(a, b)^R(a, b) \forall\ a, b \in S$ because ab = ba

$\therefore$ R is reflexive

(b) Let $(a, b)\ ^R\ (c, d)$

$\Rightarrow ad = bc$

$\Rightarrow cb = da$

$\Rightarrow (c, d)\ ^R\ (a, b)$

$\Rightarrow$ R is symmetric.

(c) Let $(a, b)\ ^R\ (c, d)$ adn $(c, d)\ ^R\ (e, f)$

$\Rightarrow ad = bc$ and $cf = de$

$\Rightarrow \frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$

$\Rightarrow \frac{a}{b} = \frac{e}{f}$

$\Rightarrow af = be$

$\Rightarrow (a, b)\ ^R\ (e, f)$

$\Rightarrow$ R is transitive

(d) R is not antisymmetric

For ex. $(1, 2)\ ^R\ (2, 4)$ and $(2, 4)\ ^R\ (1, 2)$

17. Ans: (d)

Sol: Equivalence class of x = [x]

= {y | (x ÷ y) is a rational number}

= set of all non zero relational numbers

18. Ans: 26

Sol: If A is set with n elements, then number of asymmetric relations possible on $A = 3^{\frac{n(n-1)}{2}}$

The only relation which is symmetric and asymmetric is the empty relation on A.

$\therefore$ Required number of relations $= 3^{\frac{n(n-1)}{2}} - 1$

Where n = 3

$= 3^3 - 1 = 26$

19. Ans: (c)

Sol: If R is antisymmetric, then $(R \cap R^{-1}) \subseteq \Delta_A$.

Any subset of diagonal relation is symmetric, antisymmetric and transitive.

But, $(R \cap R^{-1})$ is not asymmetric, because diagonal pairs are not allowed in asymmetric relation.

20. Ans: (c)

Sol: The equivalence class of 1 = [1] = {1, 2, 3}

[2] = {1, 2, 3}

[3] = {1, 2, 3}

[4] = {4, 6}

[5] = {5}

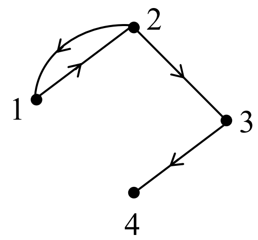
[6] = {4, 6}

The partition of A with respect to R = set of all distinct equivalence classes of the elements of A

= {{1, 2, 3}, {4, 6}, {5}}

21. Ans: (c)

Sol: The digraph for R is shown below



We see that from vertex 1, we have paths to vertices 2, 3, 4 and 1.

From vertex 2, we have paths to vertices 1, 2, 3 and 4.

From vertex 3, we have path to vertex 4.

$\therefore$ Transitive closure of R = {(1, 1), (1, 2), (1, 3), (1, 4), (2,1), (2,1), (2,3), (2,4), (3, 4)}.

22. Ans: 392

Sol: Number of relations on A which are symmetric or reflexive

$= 2^{n(n-1)} + 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n-1)}{2}}$

Where n = 3

$= 2^6 + 2^6 - 2^3 = 120$

$\therefore$ Required number of relations

$= 2^{(n^2)} - 120$ where n = 3

= 512 – 120 = 392

23. Ans: (c)

Sol: For a, b, $\in$ A

Let $a^R b$

$\Rightarrow a^R b$ and $b^R b$ ($\therefore$ R is reflexive)

$\Rightarrow b^R a$ (By definition of R)

$\Rightarrow$ R is symmetric

For any three elements a, b, c $\in$ A

Let $a^R b$ and $b^R c$

$\Rightarrow c^R b$ and $b^R a$ ($\because$ By definition of R and R is symmetric)

$\Rightarrow a^R c$ (By definition of R)

$\therefore$ R is an equivalence relation

24. Ans: (c)

Sol: we have, $a = a^1$

$\Rightarrow a\ R\ a \ \forall a \in N$

$\Rightarrow$ R is reflexive on N

Let $a\ R\ b$ and $b\ R\ a$

$\Rightarrow a = b^{K_1}$ and $b = a^{K_2}$ .... (1)

$\Rightarrow b = \left(b^{K_1}\right)^{K_2} = b^{K_1 K_2}$ (from (1))

$\Rightarrow k_1 k_2 = 1$

$\Rightarrow k_1 = 1$ and $k_2 = 1$

$\Rightarrow a = b$ from (1)

$\therefore$ R is antisymmtric.

Let $a\ R\ b$ and $b\ R\ c$

$\Rightarrow a = b^{K_1}$ and $b = c^{K_2}$ .... (2)

$\Rightarrow a = \left(c^{K_2}\right)^{K_2} = c^{K_1 K_2}$

$\Rightarrow a = c^{K_3}$ where $k_3 = k_1 . k_2$

$\Rightarrow a\ R\ c$

$\therefore$ R is transitive

R is not symmetric. For example $8\ R\ 2$ but $2\ \not R\ 8$

R is not a total order. For example $2\ \not R\ 3$ and $3\ \not R\ 2$

$\therefore$ only option (c) is correct.

25. Ans: 32

Sol: The Hasse diagram of the poset is 4-cube.

Number of edges in n-cube = $n.2^{n-1}$

$\therefore$ Number of edges in the Hasse diagram = $4.\ 2^{4-1} = 32$

26. Ans: (d)

Sol: Let x, y, z be elements in a poset. If ($x^R z$ and $z^R y$) then $x^R y$.

In this case, there is no edge between x and y in the poset diagram even though x is related to y.

$\therefore S_1$ is false and S_2 is also false.

27. Ans: 49

Sol: The maximal elements are {51, 52,....., 100}

$\therefore$ Number of maximal elements = 50

Note: An element x is said to be maximal, if x is not related to any other element of the poset.

28. Ans: (b)

Sol: In the given lattice, y and x are upper and lower bounds.

For the elements b & c, $b \vee c = y$ & $b \wedge c = x$

$\therefore$ b and c complements of each other.

Similarly, a and c are complements of each other.

For the element c we have two complements.

$\therefore$ The given lattice is not distributive.

In the given lattice, each element has atleast one complement.

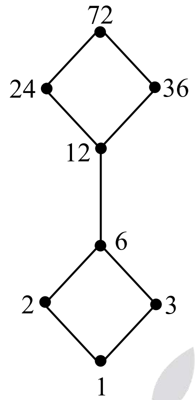
$\therefore$ The given lattice is a complemented lattice.

Hence, only S3 is false.

29. Ans: (A)

Sol: The Hasse diagram of the poset is shown below.

The number of edges in the poset = 9



In the poset, for every pair of elements lub and glb exist

∴ The poset is a lattice.

30. Ans: (b)

Sol: For the given lattice,

Complement of a = e

Complement of b = c

Complements of c are <u>b and d</u>

Complement of d = c

Complement of e = a

Since each element of L has a complement in L, L is a complemented lattice.

Since, the element c has two complements, L is not a distributive lattice.

31. Ans: (b)

Sol: Let us give the complements of each element.

complement of a = e

complement of b = c

complements of c are b and d

complement of d = c

complements of e = a

Since, each element has atleast one complement, the given Lattice is a complemented Lattice.

In a distributive Lattice, each element has atmost one complement.

Since, the element c has two complements, the given Lattice is not a distributive Lattice.

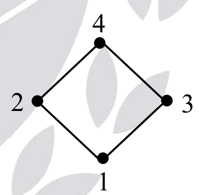
32. Ans: (d)

Sol: The greatest lower bound of c and d does not exist in the poset.

∴ The poset is not a lattice.

33. Ans: (d)

Sol: R is a partial order on A.



The Hasse diagram is a 2-cube.

∴ [A; R] is a boolean algebra.

34. Ans: 9

Sol: D_{36} = {1, 2, 3, 4, 6, 9, 12, 18, 36}

Let x be the complement of 4.

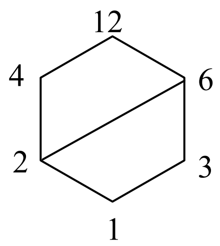
lub of 4 and x = l.c.m of 4 and x = 36

and glb of 4 and x = g.c.d of 4 and x = 1

⇒ x = 9

35. Ans: (d)

Sol: The Hasse diagram of the poset is shown below



The elements 1 and 12 are complement of each other.

The elements 3 and 4 are complements of each other.

For the elements 2 and 6 complements do not exist.

36. Ans: (b)

Sol: If $x^R y$ then $x \vee y = y$ and $x \wedge y = x$

$\therefore$ only option (b) is true

37. Ans: (d)

Sol: we have f(1) = f(2) = 50

$\Rightarrow$ f is not one-to-one

In the co-domain, the integers 1, 2, 3...... 49 are not mapped by any integers of the domain.

$\Rightarrow$ f is not on-to

38. Ans: (b)

Sol: Inverse of a functions exists iff f is a bijection.

Here, f is not a bijection

for example f(1) = f(2) = a

$\Rightarrow$ f is not 1 – 1

g is a bijection

$\Rightarrow g^{-1}$ exists.

