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ENGINEERING MATHEMATICS

Volume - I : Study Material with Classroom Practice Questions

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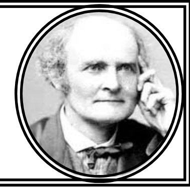
Classroom Practice solutions To Engineering Mathematics

CONTENTS

Chapter **1** # *Linear Algebra*



Arthur Cayley
(1821 – 1895)

01. Ans: 1500

Sol: Given that P is 10×5 matrix.

Q is 5×20 matrix

and R is 20×10 matrix

Now PQR is 10×10 matrix. Total number of elements in PQR = 100. Here, we can find the product PQR only in two ways i.e., (PQ)R and P(QR) because $PQ \neq QP$.

So, to find the product matrix PQR first we find PQ and then find (PQ)R (or) we, first find QR and then find P(QR)

For the product $(PQ)_{10\times 20}$

Number of elements in PQ = 200.

To compute each element of the matrix PQ, we require '5' multiplications.

$\therefore$ Number of multiplications = 200×5

= 1000

For the product $[(PQ)R]_{10\times10}$

Number of elements in (PQ) R = 100

To compute each element of the matrix (PQ)R, we require 20 multiplications.

$\therefore$ Number of multiplications = 100×20

= 2000

Hence, the total number of multiplication operations to find the product $[(PQ)R]_{10\times10}$

= 1000 + 2000

= 3000

Similarly, if we find the product $[P(QR)]_{10\times10}$ by above method, the total number of multiplication operations to find the product $[P(QR)]_{10\times10}$ = 1000 + 500

= 1500

$\therefore$ The minimum number of multiplication operations to find PQR = 1500.

02. Ans: (d)

Sol: Giving that

$(I - A + A^2 - \ldots + (-1)^n A^n) = O$ .... (i)

multiplying by A^{-1}

$A^{-1} - I + A - A^2 + \ldots + (-1)^{n-1}A^{n-1} = 0$ ...... (ii)

Adding (i) & (ii), we get

$A^{-1} + (-1)^n A^n = O$

$\therefore A^{-1} = (-1)^{(n-1)}. A^n$

03. Ans: (b)

Sol: Here determinant of A = –8

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$\Rightarrow c = \frac{-1}{8}$ (cofactor of the element 6 in A)

Arthur Cayley was probably the first mathematician to realize the importance of the notion of a matrix and in 1858 published book, showing the basic operations on matrices. He also discovered a number of important results in matrix theory.

$= \frac{-1}{8}.\left(-1^{3+1}\right)\begin{vmatrix}2 & 3\\0 & 4\end{vmatrix}$

$= -1$

04. Ans: 324

Sol: Det $M_r = 2r - 1$

Det M_1 + Det M_2 + ... Det M_{18}

$= 1 + 3 + 5 + ...+ 37$

$= 324$

05. Ans: –3

Sol: Given that $|A|^{10} = 2^{10}$

$\Rightarrow |A| = \pm 2$

$\Rightarrow -\alpha^3 - 25 = \pm 2$

$\Rightarrow \alpha^3 = -27$ or $\alpha^3 = -23$

$\Rightarrow \alpha = -3$ or $\alpha = (-23)^{\frac{1}{3}}$

06. Ans: 8

Sol: Given that $\sum_{n=1}^{k} A_n = 72$

$$\Rightarrow \begin{vmatrix} k & k & k \\ k^2+k & k^2+k+1 & k^2+k \\ k^2 & k^2 & k^2+k+1 \end{vmatrix} = 72$$

$C_2 \to (C_2 - C_1)$, $C_3 \to C_3 - C_1$

$$\Rightarrow \begin{vmatrix} k & 0 & 0 \\ k^2+k & 1 & 0 \\ k^2 & 0 & k+1 \end{vmatrix} = 72$$

$\Rightarrow k(k+1) = 72$

$\Rightarrow k = 8$

07. Ans: 0.5

Sol: Given that $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

$R_2 - R_1$, $R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 1 \\ 0 & 0 & \cos\theta \end{vmatrix}$$

$= \sin\theta . \cos\theta$

$= \frac{\sin 2\theta}{2}$

$\therefore$ maximum value of $\Delta = \frac{1}{2}$

08. Ans: 0

Sol: Given that

$$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & x \end{vmatrix}$$

applying $\frac{R_2}{x}$ and $\frac{R_3}{x}$

$$\frac{f(x)}{x^2} = \begin{vmatrix} \cos x & x & 1 \\ \frac{2\sin x}{x} & x & 2 \\ \frac{\tan x}{x} & 1 & 1 \end{vmatrix}$$

$$\underset{x\to 0}{\text{Lt}} \frac{f(x)}{x^2} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$= 0$

09. Ans: (c)

Sol: In a skew symmetric matrix, the diagonal elements are zero and $a_{ij} = -a_{ji}$ for $i \neq j$.

Each element above the principal diagonal, we can choose in 3 ways (0, 1, –1).

Number of elements above the principal diagonal = $\frac{n(n-1)}{2}$

∴ By product rule,

Required number of skew symmetric matrices = $3^{\frac{n(n-1)}{2}}$.

10. Ans: (c)

Sol: Number of 2×2 determinants possible with each entry as 0 or 1 = $2^4 = 16$.

Let $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

If $\Delta > 0$ then a = d = 1 and atleast one of the entries b or c is 0.

∴ The determinants whose value is +ve are

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

∴ Required probability = $\frac{3}{16}$

11. Ans: 1

Sol: If the vectors are linearly dependent, then

$$\begin{vmatrix} 1-t & 0 & 0 \\ 1 & 1-t & 0 \\ 1 & 1 & 1-t \end{vmatrix} = 0$$

$\Rightarrow (1-t)^3 = 0$

$\Rightarrow t = 1$

12. Ans: 1

Sol: If the vectors are linearly independent, then

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & t \\ 0 & 0 & 1 & 0 \end{vmatrix} \neq 0$$

Expanding by third column

$$\Rightarrow (-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & t \end{vmatrix} \neq 0$$

$\Rightarrow (-1).\ (1-(t-1)-1) \neq 0$

$\Rightarrow t \neq 1$

13. Ans: (c)

Sol: $\begin{pmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 5 \\ 0 & 0 & a+6 \\ 0 & 0 & a+6 \end{pmatrix}$

$= \begin{pmatrix} -1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ if a = –6 and Rank = 1

If $a \neq -6$ then Rank of the matrix is 2

∴ Option (c) is correct.

14. Ans: (c)

Sol: The characteristic equation is

$|A - \lambda I| = 0$

$\Rightarrow (\lambda^2 - 4)(\lambda^2 + 4) = 0$

$\Rightarrow \lambda^4 = 16$

By Caley Hamilton's Theorem

$A^4 = 16I$

15. Ans: 4

Sol: $A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

$R_4 \to R_4 + R_1$

$$A \sim \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$A \sim \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$A \sim \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_5 \to R_5 - R_4$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

= Echelon form of A

$\therefore$ Rank of A = number of non-zero rows in Echelon form of 'A' = 4

16. Ans: (b)

Sol: The augmented matrix of the given system is

$$[A|B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_2 - R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_3 + R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Rank of coefficient matrix A = 2

Rank of [A|B] = 3

$\therefore$ The system has no solution

17. Ans: (c)

Sol: Let the given system be AX = B

The augmented matrix of the system

$$=[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3 - R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_4 + R_3$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Here $\rho[A] = 3$ and

$\rho[A|B] = 4$

$\therefore$ The system has no solution.