39. Ans: (c)

Sol: Let f(a, b) = f(c, d)

$\Rightarrow (2a - b, a - 2b) = (2c - d, c - 2d)$

$\Rightarrow 2a - b = 2c - d$ and $a - 2b = c - 2d$

$\Rightarrow a = c$ and $b = d$

$\Rightarrow (a, b) = (c, d)$

$\Rightarrow$ f is 1 – 1

Let f(x, y) = (u, v)

$\Rightarrow (2x - y, x - 2y) = (u, v)$

$\Rightarrow 2x - y = u$ and $x - 2y = v$

$\Rightarrow x = \dfrac{2u - v}{3}$ and $y = \dfrac{u - 2y}{3}$

If u and v are any two real numbers, then

$\left(\dfrac{2u - v}{3}, \dfrac{u - 2v}{3}\right) \in (RxR)$

$\Rightarrow$ f is onto

Hence, f is a bijection.

40. Ans: (b)

Sol: Let G = set of all odd integers.

G is closed with respect to multiplication.

Multiplication of odd integers is associative.

1 is identity element of G with respect to multiplication.

$\therefore$ G is a monoid w.r.t multiplication.

G is not a group w.r.t multiplication. For example, inverse of 3 = $\dfrac{1}{3}$

But $\dfrac{1}{3} \notin G$.

41. Ans: (a)

Sol: The identity element of the group (P(S), *) is ϕ.

Inverse of A = A $\forall$ A $\in$ P(S)

Because, (A*A) = ϕ $\forall$ A $\in$ P(S)

$\therefore$ Inverse of {1} = {1}

42. Ans: 54

Sol: Number of generators = ϕ (81), where ϕ is Euler function.

$$= 81\left(\frac{3-1}{3}\right) = 54$$

43. Ans: (a)

Sol: Any group with 4 elements is abelian.

$\Rightarrow$ The rows and columns of the table are identical

$\Rightarrow$ First row is a b c d

and second row is b d a c

In the composition table of a group, in each row and each column the entries are distinct.

$\therefore$ The entry in the third row and third column is c.

Hence, the fourth column is d
c
b
a

44. Ans: 6

Sol: The identity element of the group is 1.

If x is inverse of 2, then $2 \otimes_{11} x = 1$

$\Rightarrow$ x = 6

45. Ans: (b)

Sol: Let 'e' be the identity element of Z w.r.t. *

Now, a * e = a

$\Rightarrow$ a + e + 1 = a

$\Rightarrow$ e = –1

Let x be the inverse of –5

$\Rightarrow$ –5 * x = e

$\Rightarrow$ –5 + x + 1 = –1

$\Rightarrow$ x = 3

46. Ans: (d)

Sol: Let a, b, c are any 3 integers.

$3^a \cdot 3^b = 3^{a+b} \in G \quad \forall$ a, b, $\in$ Z

$\Rightarrow$ Multiplication is a closed operation on G.

The elements of G are relational numbers and multiplication of rational numbers is associative.

we have, $1 = 3^0 \in G$ and

$3^a \cdot 3^0 = 3^a \quad \forall a \in Z$

$\therefore$ Identity element exists.

If n $\in$ Z then –n $\in$ Z.

For each element $3^n \in G$ we have $3^{-n} \in G$ such that

$3^n \cdot 3^{-n} = 3^0$

$\Rightarrow$ Each element of G has inverse in G.

Further, multiplication of rational numbers is commutative.

$\therefore$ G is an abelian group w.r.t multiplication

47. Ans: (c)

Sol: (a) 2 * 6 = 12 $\notin$ S

* is not a closed operation on S

(b) 1 * (–4) = |– 4 | = 4 $\notin$ S

* is not a closed operation on S

(C) $a * b = (a + b^2) \in S \ \forall a, b \in S$

$\Rightarrow$ * is a closed operation on S

(D) $2 * 3 = 2 - 3 = -1 \notin S$

* is not a closed operation on S

48. Ans: (c)

Sol: (a) The set {1, 4} is closed w.r.t. the given binary operation.

$\therefore$ It is subgroup of G

(b) The set {1, 11} is closed w.r.t. the given binary operation.

$\therefore$ It is a subgroup of G.

(c) The set {1, 13} is not closed w.r.t. the given binary operation.

$\therefore$ The set is not a subgroup of G.

(d) The set {1, 14} is closed w.r.t. the given binary operation.

$\therefore$ It is a sub group of G.

49. Ans: (b)

Sol: We have

$a * b = \text{g.c.d of } \{a, b\} \in D_{12} \quad \forall a, b \in D_{12}$

$\therefore$ * is a closed operation on D_{12}

* is associative on D_{12}

We have $a * 12 = a \quad \forall a$

The identity element = 12

The inverse of any element in D_{12} except 12 does not exist.

$\therefore (D_{12};*)$ is a monoid but not a group.

50. Ans: (d)

Sol: The identity element = 6

We have, $2^2 = 2 \otimes_{10} 2 = 4$

$2^3 = 2^2 \otimes_{10} 2 = 4 \otimes_{10} 2 = 8$

$2^4 = 2^3 \otimes_{10} 2 = 6$

$\therefore$ 2 is a generator

5. Probability

01. Ans: (c)

Sol: ***Proof by contradiction:***

S_1: Suppose, A and B are mutually exclusive and independent

$P(A \cap B) = P(A) . P(B)$

$\Rightarrow 0 = P(A) . P(B)$

($\because$ A and B are mutually exclusive)

$\Rightarrow P(A) = 0$ or $P(B) = 0$.

This is a contradiction, because A and B are possible events.

Hence, A and B are not independent.

S_2: Suppose A and B are independent and mutually exclusive.

Then, $P(A) . P(B) = P(A \cap B) = 0$

$\Rightarrow P(A) = 0$ or $P(B) = 0$

Which is a contradiction, because A and B are possible events.

Hence, A and B are not mutually exclusive.

02. Ans: (b)

Sol: **Case 1:** The first 5 cars sold are not defective and 6^{th} car is defective.

The Probability for this event $= \frac{C(8,5)}{C(10,5)} . \frac{2}{5}$

Case 2: The first 5 cars sold have one defective car and 6^{th} car sold is defective.

The probability for this event $= \frac{C(8,4).C(2,1)}{C(10,5)} . \frac{1}{5}$

The required probability

$$= \frac{C(8,5)}{C(10,5)} . \frac{2}{5} + \frac{C(8,4).C(2,1)}{C(10,5)} . \frac{1}{5}$$

$$= \frac{1}{5} = 0.2$$

03. Ans: (c)

Sol: Required probability $= P(\overline{A} \cap \overline{B})$

$= P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - \{P(A) + P(B) - P(A \cap B)\}$

$= 1 - \{0.25 + 0.5 - 0.14\}$

$= 0.39$

04. Ans: 0.3

Sol: $P\{(A \cap \overline{B}) \cup (B \cap \overline{A})\}$

$= P(A) + P(B) - 2P(A \cap B)$

$= 0.7 - 2(0.2)$

$= 0.3$

05. Ans: 0.83 range 0.83 to 0.84

Sol: $A \cup B^C = A \cup (A^C \cap B^C)$

$P(A \cup B^C) = P(A) + P(A^C \cap B^C) = \frac{1}{2} + \frac{1}{3}$

$= \frac{5}{6} = 0.83$

06. Ans: (a)

Sol: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.5 = P(A) + P(B) - P(A).\ P(B)$

$\Rightarrow 0.5 = P(A) + P(A) - P(A)\ .\ P(A)$

$\Rightarrow P(A) = 0.2929$

07. Ans: (a)

Sol: Number of bit strings of length 6 = $2^6 = 64$

Number of substrings with atleast four ones

$= C(6, 4) + C(6, 5) + C(6, 6)$

$= 15 + 6 + 1 = 22$

Required probability $= \frac{22}{64} = \frac{11}{32}$

08. Ans: 0.295

Sol: Number of ways we can choose two socks from the bag = C (15, 2) = 105

Number of ways we can choose 2 socks of same colour= C(4,2) + C(6,2)+ C(5,2) = 31

$\therefore$ Required probability $= \frac{31}{105} \cong 0.295$

09. Ans: 0.14285

Sol: Required probability

$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + ...\infty$

$= \frac{1}{2^3}\left(1 + \left(\frac{1}{2^3}\right) + \left(\frac{1}{2^3}\right)^2 + ...\infty\right)$

$= \frac{1}{8} . \left(\frac{1}{1 - \frac{1}{8}}\right) = \frac{1}{7} = 0.14285$

10. Ans: (a)

Sol: We know that

$P(A \cap B) \le$ min. of $\{P(A), P(B)\}$

$\Rightarrow P(A \cap B) \le 0.25$ .... (1)

we have, $P(A \cup B) \le P(S)$

$\Rightarrow \{P(A) + P(B) - P(A \cap B)\} \le 1$

$\Rightarrow \{0.25 + 0.8 - P(A \cap B)\} \le 1$

$\Rightarrow 0.05 \le P(A \cap B)$ .... (2)

From (1) and (2), we have

$0.05 \le P(A \cap B) \le 0.25$

11. Ans: 0.3956 range 0.39 to 0.4

Sol: Anita's chances of winning are $\frac{1}{6}, \left(\frac{5}{6}\right)^3.\frac{1}{6}, \left(\frac{5}{6}\right)^6.\frac{1}{6}, \left(\frac{5}{6}\right)^9.\frac{1}{6}, ...$

P(Anita winning game)

$= \frac{1}{6}+\frac{1}{6}\left(\frac{5}{6}\right)^3+\frac{1}{6}.\left(\frac{5}{6}\right)^6+\frac{1}{6}.\left(\frac{5}{6}\right)^9+........$

$= \frac{1}{6}\left[1+x+x^2+x^3+........\infty\right]$

here $x = \left(\frac{5}{6}\right)^3 = \frac{1}{6}(1-x)^{-1} = \frac{1}{6}\left[1-\left(\frac{5}{6}\right)^3\right]^{-1}$

$= \frac{1}{6}\left[1-\frac{125}{216}\right]^{-1} = \frac{1}{6}\left(\frac{216}{91}\right)$

$= \frac{36}{91} = 0.3956$

12. Ans: 0.125

Sol: Let x_1, x_2 and x_3 be the numbers on the 3 dice.

Number of outcomes possible with 3 dice = $6^3 = 216$

Number of outcomes in which sum is 10 = number of non negative integer solutions to the equation $x_1 + x_2 + x_3 = 10$ where $1 \le x_i \le 6$ (i = 1, 2, 3)

= coefficient of x^{10} in the expansion of $(x + x^2 + ……+ x^6)^3$

$= (x + x^2 +..+x^6)^3 = x^3(1 + x + x^2 +….. + x^5)^3$

$= x^3\frac{(1-x^6)^3}{(1-x)^3}$

$= x^3(1 - 3x^6 + 3x^{12} - x^{18})(1 - x)^{-3}$

$= \left(x^3-3x^9+3x^{15}-x^{21}\right)\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}.x^n$

Coefficient of $x^{10} = 36 - 3(3) = 27$

Required probability $= \frac{27}{216} = 0.125$

13. Ans: 0.44 range 0.43 to 0.45

Sol: Let A = Event of getting a sum of 5 and

B = Event of getting a sum of 8

The probability that a sum of 5 is rolled before a sum of 8 is rolled = conditional probability for a sum of 5, given that a sum of 5 or 8 has occurred

$= P(A| A \cup B) = \frac{P(A\cap(A\cup B))}{P(A\cup B)}$

$= \frac{P(A)}{P(A\cup B)} = \frac{n(A)}{n(A)+n(B)} = \frac{4}{4+5} = \frac{4}{9}$

= 0.44

14. Ans: 0.9677 range 0.96 to 0.97

Sol: Let A = Event that the family has atleast one girl

B = Event that the family has atleast one boy

Required probability = $P(A|B) = \frac{P(A\cap B)}{P(B)}$

P(B) = 1 – P(The family has no boy)

$= 1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$

$P(A\cap B) = 1-P(\overline{A\cap B}) = 1-\{P(\overline{A})+P(\overline{B})\}$

$= 1 - \left\{\frac{1}{32}+\frac{1}{32}\right\} = \frac{15}{16}$

Required probability $= \frac{\left(\frac{15}{16}\right)}{\left(\frac{31}{32}\right)} = \frac{30}{31} = 0.9677$

15. Ans: 0.333 (range 0.33 to 0.34)

Sol: Let A = one of the face is 4

and B = The faces are different numbers

Number of cases favourable to $A \cap B$

$= n(A \cap B) = 10$

Number of cases favourable to B

$= n(B) = 30$

$\therefore$ Required probability = P(A|B)

$$= \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)} = \frac{10}{30} = \frac{1}{3}$$

16. Ans: (b)

Sol: Let A = Die is actually a six and

B = The outcome of die is reported as six.

Required probability = P (A|B)

$$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}.\frac{3}{4}}{\frac{1}{6}.\frac{3}{4} + \frac{5}{6}.\frac{1}{4}} = \frac{3}{8}$$

17. Ans: (d)

Sol: A mode of the distribution of a continuous random variable X, is the value of x where the probability density function attains a relative maximum.

consider, $\frac{df}{dx} = 0$

$\Rightarrow 2x\, e^{-bx} - b\, x^2\, e^{-bx} = 0$

$\Rightarrow (2 - bx)\, x = 0$

$\Rightarrow x = 0$ or $x = \frac{2}{b}$

Thus the mode of X is $\frac{2}{b}$.

18. Ans: 2525

Sol: Given that, E(X) = 50 and $\sigma = 5$

$\sigma^2 = E(X^2) - \{E(X)\}^2$

$E(X^2) = \{E(X)\}^2 + \sigma^2$

$= 2500 + 25 = 2525$

19. Ans: (c)

Sol: Let X be the number of heads in first 3 throws. X has the following probability distribution.

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Required probability

$$= \left(\frac{1}{8}\right)\!.\!\left(\frac{1}{8}\right) + \left(\frac{3}{8}\right)\!.\!\left(\frac{3}{8}\right) + \left(\frac{3}{8}\right)\!.\!\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)\!.\!\left(\frac{1}{8}\right)$$

$$= \frac{5}{16}$$

20. Ans: 280

Sol: The probability density function of

$$X = f(x) = \begin{cases} \frac{1}{8-2} & \text{for } 2 \le x \le 8 \\ 0\ , & \text{otherwise} \end{cases}$$

V= volume of the box = 10 X^2

$E(V) = E\,(10X^2) = 10.\ E(X^2)$

$$= 10\int_2^8 x^2\ .f(x)\ dx$$

$$= 10\int_2^8 x^2 . \frac{1}{6} dx$$

$= 280$

21. Ans: 0.416 (range 0.41 to 0.42)

Sol: Required probability $= \dfrac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)}$

$$= \frac{\int_{1/3}^{1/2} 2x\,dx}{\int_{1/3}^{2/3} 2x\,dx} = \frac{\left(\frac{5}{36}\right)}{\left(\frac{1}{3}\right)} = \frac{5}{12}$$

22. Ans: 0.4285 range 0.42 to 0.43

Sol: P(j dots turning up) $\propto$ j

$\Rightarrow$ P(j dots turning up) = kj where k is proportionality constant

P(S) = P(1) + P(2) + ... + P(6) = 1

$\Rightarrow$ k + 2k +........ + 6k = 1

$\Rightarrow k = \dfrac{1}{21}$

Required probability = P(1) + P(3) + P(5)

$$= \frac{1}{21} + \frac{3}{21} + \frac{5}{21} = \frac{3}{7} = 0.4285$$

23. Ans: (b)

Sol: If f(x) is a probability density function, then

$\int_{-\infty}^{\infty} f(x)dx = 1.$

$\int_{-\infty}^{\infty} f(x)dx = \int_{1}^{2} 2x^{-2}\,dx = 1$

$\Rightarrow$ f(x) is probability density function.

$\int_{-\infty}^{\infty} g(x)dx = \int_{-1}^{1} (1+|x|)dx$

$= \int_{-1}^{0}(1-x)dx + \int_{0}^{1}(1+x)\,dx$

$= \left(x - \frac{x^2}{2}\right)_{-1}^{0} + \left(x + \frac{x^2}{2}\right)_{0}^{1} = 3$

$\Rightarrow$ g(x) is not a probability density function.

24. Ans: 209

Sol: $E(2X+1)^2 = \sum(2X+1)^2 . P(X)$

$= \{2(-3)+1\}^2 \frac{1}{6} + \{2(6)+1\}^2 \frac{1}{2}$

$+ \{2(9)+1\}^2 . \frac{1}{3}$

$= 209$

25. Ans: 0.22 (range 0.21 to 0.23)

Sol: $P(X > 30) = \int_{30}^{\infty} e^{\frac{-x}{20}}\,dx$

$= e^{-1.5} = 0.22$

26. Ans: 0.035 range 0.034 to 0.036

Sol: Probability of getting 2 heads, in first 9 tosses $= C(9, 2) . \left(\frac{1}{2}\right)^9$

Required probability

$$= C(9, 2). \left(\frac{1}{2}\right)^9 . \left(\frac{1}{2}\right) = \frac{9}{256} = 0.035$$

27. Ans: 11

Sol: Let X = number of bombs striking the target

Suppose n bombs are dropped.

Probability of not destroying the target completely = P(X < 2)

= P(X = 0) + P(X = 1)

$= \left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n = \frac{n+1}{2^n}$

Probability of destroying the target

= 1 – P(X < 2)

$= 1 - \left(\frac{n+1}{2^n}\right)$

we have to find the positive integer n such that

$$1-\left(\frac{n+1}{2^n}\right) \geq 0.9$$

$$\Rightarrow 1-0.99 \geq \frac{n+1}{2^n}$$

$$\Rightarrow \frac{2^n}{100} \geq n+1$$

$$\Rightarrow 2^n \geq (100n + 100)$$

By trial method, $2^{11} \geq 100(1 + 11)$

Hence, the required number of bombs = 11.

28. Ans: (b)

Sol: For Poisson distribution, standard deviation $= \sqrt{\lambda} = 2$

$\Rightarrow \lambda = 4$

$$P(X = K) = \frac{e^{-\lambda}.\lambda^k}{\angle K} \left(K = 0,1,2,....\right)$$

$P(X = 1) = \lambda\, e^{-\lambda} = 4e^{-4}$

$P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0) = 1 - e^{-\lambda} = 1 - e^{-4}$

Required probability = $P(X = 1) \mid (X \geq 1)$

$$= \frac{P(X=1)}{P(X\geq 1)} = \frac{4e^{-4}}{1-e^{-4}}$$

$$= \frac{4}{e^4-1}$$

29. Ans: (c)

Sol: $P(X = 0) = \frac{\lambda^0 . e^{-\lambda}}{\angle 0} = \frac{1}{3}$

$$\Rightarrow -\lambda = \log_e\left(\frac{1}{3}\right)$$

$$\Rightarrow \lambda = \log_e 3$$

Required probability = $P(X \geq 2)$

$= 1 - P(X < 2)$

$= 1 - \{P(X = 0) + P(X = 1)\}$

$= 1 - (e^{-\lambda} + \lambda\, e^{-\lambda})$

$$= 1 - \left\{\frac{1}{3} + \frac{1}{3}\log_e 3\right\} = 0.3005$$

30. Ans: (c)

Sol: Average number of errors per page $= \frac{390}{520} = 0.75$

Expected number of errors in a random sample of 5 pages = 5 (0.75)

= 3.75

Let X = Number of errors in a sample of 5 pages.