18. Ans: (d)

Sol: $A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$

$R_2 - 5R_1$

$R_3 + 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -3 & -9 \\ -0 & 1 & 3 \end{pmatrix}$$

$3R_3 + R_2$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -3 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

Here $\rho[A] = 2$

If B is a linear combination of columns of A, then $\rho[A] = \rho[A|B]$

$\therefore$ The system has infinitely many solutions

19. Ans: (c)

Sol: If the system has non trivial solution, then

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = 0$$

$R_2 - R_1, R_3 - R_1$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$$

$$\Rightarrow a+b+c = 0 \text{ or } a = b = c$$

20. Ans: (b)

Sol: Let the given system be AX = B

The augmented matrix of the system

$$= [A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right]$$

$R_2 - 2R_1$

$R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-b-3a \end{array}\right]$$

The system is inconsistent

if $c - b - 3a \neq 0$

$\Rightarrow 3a + b - c \neq 0$

21. Ans: (c)

Sol: Let the given system be AX = B

The augmented matrix of the system

$$= [A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 - R_1$, $R_2 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-6 \end{array}\right]$$

The system has unique solution if $\lambda \neq 3$.

22. Ans: (d)

Sol: If the system has non trivial solution then

$$\begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow 1(k - 3) + k.2k + (k-9) = 0$

$\Rightarrow 2k^2 + 2k - 12 = 0$

$\Rightarrow k = 2, \ -3$

23. Ans: 2

Sol: The characteristic equation is

$|A - \lambda I| = 0$

A real eigen value of A is $\lambda = 5$

The eigen vectors for $\lambda = 5$ are given by

$[A - 5I]\, X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_4 = 0, x_3 = 0$

$\Rightarrow \rho[A] = 2$ and n = 4=number of variables

$\therefore$ The number of linear independent eigen vectors corresponding to $\lambda = 5$ are 2.

24. Ans: 0

Sol: Let a = $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = \pm 5$$

The eigen vectors for $\lambda = 5$ are given by
$[A - 5I]\,X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x - 2y = 0$$

$$\therefore X_1 = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The eigen vectors for $\lambda = -5$ are given by

$[A + 5I]\,X = 0$

$$\Rightarrow \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x + y = 0$$

$$\therefore X_2 = c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\therefore a + b = 0$$

25. Ans: (a)

Sol: Let A = $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

The eigen values of A are 1, 2

The eigen vectors for $\lambda = 1$ are given by

$[A - I]\,X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y = 0$$

$$\therefore X_1 = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigen vectors for $\lambda = 2$ are given by

$[A - 2I]\,X = 0$

$$\Rightarrow \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x + y = 0$$

$$\therefore X_2 = c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector pair is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

26. Ans: (c)

Sol: If A is singular then 0 is an eigen value of A.

$\therefore$ The minimum eigen value of A is 0.

The eigen vectors corresponding to the eigen value $\lambda = 0$ is given by

$$[A - 0I]X = 0$$

$$\Rightarrow \begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying cross multiplication rule for first and second rows of A, we have

$$\Rightarrow \frac{x}{11} = \frac{y}{-11} = \frac{z}{11}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore$ The eigen vectors are

$$X = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

27. Ans: (b)

Sol: Here, A is the elementary matrix obtained given I_3 with elementary operation $R_1 \leftrightarrow R_3$

$$\therefore A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation is

$|A - \lambda I| = 0$

$$\Rightarrow \lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)(\lambda^2 - 1) = 0$

$\Rightarrow \lambda = 1, 1, -1$

28. Ans: –6

Sol: The given matrix has rank 2

$\therefore$ There are only 2 non zero eigen values

The characteristic equation is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & -1-\lambda & -1 & -1 & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 1 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_5$ and

$R_2 \to R_2 + R_3 + R_4$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 & 0 & 2-\lambda \\ 0 & -3-\lambda & -3-\lambda & -3-\lambda & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 1 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda).(-3 - \lambda).$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1-\lambda & -1 & 0 \\ 1 & -1 & -1 & -1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 2, -3$

$\therefore$ product of the non zero eigen values = –6

29. Ans: 3

Sol: If λ is eigen value then $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ 17 & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$$

$\Rightarrow 6 + 2k = \lambda$

$21 + k = 2\lambda$

$\Rightarrow 42 + 2k = 4\lambda$

$\lambda = 12$ and $k = 3$

30. Ans: 3

Sol: Sum of the eigen values = Trace of A = 14

$\Rightarrow a + b + 7 = 14$ ...... (i)

product of eigen values = $|A| = 100$

$\Rightarrow 10ab = 100$

$\Rightarrow ab = 10$ .....(ii)

solving (i) & (ii), we have

$\Rightarrow a = 5$ and $b = 2$

$\therefore |a - b| = 3$

31. Ans: 1

Sol: Product of eigen values = |A| = 0

$$\Rightarrow \begin{vmatrix} 3 & 4 & 2 \\ 9 & 13 & 7 \\ -6 & -9+x & -4 \end{vmatrix} = 0$$

$R_2 - 3R_1$, $R_3 + 2R_1$

$$\Rightarrow \begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & x-1 & 0 \end{vmatrix} = 0$$

$\Rightarrow 3(1 - x) = 0 \quad \Rightarrow x = 1$

32. Ans: (d)

Sol: The characteristic equation is

$\lambda^4 = \lambda$

$\Rightarrow \lambda^4 - \lambda = 0$

$\Rightarrow \lambda(\lambda^3 - 1) = 0$

$\Rightarrow \lambda = 0, 1, -1 \pm \sqrt{3}\, i$

$\Rightarrow \lambda = 0, 1, -0.5 \pm (0.866)i$

33. Ans: 3

Sol: Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Here, A is upper triangular matrix

The eigen values are $\lambda = 2, 2, 3$

The eigen vectors for $\lambda = 2$ are given by

$[A - 2I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Here Rank of $[A - 2I] = 1$

$\therefore$ Number of Linearly independent eigen vectors for $\lambda = 2$ is $n - r$

$= 3 - 1 = 2$

For since, $\lambda = 3$ is not a repeated eigen value, there will be only one independent eigen vector for $\lambda = 3$.

$\therefore$ The number of linearly independent eigen vectors of A = 3.

34. Ans: (d)

Sol: The characteristic equation is

$(\lambda^3 - 6\lambda^2 + 9\lambda - 4) = 0$

|A| = product of the roots of the characteristic equation = 4

Trace of A = sum of the roots of characteristic equation = 6

35. Ans: (b)

Sol: A is symmetric matrix.

The eigen vectors of A are orthogonal.

For the given eigen vector, only the vector given in option (b) is orthogonal.

$\therefore$ option (b) is correct.

36. Ans: (c)

Sol: The characteristic equation is

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda = 0, 3, 15$

The eigen vector for $\lambda = 15$ are given by

$[A - 15I]\, X = 0$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x}{40} = \frac{y}{-40} = \frac{z}{20}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$\therefore$ The eigen vectors for $\lambda = 15$ are

$$X = k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad (k \neq 0)$$

37. Ans: (c)

Sol: The characteristic equation is

$$\begin{vmatrix} a-\lambda & 1 & 0 \\ 1 & a-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a - \lambda)\, [\{(a - \lambda)^2 - 1\} - (a - \lambda)] = 0$

$\Rightarrow \lambda = a, a \pm \sqrt{2}$

38. Ans: $\lambda^2 - 3\lambda + 2$

Sol: The characteristic equation is

$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda - 1)\, (\lambda - 2)^2 = 0$

$\therefore$ Either $(\lambda - 1)\, (\lambda - 2)$ or $(\lambda - 1)\, (\lambda - 2)^2$ is the minimal polynomial

$(A - I)\, (A - 2I)$

$$= \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} = O$$

$\therefore$ The minimal polynomial of A

$= (\lambda - 1)\, (\lambda - 2)$

$= \lambda^2 - 3\lambda + 2$

39. Ans: (a)

Sol: Let the given vectors $X_1 = [2, 2, 0]$, $X_2 = [3, 0, 2]$ and $X_3 = [2, -2, 2]$

suppose $X_1 = a\, X_1 + b\, X_2$

$\Rightarrow [2, 2, 0] = a[3, 0, 2] + b[2, -2, 2]$

$\Rightarrow$ $2 = 3a + 2b$ .... (i)

$2 = -2b$ .... (ii)

$0 = 2a + 2b$ ....(iii)

From (i) and (ii), we get

$a = 0$ and $b = -1$

But, equation (iii) is not satisfied for these values.

$\therefore$ The given vectors are linearly independent

40. Ans: $k \neq 0$

Sol: If the given vectors form a basis, then they are linearly independent

$$\therefore \begin{vmatrix} k & 1 & 1 \\ 0 & 1 & 1 \\ k & 0 & k \end{vmatrix} \neq 0$$

$\Rightarrow k^2 + k - k \neq 0$

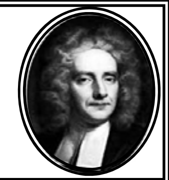
$\Rightarrow k \neq 0$

Chapter **2**

Calculus

(With Vector Calculus & Fourier Series)

Sir Isaac Newton (1643 – 1727)



G. W. Von Leibniz (1646 – 1716)

01. Ans: 7

Sol: $\underset{n\to\infty}{\text{Lt}}\left(7^n+5^n\right)=\underset{n\to\infty}{\text{Lt}}\ 7\left(1+\left(\frac{5}{7}\right)^n\right)^{\frac{1}{n}}$

$= 7 \quad \left[\because \underset{n\to\infty}{\text{Lt}}\ r^n = 0,\ |r|<1\right]$

02. Ans: ∞

Sol: $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\left(\sec x-\frac{1}{1-\sin x}\right)$

$= \underset{x\to\frac{\pi}{2}}{\text{Lt}}\left[\frac{1-\sin x-\cos x}{\cos x(1-\sin x)}\right] \qquad \left[\frac{0}{0}\text{ form}\right]$

$= \underset{x\to\frac{\pi}{2}}{\text{Lt}}\left[\frac{1-\cos x+\sin x}{-\sin x-\cos 2x}\right]$

(by L' Hospital's Rule)

$= \infty$

03. Ans: 1

Sol: $\underset{x\to 0}{\text{Lt}}\left[\frac{e^x-e^{\sin x}}{x-\sin x}\right] \qquad \left[\frac{0}{0}\text{ form}\right]$

$= \underset{x\to 0}{\text{Lt}}\left[\frac{e^x-e^{\sin x}.\cos x}{1-\cos x}\right]$

(by L' Hospital's Rule)

= 1 (applying L' Hospital's Rules two times)

04. Ans: (a)

Sol: $\underset{x\to 0}{\text{Lt}}\frac{\sin 2x+a\sin x}{x^3}=b$

By L' Hospital's rule

$\underset{x\to 0}{\text{Lt}}\frac{2\cos 2x+a\cos x}{3x^2}=b$

$\Rightarrow 2 + a = 0 \quad (\because$ b is finite)

$\therefore\ a = -2$

By L' Hospital's rule

$\underset{x\to 0}{\text{Lt}}\frac{-4\sin 2x+2\sin x}{6x}=b$

again, by L' Hospital's rule

$\underset{x\to 0}{\text{Lt}}\frac{-8\cos 2x+2\cos x}{6}=b$

$\Rightarrow b = -1$

$\therefore\ a = -2\ \&\ b = -1$

05. Ans: 1

Sol: $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x}\right)^{\tan x} \quad (\infty)^0$

Let $y=\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x}\right)^{\tan x}$

Taking Logarithms on both sides

$\log y=\underset{x\to 0}{\text{Lt}}\ \tan x\log\left(\frac{1}{x}\right)$ ……… 1 $(0\times\infty)$

$= \underset{x\to 0}{\text{Lt}}\frac{-\log x}{\cot x} \qquad \left(\frac{\infty}{\infty}\right)$

Issac Newton and Leibniz independently developed calculus which leads to the development of differential and integral equations of mathematical physics

By L' Hospital's rule

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{-1}{x}}{-\cos^2 x}$$

$$= \underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right) \underset{x\to 0}{\text{Lt}} \sin x = 0$$

log y = 0

$\Rightarrow y = e^0 = 1$

06.

Sol: $\underset{x\to 0^-}{\text{Lt}} f(x) = \underset{x\to 0^-}{\text{Lt}} f(x) = f(0)$

$$\underset{x\to 0^-}{\text{Lt}} f(x) = \underset{h\to 0^-}{\text{Lt}} f(0-h) = \underset{h\to 0^-}{\text{Lt}} f(-h)$$

$$= \underset{h\to 0^-}{\text{Lt}} \frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h}$$

$$= \underset{h\to 0^-}{\text{Lt}} \frac{(1-ph)-(1+ph)}{-h\left(\sqrt{1-ph}+\sqrt{1+ph}\right)}$$

$$= \underset{h\to 0^-}{\text{Lt}} \frac{2p}{\sqrt{1-ph}+\sqrt{1+ph}}$$

$$= \frac{2p}{2}$$

$$= p$$

Now $f(0) = \frac{2(0)+1}{0-2} = \frac{-1}{2}$

$\therefore p = \frac{-1}{2}$

07. Ans: (a)

Sol: If f(x) is continuous at x = 0, then

$$\underset{x=0}{\text{Lt}} f(x) = f(0)$$

$$\Rightarrow \underset{x\to 0}{\text{Lt}} \left(\frac{1-x}{1+x}\right)^{\frac{1}{x}} = f(0)$$

$$\Rightarrow \underset{x\to 0}{\text{Lt}} \left[\frac{(1-x)^{\frac{1}{x}}}{(1+x)^{\frac{1}{x}}}\right] = f(0)$$

$$\Rightarrow \frac{e^{-1}}{e} = f(0)$$

$$\Rightarrow f(0) = e^{-2}$$

08. Ans: (c)

Sol: $f^1(x) = 2ax,\ x \le 1$

$\qquad = 2x + a,\ x > 1$

$f^1(1^-) = f'(1^+)$

($\because$ since f(x) is differentiable at x = 1)

$2a = a + 2 \Rightarrow a = 2$

$f(1^-) = f(1^+)$ ($\because$ f(x) is continuous at x = 1)

a + 1 = 1 + a + b

$\Rightarrow b = 0$

09. Ans: –1

Sol: Let $f(x, y) = x^y + y^x = C$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

$$= -\frac{y\,x^{y-1} + y^x.\log y}{x^y.\log x + x\,y^{x-1}}$$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