X has poisson distribution with parameter λ = 3.75.

$$P(X = k) = \frac{e^{-\lambda}.\lambda^k}{\angle k} (k = 0,1,2...)$$

Required probability = $P(X=0) = e^{-\lambda} = e^{-3.75}$

31. Ans: 0.2615 range 0.25 to 0.27

Sol: Required probability = $P(X \geq 2 \mid X \leq 4)$

$$= \frac{P(2\leq X\leq 4)}{P(X\leq 4)}$$

$$P(2 \leq X \leq 4) = \sum_{x=2}^{4} \frac{\lambda^x.e^{-\lambda}}{\angle x}$$

$$= e^{-1}\left(\frac{1}{\angle 2}+\frac{1}{\angle 3}+\frac{1}{\angle 4}\right) = \frac{17}{24}e^{-1}$$

$$P(X \leq 4) = \sum_{x=0}^{4} \frac{\lambda^x.e^{-\lambda}}{\angle x}$$

$$= e^{-1}\left(1+\frac{1}{\angle 1}+\frac{1}{\angle 2}+\frac{1}{\angle 3}+\frac{1}{\angle 4}\right) = \frac{65}{24}e^{-1}$$

$\therefore$ Required probability $= \frac{17}{65} = 0.2615$

32. Ans: (b)

Sol: Let X = number of cashew nuts per biscuit.

We can use Poisson distribution with mean

$$= \lambda = \frac{2000}{1000} = 2$$

$$P(X = k) = \frac{e^{-\lambda}.\lambda^{k}}{\angle k} \quad (k = 0,1,2...)$$

Required probability =P (X =0)

$$= e^{-\lambda} = e^{-2} = 0.135$$

33. Ans: (a)

Sol: f(x) is an even function. The probability density function is symmetric about y axis.

Thus, the median of X = 0

34. Ans: 0.1974 (range 0.19 to 0.20)

Sol: The standard normal variable $Z = \frac{X-4}{\sigma}$

$$= \frac{X-3}{4}$$

$$X = 2 \Rightarrow Z = -\frac{1}{4}$$

$$X = 4 \Rightarrow Z = \frac{1}{4}$$

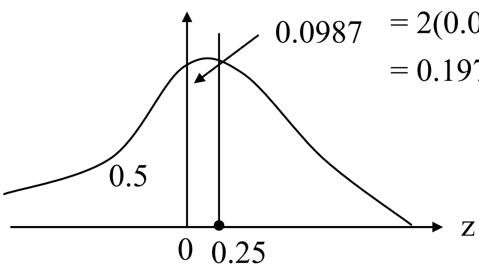
Required probability = P(2 < X < 4)

$$= P\left(-\frac{1}{4} < Z < \frac{1}{4}\right)$$

$$= 2P\left(0 < Z < \frac{1}{4}\right)$$

$$= 2(0.0987)$$

$$= 0.1974$$



35. Ans: 0.416 (range 0.41 to 0.42)

Sol: Let X represent the co-ordinate of the chosen point. The probability density function of X is

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < X < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$P\left(\frac{2}{3} < X < \frac{3}{2}\right) = \int_{\frac{2}{3}}^{\frac{3}{2}} \frac{1}{2} dx$$

$$= \frac{1}{2}\left(\frac{3}{2} - \frac{2}{3}\right) = \frac{5}{12}$$

36. Ans: 0.7

Sol: The probability density function of

$$X = f(x) = \begin{cases} \frac{1}{10} & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P\left\{\left(X + \frac{10}{X}\right) \ge 7\right\} = \left\{P\left(X^2 + 10 \ge 7X\right)\right\}$$

$$= P(X^2 - 7X + 10 \ge 0)$$

$$= P\{(X-5)(X-2) \ge 0\}$$

$$= P(X \le 2 \text{ or } X \ge 5)$$

$$= 1 - P(2 \le X \le 5)$$

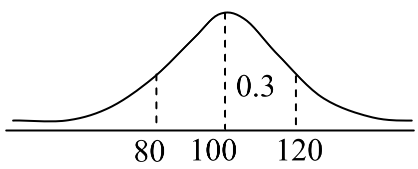
$$= 1 - \int_2^5 f(x)\, dx$$

$$= 1 - \int_2^5 \frac{1}{10} dx$$

$$= 1 - \frac{3}{10} = 0.7$$

37. Ans: 0.2

Sol: The area under normal curve is 1 and the curve is symmetric about mean.



$\therefore$ P(100 < X < 120) = P(80 < X < 120)

= 0.3

Now, P(X < 80) = 0.5 – P(80 < X < 120)

= 0.5 – 0.3 = 0.2

38. Ans: (i) 28 (ii) 28 (iii) 205

Sol: The parameters of normal distribution are $\mu = 68$ and $\sigma = 3$

Let X = weight of student in kgs

Standard normal variable = $Z = \dfrac{X-\mu}{\sigma}$

(i) When X = 72, we have Z = 1.33

Required probability = P(X > 72)

= Area under the normal curve to the right of Z = 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh greater than 72 kgs = 300 × 0.0918 = 28

(ii) When X = 64, we have Z = –1.33

Required probability = $P(X \le 64)$

= Area under the normal curve to the left of Z = – 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

(By symmetry of normal curve)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh less than 68 kgs = 300 × 0.0918

= 28

(iii) When X = 65, we have Z = –1

When X = 71, we have Z = +1

Required probability = P(65< X < 71)

= Area under the normal curve to the left of Z = – 1 and Z = +1

= 0.6826

(By Property of normal curve)

Expected number of students who weighs between 65 and 71 kgs

= 300 × 0.6826

≈ 205

39. Ans: 0.393

Sol: The probability density function of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$P(X<5) = \int_0^5 f(x)\,dx$$

$$= \int_0^5 \frac{1}{10} e^{\frac{-x}{10}}\,dx \approx 0.393$$

40. Ans: (a)

Sol: Mean $\dfrac{1}{\theta} = 0.5 \Rightarrow \theta = \dfrac{1}{0.5} = 2$

The probability function of exponential distribution is $f(x) = 2e^{-2x}$, $x \ge 0$.

$$P(X > \tfrac{1}{2}) = \int_{\frac{1}{2}}^{\infty} 2e^{-2x}\,dx = \left(-e^{-2x}\right)_{\frac{1}{2}}^{\infty}$$

$$= (0) - \{-e^{-1}\} = e^{-1}$$

41. Ans: (d)

Sol: The density function $f(x) = \frac{1}{5}e^{\frac{-1}{5}x}$

We require $P(x>8) = \int_8^\infty f(x)\,dx = e^{-8/5}$

$= 0.2$

42. Ans: Mean = 34, Median = 35, Modes = 35, 36 & SD = 4.14

Sol: Mean $= \frac{\sum x_i}{n} = 34$

Median is the middle most value of the data by keeping the data points in increasing order or decreasing order.

Mode = 36

S.D = 4.14

43. Ans: 1.095

Sol: $\mu = \text{Mean} = \sum_{k=1}^{5}\{x_k.P(X=k)\}$

$= 1(0.1) + 2(0.2) + 3(0.4) + 4(0.2) + 5(0.1)$

$= 3$

$P(X \le 2) = 0.1 + 0.2 = 0.3$

$P(X \le 3) = 0.1 + 0.2 + 0.4 = 0.7$

$\therefore$ Median $= \frac{2+3}{2} = 2.5$

Mode = The value of X at which P(X) is maximum = 3

Variance $= \sum_{k=1}^{5} x_k^2.P(X=k) - \mu^2$

$= 10.2 - 9 = 1.2$

Standard deviation $= \sqrt{1.2} = 1.095$

44. Ans: k = 6, Mean = $\frac{1}{2}$, Median = $\frac{1}{2}$, Mode = $\frac{1}{2}$ and S.D = $\frac{1}{2\sqrt{5}}$

Sol: we have $\int_{-\infty}^{\infty} f(x)dx = 1$

$\int_0^1 k(x - x^2)dx = 1$

$\Rightarrow k\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1\right] = 1$

$\Rightarrow k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$

$\Rightarrow k\left(\frac{3-2}{6}\right) = 1 \Rightarrow k = 6$

Mean $= \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 6(x^2 - x^3)dx$

$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{2}$

Median is that value 'a' for which

$P(X \le a) = \frac{1}{2}$ $\int_0^a 6(x - x^2)dx = \frac{1}{2}$

$\Rightarrow 6\left(\frac{a^2}{2} - \frac{a^3}{3}\right) = \frac{1}{2}$

$\Rightarrow 3a^2 - 2a^3 = \frac{1}{2} \Rightarrow a = \frac{1}{2}$

Mode a that value at which f(x) is max/min

$\therefore f(x) = 6x - 6x^2$

$f^1(x) = 6 - 12x$

For max or min $f^1(x) = 0 \Rightarrow 6 - 12x = 0$

$\Rightarrow x = \frac{1}{2}$ $f^{11}(x) = -12$ $f^{11}\left(\frac{1}{2}\right) = -12 < 0$