10. Ans: 2.718

Sol: $u = x\,e^y\,z$ where $y = \sqrt{a^2 - x^2}$ and z $\sin^2 x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx} + \frac{\partial u}{\partial z}.\frac{dz}{dx}$$

$$= e^y z + x\,e^y z\,.\left(\frac{-x}{\sqrt{a^2 - x^2}}\right) + x\,e^y \sin 2x$$

$$\left(\frac{dy}{dx}\right)_{(0,1,1)} = e = 2.718$$

11. Ans: (c)

Sol: (a) Let f(x) = (x–2) in [1, 3]

Here, $f(1) \neq f(3)$

$\therefore$ Roll's theorem is not applicable

(b) Let $f(x) = 1 - (1 - x)^{-1}$ in [0, 2]

Here, f(x) is not continuous in [0, 2]

$\therefore$ Roll's theorem is not applicable

(c) Let f(x) = sin x in [0, π]

Here, f(x) is continuous in [0, π] and differentiable in (0, π). Further,

$f(0) = f(\pi)$

$\therefore$ Roll's theorem is applicable

(d) Let f(x) = Tan x in [0, 2π]

Here, f(x) is not continuous in [0, 2π]

$\therefore$ Roll's theorem is not applicable

12. Ans: (d)

Sol: Here, f(x) is neither continuous nor differentiable in the interval [–1, +1].

$\therefore$ Option (d) is correct.

13. Ans: 1.732

Sol: By Cauchy's mean value theorem

$$\frac{f'(d)}{g'(d)} = \frac{f(3) - f(1)}{g(3) - g(1)}$$

$$\Rightarrow -d = \left[\frac{\sqrt{3} - 1}{\frac{1}{\sqrt{3}} - 1}\right]$$

$$\Rightarrow d = \sqrt{3}$$

14. Ans: (d)

Sol: Let $x - \pi = t$

$$x = \pi + t$$

$$\frac{\sin x}{x - \pi} = \frac{\sin(\pi + t)}{t}$$

$$= \frac{-\sin t}{t}$$

$$= \frac{-1}{t}\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - ....\right\}$$

$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + ...$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + ...$$

15. Ans: (c)

Sol: $$e^{x+x^2} = 1 + \frac{(x + x^2)}{1!} + \frac{(x + x^2)^2}{2!} + \frac{(x + x^2)^3}{3!} + ...$$

$$= 1 + x + \frac{3x^2}{2} + \frac{7}{6}x^3 + ...\infty$$

16. Ans: (b)

Sol: Let $u = \tan^{-1}\left(\frac{x^3y^3}{x^4+y^4}\right)$

$$\Rightarrow f(u) = \tan u = \frac{x^3+y^3}{x^4+y^4}$$

Here, tan u is a homogeneous function of degree 2.

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{f(u)}{f'(u)}$$

$$= 2\left(\frac{\tan u}{\sec^2 u}\right)$$

$$= \sin(2u)$$

17. Ans: (b)

Sol: Let $u = \sqrt{x^2+y^2+z^2}$

$$u_x = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{u}$$

$$u_{xx} = \frac{u^2-x^2}{u^3}$$

similarly,

$$u_{yy} = \frac{u^2-y^2}{u^3}$$

$$u_{zz} = \frac{u^2-z^2}{u^3}$$

Adding

$$u_{xx}+u_{yy}+u_{zz} = \frac{3u^2-(x^2+y^2+z^2)}{u^3}$$

$$= \frac{2}{u}$$

18. Ans: (c)

Sol: Given

$u(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$

u(x, y) is a homogenous function of degree 2

By Euler's theorem,

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = 2(2-1)u$$

$$= 2u$$

19. Ans: (c)

Sol: Give $u = f(2x-3y, 3y-4z, 4z-2x)$

Let $r = 2x-3y$, $s = 3y-4z$ and $t = 4z-2x$

$u_x = u_r.r_x + u_s.s_x + u_t.t_x$

$= 2u_r + u_s.0 + u_t(-2)$

$u_y = u_r.r_y + u_s.s_y + u_t.t_y$

$= -3u_r + 3u_s + u_t.0$

$u_z = u_r.r_z + u_s.s_z + u_t.t_z$

$= u_r.0 + u_s(-4) + u_t(4)$

$6u_x + 4u_y = 12u_r - 12u_t - 12u_r + 12u_s$

$= 12u_s - 12u_t$

$= -3u_z$

20. Ans: (d)

Sol: Let $f(x) = x^{\frac{1}{x}}$

$$f'(x) = x^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}\right]$$

$f'(x) = 0 \quad \Rightarrow \quad x = e$

Further $f''(e) < 0$

$\therefore$ f(x) has maximum at x = 0

The maximum value = $f(e) = e^{\frac{1}{e}}$

21. Ans: 0.785

Sol: Let $y = f(x) = \tan^{-1}\left[\frac{1-x}{1+x}\right]$

$$f'(x) = \frac{-1}{(1+x^2)}$$

f(x) has no stationary points.

Further $f(0) = \frac{\pi}{4}$ and

$f(1) = 0$

$\therefore$ The maximum value of $y = \frac{\pi}{4}$

22. Ans: (b)

Sol: $f'(t) = (t-2)^2 (t-1)$

$f'(t) = 0 \Rightarrow t = 1, 2$

$f''(t) = (t-2)^2 + 2(t-1)(t-2)$

$f''(1) = 1$ and

$f''(2) = 0$

$\therefore$ f(t) has a minimum at t = 1

23. Ans: (c)

Sol: $y = a \log|x| + bx^2 + x$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=-1} = 0$

$\Rightarrow -a - 2b + 1 = 0$ .... (1)

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = 0$

$\Rightarrow \frac{a}{2} + 4b + 1 = 0$ .... (2)

solving (1) & (2), we have $a = 2$, $b = \frac{-1}{2}$

24. Ans: 2

Sol: $f(x) = 6x^2 - 18ax + 12a^2$

$= 6(x-a)(x-2a)$

$\therefore f'(x) = 0 \Rightarrow x = a$ or $2a$

If $x_1 = a$ then $x_2 = 2a$

$x_2 = x_1^2 \Rightarrow 2a = a^2 \Rightarrow a = 0$ or 2

Clearly f has a local maximum at x = 2 and a local minimum at x = 4

25. Ans: 5

Sol: $z = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= a^2 - 2a + 6$

$= (a-1)^2 + 5 \geq 5$

$\therefore$ z is least iff a = 1

least value of $z = [z]_{a=1} = 5$

26. Ans: (a)

Sol: $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

consider $f_x = 4x - 4x^3 = 0$

$\Rightarrow x = 0, 1, -1$

$f_y = -4y + 4y^3 = 0$

$\Rightarrow y = 0, 1, -1$

$r = f_{xy} = 4 - 12x^2$

$s = f_{xy} = 0$

$t = f_{yy} = -4 + 12y^2$

At (0,1), we have r > 0 and $(rt - s^2) > 0$

$\therefore$ f(x, y) has minimum at (0,1)

At (–1, 0), we have r < 0 and $(rt - s^2) > 0$

$\therefore$ f(x, y) has a maximum at (–1, 0)

27. Ans: (b)

Sol: $f(x, y) = xy + x - y$

$f_x = y + 1 = 0$

$f_y = x - 1 = 0$

$\therefore$ P(1, –1) is a stationary point

$r = f_{xx} = 0$

$s = f_{xy} = 1$

$t = f_{yy} = 0$

$rt - s^2 = -1 < 0$

$\therefore$ P(1, –1) is a saddle point

28. Ans: 0.0023

Sol: $f(x, y, z) = xy^2z^3$

$x + y + z = 1$ .... (1)

Let $f = xy^2z^3 + \lambda(x + y + z - 1)$

$f_x = y^2z^3 + \lambda = 0$ .... (2)

$f_y = 2xyz^3 + \lambda = 0$ .... (3)

$f_z = 3xy^2z^2 + \lambda = 0$ .... (4)

from (1), (2), (3) and (4)

$x = \frac{1}{6}, \quad y = \frac{1}{12}, \quad z = \frac{1}{18}$

$\therefore$ The maximum value of xy^2z^3

$$= \frac{1}{6}\cdot\left(\frac{1}{12}\right)^2\cdot\left(\frac{1}{18}\right)^3 = 0.0023$$

29. Ans: 39

Sol: $\int_4^{10}[x]\,dx$

$= \int_4^5 [x]\,dx + \int_5^6 [x]\,dx + ... + \int_9^{10} [x]\,dx$

$= \int_4^5 4\,dx + \int_5^6 5\,dx + ... + \int_9^{10} 9\,dx$

$= 4 + 5 + ...+ 9$

$= 39$

30. Ans: 0.523

Sol: We have,

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Here, $f(x) = \dfrac{1}{1+\tan^4 x}$

$$= \frac{\cos^4 x}{\cos^4 x + \sin^4 x}$$

$$f(a+b-x) = f\left(\frac{\pi}{2} - x\right) = \frac{\sin^4 x}{\sin^4 x + \cos^4 x}$$

Let $I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} f(x)\,dx = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}dx$

again

$I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} f(a+b-x)\,dx = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}dx$

adding

$2I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} dx = \frac{4\pi}{12}$

$\therefore I = \frac{\pi}{6}$

31. Ans: (d)

Sol: $\int_0^\infty t^{-3/2}\left(1-e^{-t}\right)dt$

$$= \left[\left(1-e^{-t}\right).t^{\frac{-1}{2}}(-2)\right]_0^\infty - \int_0^\infty (-2)t^{\frac{-1}{2}}.e^{-t}\,dt$$

$$= 2.\int_0^\infty e^{-t}\,t^{-\frac{1}{2}}\,dt = 2.\,\Gamma\left(\frac{1}{2}\right) = 2\sqrt{\pi}$$

32. Ans: (a)

Sol: $\int_0^{\pi} x\sin^4 x\cos^6 x\,dx$ apply property (3)

By property 9

$$I = \frac{\pi}{2}\int_0^{\pi}\sin^6 x\cos^4 x\,dx$$

$$= 2\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^6 x\cos^4 x\,dx \quad \text{(property 6)}$$

By reduction formula

$$I = \pi\left[\frac{5.3.1.3.1}{10.8.6.4.2}\frac{\pi}{2}\right] = \frac{3\pi^2}{512}$$

33. Ans: 4

Sol: $\int_0^{2\pi}[x\sin x]\,dx = k\pi$

$$\Rightarrow \int_0^{\pi} x\sin x\,dx + \int_{\pi}^{2\pi} -x\sin x\,dx = k\pi$$

$$\Rightarrow [x(-\cos x + \sin x)]_0^{\pi} - [-x\cos x + \sin x]_{\pi}^{2\pi} = k\pi$$

$$\Rightarrow \pi - [-3\pi] = k\pi$$

$$\Rightarrow k = 4$$

34. Ans: (a)

Sol: $\int_{-\infty}^{0}\sin hx\,dx = |\cos hx|_{-\infty}^{0} = \left|\frac{e^x + e^{-x}}{2}\right|_{-\infty}^{0}$

$$= \frac{2}{2} - \left(\frac{e^{-\infty} + e^{\infty}}{2}\right)$$

$$= 1 - 0 - \frac{e^{\infty}}{2} = -\infty$$

35. Ans: (b)

Sol: Length $= \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^3 \sqrt{1+x}\,dx$$

$$= \frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^3 = \frac{14}{3}$$

36. Ans: (a)

Sol: $\int_{-1}^{1}\frac{dx}{x^2} = 2\int_0^1\frac{dx}{x^2}$ ($\because \frac{1}{x^2}$ is even function)

$= 2\underset{x\to 0^+}{\text{Lt}}\int_0^1\frac{dx}{x^2}$ (since $\frac{1}{x^2}$ is not defined)

$$= 2\left(\frac{-1}{x}\right)_0^1 = 2\{(-1) - (-\infty)\}$$

$= \infty$(Divergent)

37. Ans: (a)

Sol: $\int_1^3 \frac{\sqrt{1+x^2}}{(x-1)^2}\,dx$ at x =1

Let $f(x) = \frac{\sqrt{1+x^2}}{(x-1)^2}$ $f(x)\to\infty$ as $x\to 1$

Let $g(x) = \frac{1}{(x-1)^2}$

$$\underset{x\to 1}{\text{Lt}}\frac{f(x)}{g(x)} = \underset{x\to 1}{\text{Lt}}\frac{\sqrt{1+x}}{(x-1)^2}\times(x-1)^2 = \sqrt{2}$$

But $\int_1^3\frac{1}{(x-1)^2}$ is known to be divergent.

$\therefore$ By comparison test, the given integral also divergent.

38. Ans: (a)

Sol: $\int_1^2 \frac{x^3+1}{\sqrt{2-x}}dx$

Let $f(x) = \frac{x^3+1}{\sqrt{2-x}}$

$f(x) \to \infty$ an $x \to 2$

Let $g(x) = \frac{1}{\sqrt{2-x}}$

$$\underset{x\to 2}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x\to 2}{\text{Lt}}\left(\frac{x^3+1}{\sqrt{2-x}} \times \sqrt{2-x}\right) = 9 \text{ finite}$$

But $\int_1^2 g(x)\, dx$ in known to be convergent

∴ By comparison test, the given integral also convergent.

39. Ans: (d)

Sol: $\int_1^\infty \frac{e^{-x}}{x^2}dx$

Let $f(x) = \frac{e^{-x}}{x^2}$

Choose $g(x) = \frac{1}{x^2}$

$\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^2}dx = -1$ is known to be convergent.

∴ By comparison test, the given integral also convergent.

40. Ans: (c)

Sol: $\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r\sin\theta \, dr\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \left(\frac{r^2}{2}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \; 2a^2 \cos^2\theta \, d\theta$$

$$= -2a^2 \int_1^0 t^2 \, dt \qquad \text{Put } \cos\theta = t$$

$$= 2a^2 \int_0^1 t^2 \, dt$$

$$= \frac{2a^2}{3}$$

41. Ans: 0.1143

Sol: $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz\, dx\, dy$

$$= \int_0^1 \left(\int_{y^2}^1 x(1-x)dx\right) dy$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{y^2}^1 dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy = \frac{4}{35}$$

42. Ans: 1.047

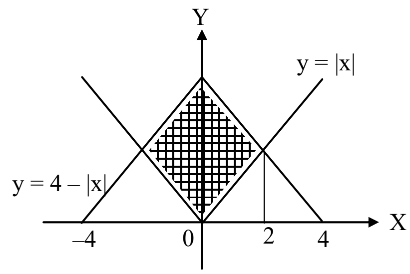
Sol: $\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 r^2 \sin\phi \, dr d\phi d\theta$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left(\frac{r^3}{3}\right)_0^1 \sin\phi \, d\phi d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}(-\cos\phi)_0^{\frac{\pi}{3}} \, d\theta = \frac{\pi}{3}$$