$\therefore$ maximum at $x = 1/2$

$\therefore$ mode is 1/2

S.D $= \sqrt{E(x^2) - (E(x))^2}$

$= \frac{1}{2\sqrt{5}}$

6. Linear Algebra

01. Ans: 46656

Sol: $|A| = \begin{vmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 5 & 2 & 3 & -5 \\ -1 & -2 & 0 & 7 \end{vmatrix}$

Expanding by second row

$|A| = \begin{vmatrix} 1 & 0 & 5 \\ 5 & 3 & -5 \\ -1 & 0 & 7 \end{vmatrix}$

Expanding by second column,

$|A| = 3\begin{vmatrix} 1 & 5 \\ -1 & 7 \end{vmatrix} = 36$

$|\text{adj } A| = |A|^3 = (36)^3 = 46656$

Note: If A is n × n matrix,

then $|\text{adj } A| = |A|^{n-1}$

02. Ans: 625

Sol: If A is n × n matrix,

then $|\text{adj (adj } A)| = |A|^{(n-1)^2}$

$|A| = -5$

$|\text{adj (adj } A)| = (-5)^4 = 625$

03. Ans: (b)

Sol: Here determinant of A = –8

$A^{-1} = \dfrac{\text{adj } A}{|A|}$

$\Rightarrow c = \dfrac{-1}{8}$ (cofactor of the element 6 in A)

$= \dfrac{-1}{8}.\left(-1^{3+1}\right)\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} = -1$

04. Ans: 324

Sol: Det $M_r = 2r - 1$

Det M_1 + Det M_2 + ... Det M_{18}

$= 1 + 3 + 5 + \ldots\ldots\ldots + 37$

$= 324$

05. Ans: –3

Sol: Given that $|A|^{10} = 2^{10}$

$\Rightarrow |A| = \pm 2$

$\Rightarrow -\alpha^3 - 25 = \pm 2$

$\Rightarrow \alpha^3 = -27$ or $\alpha^3 = -23$

$\Rightarrow \alpha = -3$ or $\alpha = (-23)^{\frac{1}{3}}$

06. Ans: 8

Sol: Given that $\sum_{n=1}^{k} A_n = 72$

$\Rightarrow \begin{vmatrix} k & k & k \\ k^2+k & k^2+k+1 & k^2+k \\ k^2 & k^2 & k^2+k+1 \end{vmatrix} = 72$

$C_2 \to (C_2 - C_1)$, $C_3 \to C_3 - C_1$

$\Rightarrow \begin{vmatrix} k & 0 & 0 \\ k^2+k & 1 & 0 \\ k^2 & 0 & k+1 \end{vmatrix} = 72$

$\Rightarrow k(k+1) = 72u$

$\Rightarrow k = 8$

07. Ans: $\frac{1}{2}$ (or) 0.5

Sol: Given that $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

$R_2 - R_1$, $R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 1 \\ 0 & 0 & \cos\theta \end{vmatrix}$$

$= \sin\theta . \cos\theta$

$= \dfrac{\sin 2\theta}{2}$

$\therefore$ maximum value of $\Delta = \dfrac{1}{2}$

08. Ans: 0

Sol: Given that

$$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & x \end{vmatrix}$$

applying $\dfrac{R_2}{x}$ and $\dfrac{R_3}{x}$

$$\frac{f(x)}{x^2} = \begin{vmatrix} \cos x & x & 1 \\ \dfrac{2\sin x}{x} & x & 2 \\ \dfrac{\tan x}{x} & 1 & 1 \end{vmatrix}$$

$$\underset{x\to 0}{\text{Lt}} \frac{f(x)}{x^2} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

09. Ans: (c)

Sol: In a symmetric matrix, the diagonal elements are zero and $a_{ij} = -a_{ji}$ for $i \neq j$.

Each element above the principal diagonal, we can choose in 3 ways (0, 1, –1).

Number of elements above the principal diagonal = $\dfrac{n(n-1)}{2}$

$\therefore$ By product rule,

Required number of skew symmetric matrices = $3^{\frac{n(n-1)}{2}}$.

10. Ans: (c)

Sol: Number of 2×2 determinants possible with each ,entry as 0 or 1 = $2^4 = 16$.

Let $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

If $\Delta > 0$ then $a = d = 1$ and atleast one of the entries b or c is 0.

$\therefore$ The determinants whose value is +ve are

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$\therefore$ Required probability = $\dfrac{3}{16}$

11. Ans: 1

Sol: If the vectors are linearly dependent, then

$$\begin{vmatrix} 1-t & 0 & 0 \\ 1 & 1-t & 0 \\ 1 & 1 & 1-t \end{vmatrix} = 0$$

$\Rightarrow (1 - t)^3 = 0$

$\Rightarrow t = 1$

12. Ans: 1

Sol: If the vectors are linearly independent, then

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & t \\ 0 & 0 & 1 & 0 \end{vmatrix} \neq 0$$

Expanding by third column

$$\Rightarrow (-1) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & t \end{vmatrix} \neq 0$$

$\Rightarrow (-1). (1 -(t-1)-1) \neq 0$

$\Rightarrow t \neq 1$

13. Ans: (c)

Sol: $\begin{pmatrix}-1 & 2 & 5\\ 2 & -4 & a-4\\ 1 & -2 & a+1\end{pmatrix} \sim \begin{pmatrix}-1 & 2 & 5\\ 0 & 0 & a+6\\ 0 & 0 & a+6\end{pmatrix}$

$= \begin{pmatrix}-1 & 2 & 5\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ if a = –6 and Rank = 1

If a ≠ –6 then Rank of the matrix is 2

∴ Option (c) is correct.

14. Ans: (c)

Sol: $[A - A^T]$ is a 3×3 skew-symmetric matrix

$\Rightarrow |A - A^T| = 0$

$\Rightarrow \rho(A) \neq 3$

$[A - A^T]$ is not a zero matrix ($\because A \neq A^T$)

Rank of $[A - A^T]$ is not zero.

Further, rank of a skew-symmetric matrix cannot be 1.

$\therefore \rho(A - A^T) = 2$

15. Ans: 4

Sol: $A = \begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & -1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$

$R_4 \to R_4 + R_1$

$A \sim \begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$

$R_2 \leftrightarrow R_3$

$A \sim \begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & +1 & 0\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$

$R_4 \to R_4 + R_2$

$A \sim \begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & +1 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$

$R_4 \to R_4 + R_3$

$A \sim \begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$

$R_5 \to R_5 - R_4$

$A = \begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

= Echelon form of A

∴ Rank of A = number of non-zero rows in Echelon form of 'A' = 4

16. Ans: (b)

Sol: The augmented matrix of the given system is

$[A|B] = \left[\begin{array}{cccc|c}1 & -1 & 2 & -1 & 1\\ 1 & 0 & 1 & 1 & 0\\ 0 & -1 & 1 & -2 & -1\end{array}\right]$

$R_2 - R_1$

$$\sim\left[\begin{array}{cccc|c}1 & -1 & 2 & -1 & 1\\0 & 1 & -1 & 2 & -1\\0 & -1 & 1 & -2 & -1\end{array}\right]$$

$R_3 + R_2$

$$\sim\left[\begin{array}{cccc|c}1 & -1 & 2 & -1 & 1\\0 & 1 & -1 & 2 & -1\\0 & 0 & 0 & 0 & -2\end{array}\right]$$

Rank of coefficient matrix A = 2

Rank of [A|B] = 3

$\therefore$ The system has no solution

17. Ans: – 2

Sol: Let AX = B be the given system

If the system has no solution, then |A| = 0

$$\Rightarrow \begin{vmatrix}k & 1 & 1\\1 & k & 1\\1 & 1 & k\end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix}k+2 & 1 & 1\\k+2 & k & 1\\k+2 & 1 & k\end{vmatrix} = 0$$

$R_2 - R_1,\ R_3 - R_1$

$$\Rightarrow \begin{vmatrix}k+2 & 1 & 1\\0 & k-1 & 0\\0 & 0 & k-1\end{vmatrix} = 0$$

$\Rightarrow (k + 2)(k - 1)^2 = 0$

$\Rightarrow k = -2, 1$

when k = –2, the augmented matrix of the system is

$$[A|B] = \left[\begin{array}{ccc|c}-2 & 1 & 1 & 1\\1 & -2 & 1 & 1\\1 & 1 & -2 & 1\end{array}\right]$$

$2R_2 + R_1,\ 2R_3 + R_1$

$$\sim\left[\begin{array}{ccc|c}-2 & 1 & 1 & 1\\0 & -3 & 3 & 3\\0 & 3 & -3 & 3\end{array}\right]$$

$R_3 + R_2$

$$\sim\left[\begin{array}{ccc|c}-2 & 1 & 1 & 1\\0 & -3 & 3 & 3\\0 & 0 & 0 & 6\end{array}\right]$$

$\rho(A|B) = 3$ and $\rho(A) = 2$

$\therefore$ The system has no solution when k = – 2

18. Ans: (d)

Sol: $A = \begin{pmatrix}1 & 1 & 3\\5 & 2 & 6\\-2 & -1 & -3\end{pmatrix}$

$R_2 - 5R_1$

$R_3 + 2R_1$

$$\sim\begin{pmatrix}1 & 1 & 3\\0 & -3 & -9\\-0 & 1 & 3\end{pmatrix}$$

$3R_3 + R_2$

$$\sim\begin{pmatrix}1 & 1 & 3\\0 & -3 & -9\\0 & 0 & 0\end{pmatrix}$$

Here $\rho[A] = 2$

If B is a linear combination of columns of A, then

$\rho[A] = \rho[A|B]$

$\therefore$ The system has infinitely many solutions

19. Ans: (c)

Sol: If the system has non trivial solution, then

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = 0$$

$R_2 - R_1, R_3 - R_1$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

$\Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow a + b + c = 0$ or

$a = b = c$

20. Ans: (b)

Sol: Let the given system be AX = B

The augmented matrix of the system

$$= [A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right]$$

$R_2 - 2R_1$
$R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-b-3a \end{array}\right]$$

The system is inconsistent

if $c - b - 3a \neq 0$

$\Rightarrow 3a + b - c \neq 0$

21. Ans: (c)

Sol: Let the given system be AX = B

The augmented matrix of the system =

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 - R_1, R_2 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-6 \end{array}\right]$$

The system has unique solution if $\lambda \neq 3$.