43. Ans: 8

Sol:



On solving the two curves in the first Quadrant, we get x = 2. Therefore, the area bounded by the curves is

$$= 2\left(\int_0^2 (4-x)\,dx - \int_0^2 x\,dx\right)$$

$$= 2\left(\left(4x-\frac{x^2}{2}\right)_0^2 - \left(\frac{x^2}{2}\right)_0^2\right)$$

$= 2(8 – 2 – 2)$

$= 8$ sq. units

44. Ans: 0.0536

Sol: $\iint_s (x^2y + xy^2)\,dxdy$

$$= \int_0^1 \left[\int_{x^2}^{x} (x^2 y + xy^2)\,dy\right] dx$$

$$= \int_0^1 \left[x^2\left(\frac{y^2}{2}\right)_{x^2}^{x} + x\left(\frac{y^3}{3}\right)_{x^2}^{x}\right] dx$$

$$= \int_0^1 \left[\frac{x^2}{2}(x^2 - x^4) + \frac{x}{3}(x^3 - x^6)\right] dx$$

≈ 0.054

45. Ans: (c)

Sol: $\int_0^a \int_{\sqrt{ax}}^{a} \phi(x, y)\,dydx$

By changing the order of integration the above integral becomes

$$= \int_0^a \int_0^{\frac{y^2}{a}} \phi(x, y)\,dydx$$

Now,

$$\int_p^q \int_r^s \phi(x, y)\,dxdy = \int_0^a \int_0^{\frac{y^2}{a}} \phi(x, y)\,dydx$$

$$\therefore\ q.s = a.\left(\frac{y^2}{a}\right) = y^2$$

46. Ans: 32

Sol: The volume $= \iint_R z\ dx\ dy$

$$= \int_0^6 \int_0^2 (4 - x^2)\,dx\ dy$$

$$= \int_0^6 \left[4x - \frac{x^3}{3}\right]_0^2 dy$$

$= 32$

47. Ans: 25.12

Sol: Volume $= \int_0^4 \pi y^2\ dx$

$$= \int_0^4 \pi x\ dx$$

$= 8\pi$ cubic units

48. Ans: (d)

Sol: Length $= \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$

$$= \int_0^3 \sqrt{1+x}\ dx$$

$$= \frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^3 = \frac{14}{3}$$

49. Ans: 1.88

Sol: Volume $= \int_0^1 \pi\ x^2\ dy$

$$= \pi \int_0^1 y^{\frac{2}{3}}\ dy \approx 1.88$$

50.

Sol: $T = xy + yz + zx$

$\Rightarrow \nabla T = \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}(x+y)$

at (1, 1, 1), $\nabla T = 2\bar{i} + 2\bar{j} + 2\bar{k}$

Given $\bar{a} = 3\bar{i} - 4\bar{k}$

$\therefore$ Directional Derivative

$$= \nabla T.\ \frac{\bar{a}}{|\bar{a}|}$$

$$= \left(2\bar{i}+2\bar{j}+2\bar{k}\right).\frac{\left(3\bar{i}-4\bar{k}\right)}{\sqrt{9+16}}$$

$$= \frac{-2}{5}$$

51.

Sol: $f(x, y, z) = x^2 + y^2 + 2z^2$

$$\nabla f = 2x\bar{i} + 2y\bar{j} + 4z\bar{k}$$

at (1,1,2), $\nabla f = 2\bar{i} + 2\bar{j} + 8\bar{k}$

Given $\bar{a} = \nabla f$

Directional Derivative $= \nabla f.\ \frac{\bar{a}}{|\bar{a}|}$

$$= \nabla f.\ \frac{\nabla f}{|\nabla f|}$$

$$= |\nabla f| = \sqrt{4+4+64}$$

$$= \sqrt{72}$$

$$= 2\sqrt{18}$$

52. Ans: (a)

Sol: The Directional Derivative is maximum in the direction of $\nabla\phi$

Given $\phi(x, y, z) = x^2y^2z^4$

$\Rightarrow \nabla\phi = (2xy^2z^4)\ \bar{i} + (2x^2yz^4)\ \bar{j} + (4x^2y^2z^3)\ \bar{k}$

At (3, 1,–2), $\nabla\phi = 96\,\bar{i} + 288\,\bar{j} - 288\,\bar{k}$

$$= 96\left(\bar{i} + 3\bar{j} - 3\bar{k}\right)$$

53. Ans: (c)

Sol: Given $f = x^2 + y^2 + z^2$, $\bar{r} = x\bar{i} + y\bar{j} + 3\bar{k}$

$\Rightarrow\ f\,\bar{r} = fx\bar{i} + fy\bar{j} + fz\bar{k}$

$$\text{div}\ (f\,\bar{r}) = \frac{\partial}{\partial x}(fx) + \frac{\partial}{\partial y}(fy) + \frac{\partial}{\partial z}(fz)$$

$$= [x.(2x)+f] + [y.(2y)+f] + [z.(2y)+f]$$

$$= 2(x^2 + y^2 + z^2) + 3f = 5f$$

54. Ans: (b)

Sol: Div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$= e^x + e^{-x} + 2\sin hx$$

55. Ans: (a)

Sol: $\nabla \times \overline{V} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - x^2 + y & x(2y+1) & 0 \end{vmatrix}$

$= \overline{i}[0-0] - \overline{j}[0 - 0 + \overline{k}[(2y+1)-(2y+1)]$

$= \overline{0}$

56. Ans: (b)

Sol: Given

$$\overline{V} = (x^2 + yz)\overline{i} + (y^2 + zx)\overline{j} + (z^2 + xy)\,\overline{k}$$

Div $\overline{V}$ = 2x + 2y + 2z ≠ 0

⇒ $\overline{V}$ is not divergence free

Curl $\overline{V} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + zx & z^2 + xy \end{vmatrix}$

$= \overline{i}[x - x] - \overline{j}[y - y] + \overline{k}[z - z] = \overline{0}$

⇒ $\overline{V}$ is irrotational

57. Ans: 202

Sol: Given $\overline{F} = (2xy + z^3)\overline{i} + x^2\,\overline{j} + 3xz^2\,\overline{k}$

Curl $\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$

$= \overline{i}[0-0] - \overline{j}[3z^2 - 3z^2] + \overline{k}[2x - 2x] = \overline{0}$

⇒ $\overline{F}$ is irrotational

⇒ Work done by $\overline{F}$ is independent of path of curve

⇒ $\overline{F} = \nabla\phi$

where $\phi(x, y, z)$ is scalar potential

⇒ $(2xy + z^3)\overline{i} + x^2\,\overline{j} + 3xz^2\,\overline{k} = \frac{\partial\phi}{\partial x}\overline{i} + \frac{\partial\phi}{\partial y}\overline{j} + \frac{\partial\phi}{\partial z}\overline{k}$

⇒ $d\phi = (2xy + z^3)\,dx + x^2\,dy + 3xz^2\,dz$

⇒ $\int (2xy + z^3)dx + x^2\,dy + 3xz^2\,dz$

$= \int d(x^2y + xz^3)$

⇒ $\phi(x, y, z) = x^2y + xz^3$

∴ Workdone $= \int_C \overline{F}.d\overline{r}$

$= \phi(3,1,4) - \phi(1, -2,1)$

$= [9(1)+3(64)] - [1(-2) + 1(1)]$

$= 202$

58. Ans: 0

Sol: By Stokes' theorem,

$$\int_C \overline{F}.d\overline{r} = \iint_S (\nabla \times \overline{F}).\overline{n}\ ds$$

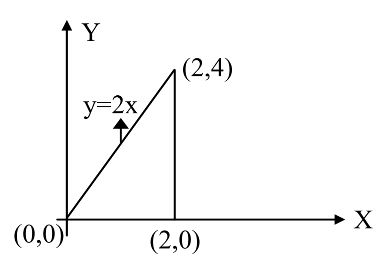
Here, $\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$

$= \overline{0}$

∴ $\int \overline{F}.d\,\overline{r} = 0$

59.