22. Ans: (c)

Sol: $A = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 1 & 4 & 3 & -1 \\ 2 & 3 & -4 & -7 \\ 3 & 8 & 1 & -7 \end{bmatrix}$

Applying $R_2 - R_1, R_3 - 2R_1, R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -1 & -2 & -1 \end{bmatrix}$$

Applying $R_3 + 3R_2$ and $R_4 + R_2$

$$A \sim \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

= Echelon form of A

$\Rightarrow$ Rank of A = 2

Number of linearly independent solutions = n – k.
Where n is the number of variables and k is the rank of A
= 4 – 2 = 2.

23. Ans: 2

Sol: The characteristic equation is

$|A - \lambda I| = 0$

A real eigen value of A is $\lambda = 5$

The eigen vectors for $\lambda = 5$ are given by

$[A - 5I]\, X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_4 = 0, x_3 = 0$

$\Rightarrow \rho[A] = 2$ and n = 4=number of variables

$\therefore$ The number of linear independent eigen vectors corresponding to $\lambda = 5$ are 2.

24. Ans: 0

Sol: Let $a = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = \pm 5$

The eigen vectors for $\lambda = 5$ are given by

$[A - 5I]\, X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - 2y = 0$

$\therefore X_1 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The eigen vectors for $\lambda = -5$ are given by

$[A + 5I]\, X = 0$

$$\Rightarrow \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x + y = 0$

$\therefore X_2 = c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\therefore a + b = 0$

25. Ans: (a)

Sol: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

The eigen values of A are 1, 2

The eigen vectors for $\lambda = 1$ are given by

$[A - I]\, X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y = 0$

$\therefore X_1 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The eigen vectors for $\lambda = 2$ are given by

$[A - 2I]\, X = 0$

$$\Rightarrow \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x + y = 0$

$\therefore X_2 = c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore$ The eigen vector pair is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

26. Ans: (c)

Sol: If A is singular then 0 is an eigen value of A.

$\therefore$ The minimum eigen value of A is 0.

The eigen vectors corresponding to the eigen value $\lambda = 0$ is given by

$$[A - 0I]X = 0$$

$$\Rightarrow \begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying cross multiplication rule for first and second rows of A, we have

$$\Rightarrow \frac{x}{11} = \frac{y}{-11} = \frac{z}{11}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore$ The eigen vectors are

$$X = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

27. Ans: (b)

Sol: Here, A is the elementary matrix obtained given I_3 with elementary operation $R_1 \leftrightarrow R_3$

$$\therefore A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)(\lambda^2 - 1) = 0 \Rightarrow \lambda = 1, 1, -1$

28. Ans: –6

Sol: The given matrix has rank 2

$\therefore$ There are only 2 non zero eigen values

The characteristic equation is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & -1-\lambda & -1 & -1 & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 1 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_5$ and

$R_2 \to R_2 + R_3 + R_4$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 & 0 & 2-\lambda \\ 0 & -3-\lambda & -3-\lambda & -3-\lambda & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 1 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda).(-3 - \lambda).$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 1 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 2, -3$

$\therefore$ product of the non zero eigen values = –6

29. Ans: 3

Sol: If λ is eigen value then $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ 17 & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$$

$\Rightarrow 6 + 2k = \lambda$

$21 + k = 2\lambda$

$\Rightarrow 42 + 2k = 4\lambda$

$\lambda = 12$ and $k = 3$

30. Ans: 3

Sol: Sum of the eigen values = Trace of A = 14

$\Rightarrow a + b + 7 = 14$ ...... (i)

product of eigen values = $|A| = 100$

$\Rightarrow 10ab = 100$

$\Rightarrow ab = 10$ .....(ii)

solving (i) & (ii), we have

$\Rightarrow a = 5$ and $b = 2$

$\therefore |a - b| = 3$

31. Ans: 1

Sol: Product of eigen values = $|A| = 0$

$$\Rightarrow \begin{vmatrix} 3 & 4 & 2 \\ 9 & 13 & 7 \\ -6 & -9+x & -4 \end{vmatrix} = 0$$

$R_2 - 3R_1$, $R_3 + 2R_1$

$$\Rightarrow \begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & x-1 & 0 \end{vmatrix} = 0$$

$\Rightarrow 3(1 - x) = 0$

$\Rightarrow x = 1$

32. Ans: (d)

Sol: The characteristic equation is

$\lambda^4 = \lambda$

$\Rightarrow \lambda^4 - \lambda = 0$

$\Rightarrow \lambda(\lambda^3 - 1) = 0$

$\Rightarrow \lambda = 0, 1, -1 \pm \sqrt{3}\, i$

$\Rightarrow \lambda = 0, 1, -0.5 \pm (0.866)i$

33. Ans: 3

Sol: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Here, A is upper triangular matrix

The eigen values are $\lambda = 2, 2, 3$

The eigen vectors for $\lambda = 2$ are given by

$[A - 2I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Here Rank of $[A - 2I] = 1$

$\therefore$ Number of Linearly independent eigen vectors for $\lambda = 2$ is $n - r$

$= 3 - 1 = 2$

For since, $\lambda = 3$ is not a repeated eigen value, there will be only one independent eigen vector for $\lambda = 3$.

$\therefore$ The number of linearly independent eigen vectors of A = 3.

34. Ans: (d)

Sol: The characteristic equation is

$(\lambda^3 - 6\lambda^2 + 9\lambda - 4) = 0$

$|A|$= product of the roots of the characteristic equation = 4

Trace of A = sum of the roots of characteristic equation = 6

35. Ans: (b)

Sol: A is symmetric matrix.

The eigen vectors of A are orthogonal.

For the given eigen vector, only the vector given in option (b) is orthogonal.

$\therefore$ option (b) is correct.

36. Ans: (c)

Sol: The characteristic equation is

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda = 0, 3, 15$

The eigen vector for $\lambda = 15$ are given by

$[A - 15I]\, X = 0$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x}{40} = \frac{y}{-40} = \frac{z}{20}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$\therefore$ The eigen vectors for $\lambda = 15$ are

$$X = k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad (k \neq 0)$$

37. Ans: (c)

Sol: The characteristic equation is

$$\begin{vmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a-\lambda)\,[\{(a-\lambda)^2 - 1\} - (a-\lambda)] = 0$

$\Rightarrow \lambda = a, a \pm \sqrt{2}$

38. Ans: $\lambda^2 - 3\lambda + 2$

Sol: The characteristic equation is

$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda - 1)\,(\lambda - 2)^2 = 0$

$\therefore$ Either $(\lambda - 1)(\lambda - 2)$ or $(\lambda - 1)(\lambda - 2)^2$ is the minimal polynomial

$(A - I)\,(A - 2I)$

$$= \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} = O$$

$\therefore$ The minimal polynomial of A

$= (\lambda - 1)(\lambda - 2)$

$= \lambda^2 - 3\lambda + 2$

LU Decomposition

39. Ans: (b)

Sol: The coefficient matrix

$$A = \begin{bmatrix} 4 & 5 \\ 12 & 14 \end{bmatrix}$$

$R_2 - 3R_1$

$$\sim \begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$$

40. Ans: (b)

Sol: $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$

$R_2 - 2R_1$

$R_3 + 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 5 \end{bmatrix}$$

$R_3 - 4R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

41. Ans: (a)

Sol: $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix}$

$R_2 - 2R_1$

$R_3 + R_1$

$\sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix}$

$R_3 + \frac{5}{2}R_2$

$\sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & \frac{-3}{2} \end{bmatrix}$

From the elementary operations used above, we can write

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \frac{-5}{2} & 1 \end{bmatrix}$

7. Calculus

01. Ans: 1

Sol: Put $x = \frac{1}{y}$. Then $y \to 0$ as $x \to \infty$.

$$\text{Given Lt} = \underset{y\to 0}{\text{Lt}}\left[\sqrt{\frac{1}{y^2}+\frac{2}{y}-1}-\frac{1}{y}\right]$$

$$= \underset{y\to 0}{\text{Lt}}\left[\frac{\sqrt{1+2y-y^2}-1}{y}\right]$$

$$= \underset{y\to 0}{\text{Lt}}\frac{1}{2\sqrt{1+2y-y^2}}(2-2y)$$

(By L-Hospital's rule)

$= 1$

02. Ans: 1

Sol: $\underset{x\to a}{\text{Lt}} = \frac{a^x - x^a}{x^x - a^a} = -1$

$\Rightarrow \underset{x\to a}{\text{Lt}} \frac{a^x \log a - a\,x^{a-1}}{x^x(1+\log x)} = -1$

(By L' Hospital's Rule)

$\Rightarrow \frac{a^a \log a - a.a^{a-1}}{a^a(1+\log a)} = -1$

$\Rightarrow \frac{\log a - 1}{\log a + 1} = -1$

$\Rightarrow \log a - 1 = -\log a - 1$

$\Rightarrow \log a = 0$

$\Rightarrow a = 1$

03. Ans: (a)

Sol: Given limit

$$= \underset{n\to\infty}{\text{Lt}}\left[\frac{(1-x)(1+x)(1+x^2)(1+x^4)........(1+x^{2n})}{1-x}\right]$$

$$= \underset{n\to\infty}{\text{Lt}}\left[\frac{1+x^{2n+1}}{1-x}\right] = \frac{1}{1-x}$$

04. Ans: –2

Sol: $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\left(\sec x - \frac{1}{1-\sin x}\right)$ $[\infty - \infty \text{ form}]$

$$= \underset{x\to\frac{\pi}{2}}{\text{Lt}}\left[\frac{1-\sin x-\cos x}{\cos x(1-\sin x)}\right] \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \underset{x\to\frac{\pi}{2}}{\text{Lt}}\left[\frac{-\cos x+\sin x}{-\sin x-\cos 2x}\right]$$

(by L' Hospital's Rule)

$= -2$

05. Ans: (c)

Sol: $y = \underset{x\to\frac{\pi}{4}}{\text{Lt}}(\tan x)^{\tan 2x}$ (1^∞ form)

$$\log y = \underset{x\to\frac{\pi}{4}}{\text{Lt}} \log(\tan x)^{\tan 2x}$$

$$\underset{x\to\frac{\pi}{4}}{\text{Lt}}((\tan 2x).\log(\tan x))$$

$$= \underset{x\to\frac{\pi}{4}}{\text{Lt}}\left[\frac{\log(\tan x)}{\cot 2x}\right]$$

$$= \underset{x\to\frac{\pi}{4}}{\text{Lt}}\left[\frac{\cot x.\sec^2 x}{-2\,\text{cosec}^2\,2x}\right]$$

$= -1$

$\Rightarrow y = e^{-1}$

06. Ans: (d)

Sol: (A) $\underset{x\to0}{\text{Lt}} f(x) = \underset{x\to0}{\text{Lt}}\left(\frac{\tan x}{x}\right) = 1 = f(0)$

$\therefore$ f(x) is continuous at x = 0

(B) $\underset{x\to0}{\text{Lt}} f(x) = \underset{x\to0}{\text{Lt}} \frac{5x^2-3x}{2x}$

$= \underset{x\to0}{\text{Lt}}\left(\frac{10x-3}{2}\right)$ (By L-Hospital's rule)

$= -\frac{3}{2} = f(0)$

$\therefore$ f(x) is continuous at x = 0

(C) $\underset{x\to0}{\text{Lt}} f(x) = \underset{x\to0}{\text{Lt}}\left[\frac{\sqrt{1+x}-1}{x}\right]$

$= \underset{x\to0}{\text{Lt}}\left[\frac{(1+x)-1}{x(\sqrt{1+x}+1)}\right]$

$= \frac{1}{2} = f(0)$

$\therefore$ f(x) is continuous at x = 0

(D) $\underset{x\to0}{\text{Lt}} f(x) = \underset{x\to0}{\text{Lt}}\left(\frac{\sin^2 x}{1-\cos x}\right)$

$= \underset{x\to0}{\text{Lt}} \frac{2\sin x.\cos x}{\sin x}$

$= 2 \neq f(0)$

$\therefore$ f(x) is not continuous at x = 0

07. Ans: (a)

Sol: $\underset{x\to2^-}{\text{Lt}} f(x) = \underset{x\to2^+}{\text{Lt}} f(x) = f(2) = 3$

$\therefore$ The above function is continuous at x = 2

but $f'(x) = \begin{cases} 1, & x \le 2 \\ -1, & x > 2 \end{cases}$

$\therefore \underset{x\to2^-}{\text{Lt}} f'(x) = 1$ and $\underset{x\to2^+}{\text{Lt}} f'(x) = -1$

i.e. f(x) is not differentiable at x = 2

08. Ans: (d)

Sol: $f'(1+) = \underset{x\to1^+}{Lt} \frac{|x-3|-|-2|}{x-1}$

$= \underset{x\to1^+}{Lt} \frac{-x+3-2}{x-1} = -1$

$f'(1-) = \underset{x\to1^-}{Lt} \frac{\frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}-2}{x-1}$

$= \frac{1}{4} \underset{x\to1^-}{Lt} \frac{(x-1)(x-5)}{(x-1)} = -1$

$\therefore$ f is continuous and differentiable at x = 1

$f'(3^+) = \underset{x\to3^+}{Lt} \frac{|x-3|-0}{(x-3)} = 1$

$f'(3^-) = \underset{x\to3^-}{Lt} \frac{-(x-3)}{(x-3)} = -1$

$\therefore$ f is not differentiable at x = 3

However, $\underset{x\to3}{Lt} f(x) = \underset{x\to3}{Lt} |x-3| = 0 = f(3)$

$\therefore$ f is continuous at x = 3

09. Ans: (a)

Sol: If f(x) is continuous at x = 0, then

$\underset{x=0}{Lt} f(x) = f(0)$

$\Rightarrow \underset{x\to0}{Lt} \left(\frac{1-x}{1+x}\right)^{\frac{1}{x}} = f(0)$

$\Rightarrow \underset{x\to0}{Lt} \left[\frac{(1-x)^{\frac{1}{x}}}{(1+x)^{\frac{1}{x}}}\right] = f(0)$

$\Rightarrow \frac{e^{-1}}{e} = f(0)$

$\Rightarrow f(0) = e^{-2}$

10. Ans: (c)

Sol: $f^1(x) = 2ax, x \le 1$

$= 2x + a, x > 1$

Since, f(x) is differentiable at x = 1

$f^1(1^-) = f'(1^+)$

$2a = a + 2 \Rightarrow a = 2$

Since, f(x) is continuous at x = 1

$f(1^-) = f(1^+)$

$a + 1 = 1 + a + b \Rightarrow b = 0$

11. Ans: (c)

Sol: (a) Let f(x) = (x–2) in [1, 3]

Here, $f(1) \ne f(3)$

$\therefore$ Roll's theorem is not applicable

(b) Let $f(x) = 1 - (1-x)^{-1}$ in [0, 2]

Here, f(x) is not continuous in [0, 2]

$\therefore$ Roll's theorem is not applicable

(c) Let f(x) = sin x in [0, π]

Here, f(x) is continuous in [0, π] and differentiable in (0, π).

Further, f(0) = f(π)

$\therefore$ Roll's theorem is applicable

(d) Let f(x) = Tan x in [0, 2π]

Here, f(x) is not continuous in [0, 2π]

$\therefore$ Roll's theorem is not applicable

12. Ans: 0.236

Sol: f(x) = x(x – 1) (x – 2)

We have,

$f'(c) = 3c^2 - 6c + 2$, a = 0 and b = 1/2

Substituting in the given relation, we get $12c^2 - 24c + 5 = 0$

$\Rightarrow$ c = 1.764 (or) 0.236

We have to choose c = 0.236, since it only lies between 0 and 1/2.

13. Ans: 1.732

Sol: By Cauchy's mean value theorem

$$\frac{f'(d)}{g'(d)} = \frac{f(3)-f(1)}{g(3)-g(1)}$$

$$\Rightarrow -d = \left[\frac{\sqrt{3}-1}{\frac{1}{\sqrt{3}}-1}\right]$$

$$\Rightarrow \quad d = \sqrt{3}$$

14. Ans: 8

Sol: $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f^1(x) = 6x^2 - 6x - 12$

$f^1(x) = 0 \Rightarrow x = -1, 2$

$f^{11}(x) = 12x - 6$

$f^{11}(-1) = -18, \quad f^{11}(2) = 18$

f(x) has maximum at x = –1

The maximum value of f(x) at x = –1

$= f(-1) = 8$

Further, At the end points of the given interval

$f(-2) = -3$ and $f\left(\frac{5}{2}\right) = \frac{-33}{2}$

∴ The greatest value of f(x) = 8

15. Ans: (c)

Sol: $f(x) = \sin x - x + \frac{x^3}{3}$

$f'(x) = \cos x - 1 + x^2$

$f''(x) = -\sin x + 2x$

$f'''(x) = -\cos x + 2$

$f'(0) = 0, f''(0) = 0$ and $f'''(0) = 1 \neq 0$

∴ f(x) has neither maximum nor minimum at x = 0

16. Ans: (b)

Sol: $f(x) = ax + \frac{b}{x}$

$f'(x) = a - \frac{b}{x^2}$

$f''(x) = \frac{2b}{x^3}$

$f'(x) = 0 \Rightarrow x = \sqrt{\frac{b}{a}}$

At $x = \sqrt{\frac{b}{a}}$, $f''(x) > 0$

∴ f(x) has minimum at $x = \sqrt{\frac{b}{a}}$

This is the only extremum (minimum in the interval $(0, \infty)$).