Sol:



By Green's Theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

where $M = x + y$, $N = x^2$ and

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 1$$

The given integral $= \int_{x=0}^{2}\int_{y=0}^{2x} (2x-1)\,dy\,dx$

$$= \int_0^2 \left[2xy - y\right]_0^{2x} dx$$

$$= \int_0^2 \left[4x^2 - 2x\right] dx$$

$$= \frac{20}{3}$$

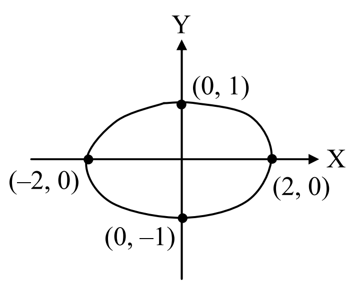
60. Ans: (c)

Sol: By Green's Theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here, $M = 2x - y$ and $N = x + 3y$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$$



The given integral $= \iint_R 2\,dx\,dy$

= 2 Area of the given ellipse

$= 2\,(\pi.\,2.\,1) = 4\pi$

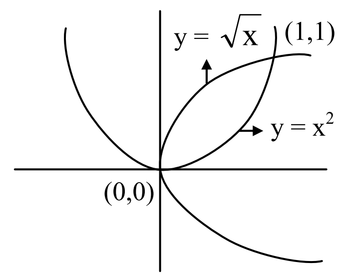
61.

Sol: By Green's Theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here, $M = x - y$ and $N = x + y$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2$$



The given integral $= \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 2\,dy\,dx$$

$$= \int_{x=0}^{1} 2\left[\sqrt{x} - x^2\right] dx$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{2}{3} - \frac{1}{3}\right]$$

$$= \frac{2}{3}$$

62. Ans: 0

Sol: Given $\overline{A} = \nabla\phi$

Curl $\overline{A} = \overline{0}$

$\Rightarrow$ $\overline{A}$ is Irrotational

∴ Line integral of Irrotational vector function along a closed curve is zero

$\Rightarrow \int_C \overline{A}.d\overline{r} = 0$

where C is $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is a closed curve.

63. Ans: (c)

Sol: By Green's Theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

C is the circle $x^2 + y^2 = 4$

Here, $M = -y^3$ and $N = x^3$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x^2 - (-3y^2) = 3(x^2 + y^2)$$

$$= \iint_R 3(x^2 + y^2)\,dx\,dy$$

Where R is $x^2 + y^2 = 4$

Using polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$ and

$x^2 + y^2 = r^2$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} 3r^2 .r\;dr\;d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[3.\frac{r^4}{4}\right]_0^2 d\theta = 12 \times 2\pi = 24\pi$$

64. Ans: 264

Sol: Using Gauss-Divergence Theorem,

$$\iint_S xy\;dy\;dz + yz\;dzdx + zx\;d\;dz = \iiint_V \text{div}\;\overline{F}\;dv$$

$$= \iiint_V (y + z + x)\;dv$$

$$= \int_{x=0}^{4} \int_{y=0}^{3} \int_{z=0}^{4} (x + y + z)\;dzdydx$$

$$= \int_{x=0}^{4} \int_{y=0}^{3} [4x + 4y + 8]\;dy\;dz$$

$$= \int_{x=0}^{4} [12x + 18 + 24]\,dx$$

$= 264$

65. Ans: (b)

Sol: Using Gauss-Divergence Theorem,

$$\int_S \overline{F}.\overline{N}\;ds = \int_V \text{div}\;\overline{F}\;dv$$

$$= \int_V 3\;dv = 3\;V$$

$$= 3 \times \frac{4}{3}\pi r^3 = 4\pi(4)^3 = 256\pi$$

66. Ans: (d)

Sol: $$\text{Curl}\;\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \overline{i}[-2yz + 2yz] - \overline{j}[0] + \overline{k}[0 + 1]$$

$\Rightarrow \text{Curl}\;\overline{F} = \overline{k}$

Using Stokes' theorem,

$$\int_C \overline{F}.d\overline{r} = \int_S \text{curl}\;\overline{F}.\overline{N}\;ds = \int_S \overline{k}.\overline{N}\;ds$$

Let R be the protection of s on xy plane

$$\Rightarrow \int_S \overline{k}.\overline{N}\;ds = \iint_R \overline{k}.\overline{N}\;\frac{dxdy}{|\overline{N}.\overline{k}|}$$

$$= \iint_R -1\;dx\,dy \quad (\overline{N} = -\overline{k})$$

= Area of Region

$= -\pi r^2 = -\pi(1)^2 = -\pi$

67. Ans: (d)

Sol: The function f(x) = x.sin x is even function

$\therefore$ The fourier series of f(x) contain only cosine terms.

The coefficient of sin 2x = 0

68. Ans: (b)

Sol: Let $f(x) = \dfrac{(\pi - x)^2}{4}$

The fourier series of f(x) in (0, 2π) is

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\}$$

$$a_o = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\ dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-12}\right]_0^{2\pi}$$

$$= \frac{-1}{12\pi}\left[-\pi^3 - \pi^3\right]$$

$$= \frac{2\pi^3}{12\pi} = \frac{\pi^2}{6}$$

The constant term $= \dfrac{a_0}{2} = \dfrac{\pi^2}{12}$

69. Ans: (b)

Sol: The given function is even in (–π, π)

$\therefore$ Fourier series of f(x) contains only cosine terms.

70. Ans: (b)

Sol: $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{k}{\pi}\left[\frac{2 - 2(-1)^n}{n}\right] \sin(nx)$

At $x = \dfrac{\pi}{2}$

$$k = \frac{k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right]$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\infty = \frac{\pi}{4}$$

71. Ans: (d)

Sol: $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$

Let $f(x) = \dfrac{a_o}{2} + \displaystyle\sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 1\ dx = 1$$

72. Ans: (c)

Sol: $f(x) = (\pi x - x^2)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx\ dx$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} [(\pi x - x^2).\sin x]\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)(-\cos x) - (\pi - 2x)(-\sin x) + (-2)\cos x\right]_0^{\pi}$$

$$= \frac{8}{\pi}$$

73. Ans: (b)

Sol: $f(x) = (x-1)^2$

The Half range cosine series is

$$(x-1)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (x-1)^2 \cos(n\pi x)\, dx$$

$$= \frac{2}{\pi}\left[(x-1)^2.\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1).\frac{\cos n\pi x}{n^2\pi^2} + 2\frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}$$

74. Ans: (b)

Sol: $f(x) = |x|$ is even function

The fourier series of $f(x)$ in $(-\pi, \pi)$ is

$f(x) = |x|$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{........(1)}$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x.\cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\left[x.\frac{\sin(nx)}{n}\right]_0^{\pi} - 1\left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{0 + \frac{\cos(n\pi)-1}{n^2}\right\}$$

substituting the values of a_0 and a_n in (1)

$$\therefore f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

Chapter 3 Probability & Statistics

C. R. Rao

01. Ans: (c)

Sol: Four numbers can be selected out of 40 in $^{40}C_4 = 37 \times 38 \times 65$ ways.