∴ The least value of f(x) occurs at $x = \sqrt{\frac{b}{a}}$

17. Ans: (c)

Sol: $f(x) = \int_0^x \frac{\sin t}{t}\, dt$

$f'(x) = \frac{\sin x}{x}$

$f'(x) = 0 \Rightarrow x = n\pi \qquad (n = 1, 2, 3..........)$

$f''(x) = \frac{x\cos x - \sin x}{x^2}$

$f'(n\pi) = \frac{1}{n\pi}\cos(n\pi) = \frac{(-1)^n}{n\pi} \neq 0$

$f''(n\pi) > 0$ if n is even

and $f''(n\pi) < 0$ if n is odd

∴ f(x) has maximum at $x = n\pi$ where n is odd

Further f(x) has minimum at $x = n\pi$ where n is even

18. Ans: (c)

Sol: Let $g(x) = 3x^4 + 8x^3 - 18x^2 + 60$

$g^1(x) = 12x^3 + 24x^2 - 36x$

$g^1(x) = 0 \Rightarrow x = -3, 0, 1$

$g^{11}(x) = 36x^2 + 48x - 36$

$g^{11}(0) = -36 < 0$

$\Rightarrow$ g(x) has a maximum at x = 0

$\Rightarrow$ f(x) has a minimum at x = 0

$g^{11}(-3) > 0 \Rightarrow$ g(x) has a minimum at x = –3

$\Rightarrow$ f(x) has a maximum at x = –3

$g^{11}(1) > 0 \Rightarrow$ g(x) has a minimum at x = 1

$\Rightarrow$ f(x) has a maximum at x = 1

19. Ans: (a)

Sol: $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

consider $f_x = 4x - 4x^3 = 0$

$\Rightarrow x = 0, 1, -1$

$f_y = -4y + 4y^{3= = 0}$

$\Rightarrow y = 0, 1, -1$

$r = f_{xy} = 4 - 12x^2$

$s = f_{xy} = 0$

$t = f_{yy} = -4 + 12y^2$

At (0,1), we have $r > 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has minimum at (0,1)

At (–1, 0), we have $r < 0$ and $(rt - s^2) > 0$

$\therefore$ f(x, y) has a maximum at (–1, 0)

20. Ans: 0.523

Sol: We have,

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Here, $f(x) = \dfrac{1}{1+\tan^4 x}$

$$= \frac{\cos^4 x}{\cos^4 x + \sin^4 x}$$

$$f(a+b-x) = f\left(\frac{\pi}{2} - x\right)$$

$$= \frac{\sin^4 x}{\sin^4 x + \cos^4 x}$$

Let $I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} f(x)\,dx$

$$= \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\,dx$$

again

$$I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} f(a+b-x)\,dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx$$

adding

$$2I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} dx = \frac{4\pi}{12}$$

$$\therefore I = \frac{\pi}{6}$$

21. Ans: (a)

Sol: Let $I = \int_0^\pi \dfrac{\sin(2kx)}{\sin x}\,dx$

$= \int_0^\pi \dfrac{\sin(2k(\pi - x))}{\sin(\pi - x)}\,dx$ (By properties of definite integrals)

$$= -\int_0^\pi \frac{\sin(2kx)}{\sin x}\,dx = -I$$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

22. Ans: (d)

Sol: $\int_a^b \left[|x-a| + |x-b|\right] dx$

$$= \int_a^b \{(x-a) + (b-x)\}\,dx$$

$= (b-a)^2$

23. Ans: 39

Sol: $\int_4^{10} [x]\, dx$

$= \int_4^5 [x]\, dx + \int_5^6 [x]\, dx + ... + \int_9^{10} [x]\, dx$

$= \int_4^5 4\, dx + \int_5^6 5\, dx + ... + \int_9^{10} 9\, dx$

$= 4 + 5 + ... + 9$

$= 39$

24. Ans: (a)

Sol: $\int_0^{\pi} x \sin^4 x \cos^6 x\, dx$

$I = \frac{\pi}{2}\int_0^{\pi} \sin^6 x \cos^4 x\, dx$ (property 9)

$= 2\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x\, dx$ (property 6)

$I = \pi\left[\frac{5.3.1.3.1}{10.8.6.4.2}\frac{\pi}{2}\right] = \frac{3\pi^2}{512}$

25. Ans: 4

Sol: $\int_0^{2\pi} [x \sin x]\, dx = k\pi$

$\Rightarrow \int_0^{\pi} x \sin x\, dx + \int_{\pi}^{2\pi} -x \sin x\, dx = k\pi$

$\Rightarrow [x(-\cos x + \sin x)]_0^{\pi} - [-x\cos x + \sin x]_{\pi}^{2\pi} = k\pi$

$\Rightarrow \pi - [-3\pi] = k\pi$

$\Rightarrow k = 4$

26. Ans: (a)

Sol: Put x = 2 Sec t

dx = 2 Sec t tan t dt

$x = 2 \Rightarrow t = 0$

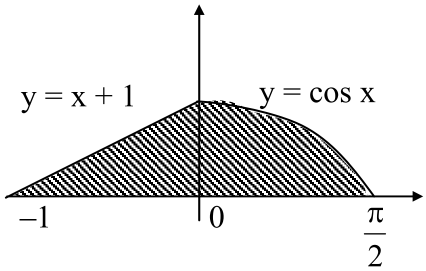
$x = 4 \Rightarrow t = \frac{\pi}{3}$

Given integral $= \frac{1}{4}\int_0^{\frac{\pi}{3}} \sin^2 t \cos t\, dt$

$= \frac{1}{12}\left[\sin^3 t\right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{32}$

27. Ans: 1.5 (range 1.4 to 1.6)

Sol: The region bounded by the curves is shown below



$y = f(x) = \begin{cases} x+1 \text{ if } -1 \le x \le 0 \\ \cos x \text{ if } 0 \le x \le \frac{\pi}{2} \end{cases}$

The require area

$= \int_{-1}^{0} (x+1)dx + \int_0^{\frac{\pi}{2}} \cos x\, dx$

$= \frac{3}{2}$

28. Ans: (a)

Sol: Put x = sin t so that dx = cos t dt

t varies from 0 to π/2.

Given integral $= \int_0^{\pi/2} \sin^4 t \cos^4 t\, dt$

$= \frac{(3.1)\times(3.1)}{8.6.4.2}.\frac{\pi}{2} = \frac{3\pi}{256}.$

29. Ans: 25.12

Sol: Volume $= \int_0^4 \pi y^2\, dx$

$= \int_0^4 \pi x\, dx = 8\pi$ cubic units

30. Ans: (d)

Sol: Length = $\int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_0^3 \sqrt{1+x}\,dx$

$= \frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^3$

$= \frac{14}{3}$

31. Ans: (b)

Sol: $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

Let $f(x) = xe^{-x^2}$

$f(-x) = -xe^{-x^2} = -f(x)$

Converges to 0

$\Rightarrow$ f(x) is odd function

$\therefore$ The value of the given integral is 0.

32. Ans: (a)

Sol: $\int_{-1}^{1}\frac{dx}{x^2} = 2\int_0^1\frac{dx}{x^2}$ ($\because$ $\frac{1}{x^2}$ is even function)

$= 2\ \underset{x\to 0^+}{\text{Lt}}\int_0^1\frac{dx}{x^2}$

(since $\frac{1}{x^2}$ is not defined)

$= 2\left(\frac{-1}{x}\right)_0^1$

$= 2\{(-1) - (-\infty)\}$

$= \infty$(Divergent)

33. Ans: (a)

Sol: $\int_1^3 \frac{\sqrt{1+x}}{(x-1)^2}dx$ at x =1

Let $f(x) = \frac{\sqrt{1+x}}{(x-1)^2}$

$f(x) \to \infty$ as $x \to 1$

Choose $g(x) = \frac{1}{(x-1)^2}$

$\underset{x\to 1}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{x\to 1}{\text{Lt}}\left(\frac{\sqrt{1+x}}{(x-1)^2}\times(x-1)^2\right) = \sqrt{2}$

$\int_1^3\frac{1}{(x-1)^2} = \left(\frac{-1}{x-1}\right)_1^3$

$= \frac{-1}{2} + \frac{1}{0} = \infty$

$\int_1^3 g(x)\,dx$ is divergent

$\therefore$ By comparison test, the given integral also diverges.

34. Ans: (a)

Sol: $\int_1^2\frac{x^3+1}{\sqrt{2-x}}dx$

Let f(x) = $\frac{x^3+1}{\sqrt{2-x}}$

$f(x) \to \infty$ an $x \to 2$

$\underset{x\to 2}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{x\to 2}{\text{Lt}}\left(\frac{x^3+1}{\sqrt{2-x}}\times\sqrt{2-x}\right) = 9$ (finite)

$\int_1^2 g(x) = \int_1^2\frac{1}{\sqrt{2-x}} = 2\left|\sqrt{t}\right|_1^2 = 2$ convergent

$\therefore$ By comparison test, the given integral also diverges.

35. Ans: (d)

Sol: $\int_1^{\infty} \frac{e^{-x}}{x^2}\, dx$

Let $f(x) = \frac{e^{-x}}{x^2}$

Choose $g(x) = \frac{1}{x^2}$

We have $0 \le f(x) \le g(x) \quad \forall \ x \ge 1$

$\int_1^{\infty} g(x)\,dx$ is known to be convergent.

$\therefore$ By compsssarison test, the given integral also converges.