E: Event that the four numbers are consecutive.

Favourable cases to E: (1 2, 3,4), (2,3,4,5), (3,4,5,).......(37,38,39,40) whose number is 37

$$\therefore P(E) = \frac{37}{^{40}C_4} = \frac{1}{2470}$$

$$\therefore \text{Required probability} = P(\overline{E})$$

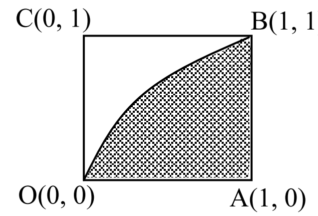
$$= 1 - P(E)$$

$$= 1 - \frac{1}{2470}$$

$$= \frac{2469}{2470}$$

02. Ans: (c)

Sol: The sample space is a square whose sides are unit segments of the coordinate axes. The figure whose set of points correspond to the outcomes favourable to the event $y^2 \le x$ is bounded by the graphs of the function and $y^2 = x$, $y = 0$ and $x = 1$ is shown below.



Required Probability = area of the shaded region $= \int_0^1 \sqrt{x}\, dx = \frac{2}{3}$

03. Ans: (d)

Sol: Let $x \in S$. Then either $x \in P$, $x \in Q$, or $x \notin P$, $x \in Q$, or $x \in P$, $x \notin Q$ or $x \notin P$, $x \notin Q$. Out of the above four cases, three cases are favourable to the event $P \cap Q = \phi$.

$$\therefore \text{The required probability} = \left(\frac{3}{4}\right)^{20}$$

04. Ans: (b)

Sol: Let

A = The event that 5 appears in first throw

B = The event that sum is 6

The cases favourable to B are

{(5, 5, 6), (5, 6, 5), (6, 5, 5), (4, 6, 6), (6, 4, 6), (6, 6, 4)}

$A \cap B$ = {(5, 5, 6), (5, 6, 5)}

Calyampudi Radhakrishna Rao, FRS known as C R Rao (born 10 September 1920) is an Indian-born, mathematician and statistician. The American Statistical Association has described him as "a living legend whose work has influenced not just statistics, but has had far reaching implications for fields as varied as economics, genetics, anthropology, geology, national planning, demography, biometry, and medicine.

Required probability $= P(A|B)$

$$= \frac{n(A \cap B)}{n(B)}$$

$$= \frac{2}{6} = \frac{1}{3}$$

05. Ans: (a)

Sol: The total number of five digit numbers formed by 1, 2, 3, 4 and 5 (without repetition) = $\angle 5 = 120$

A number is divisible by 4 if the last two digit number (i.e., tens and unit place) is divisible 4.

$\therefore$ The last two digit number must be: 12, 24, 32 and 52 (4 cases). With last two digits fixed, the other three places can be arranged in $\angle 3 (=6)$ ways.

$\therefore$ The number of favourable cases = $\angle 3 \times 4$
$= 24$

$$\therefore \text{Probability} = \frac{24}{120} = \frac{1}{5}$$

06. Ans: (c)

Sol: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c and d can take values 0 or 1.

$\therefore$ Total number of such matrices = $2^4 = 16$

Let E be the event that A is non singular.

$\therefore$ det $A \neq 0$.

i.e., atleast one of the two numbers a & d is zero or atleast one of the two numbers b & c is zero.

The matrices whose determinants are non zero are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore P(E) = \frac{6}{16} = \frac{3}{8}$$

07. Ans: (d)

Sol: Total number of triangles that can be formed by using the vertices of a regular hexagon

$$= {}^6C_3 = 20.$$

Among these, there are only two equilateral triangles.

$$\therefore \text{Required probability} = \frac{2}{20} = \frac{1}{10}$$

08. Ans: 0.4

Sol: If A and B are independent then

$P(A \cap B) = P(A). P(B) = 0.16$ .... (1)

By Addition theorem of probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.64 = P(A) + P(B) - 0.16$

$\Rightarrow$ P(A) + P(B) = 0.8 .... (2)

From (1) & (2), we get

P(A) = P(B) = 0.4

09. Ans: (a)

Sol: We have,

$$P(A) = P(B) = P(C) = \frac{18}{36} = \frac{1}{2}$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C)$$

$$= \frac{9}{36} = \frac{1}{4}$$

Thus $P(A \cap B) = \frac{1}{4} = P(A)\,P(B)$

$$P(A \cap C) = \frac{1}{4} = P(A)P(C)$$

$$P(B \cap C) = \frac{1}{4} = P(B)P(C)$$

Which indicates that A, B, and C are pair wise independent. However, since the sum of two numbers is even,

$\{A \cap B \cap C) = \phi$ and

$$P(A \cap B \cap C) \neq \frac{1}{8} = P(A)P(B)P(C)$$

which shows that A, B, and C are not independent.

10. Ans: 0.46

Sol: Let

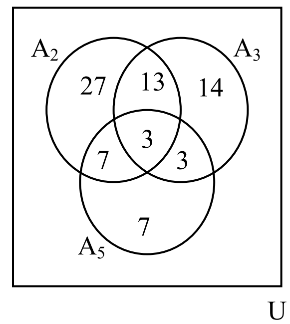
A_2 = event that the number is divisible by 2

A_3 = event that the number is divisible by 3

A_5 = event that the number is divisible by 5

Then the required probability

$$= P\{(A_3 \cup A_5) \mid A_2\}$$



$$= \frac{n[(A_3 \cup A_5) \cap A_2]}{n(A_2)}$$

$$= \frac{23}{50} = 0.46$$

11. Ans: (c)

Sol: Let E_1, E_2, E_3 be the events of selecting urns U_1, U_2, U_3 respectively and W be the event of the drawn ball is white.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

By Total theorem of probability P(W)

$$= P(E_1)\,P\left(\frac{W}{E_1}\right) + P(E_2)\,P\left(\frac{W}{E_2}\right) + P(E_3)\,P\left(\frac{W}{E_3}\right)$$

$$= \frac{1}{3}\left(\frac{2}{5}\right) + \frac{1}{3}\left(\frac{3}{5}\right) + \frac{1}{3}\left(\frac{4}{5}\right)$$

$$= \frac{9}{15}$$

$$= \frac{3}{5}$$

12. Ans: (a)

Sol: Let A, B and C denote events of a bolt manufactured by A, B and C.

Let D be the event of the drawn bolt is defective.

By Total theorem of probability P(D)

$$= P(A)\,P\left(\frac{D}{A}\right) + P(B)\,P\left(\frac{D}{B}\right) + P(C)\,P\left(\frac{D}{C}\right)$$

$$= \frac{25}{100}\left(\frac{5}{100}\right) + \frac{35}{100}\left(\frac{4}{100}\right) + \frac{40}{100}\left(\frac{2}{100}\right)$$

$$= \frac{69}{2000}$$

13. Ans: (c)

Sol: E : Correct diagnosis

$\overline{E}$: Wrong diagnosis

D : Event of death.

$$P(E) = \frac{60}{100} = \frac{3}{5},\ P(\overline{E}) = \frac{2}{5}$$

$$P\left(\frac{D}{E}\right) = \frac{70}{100} = \frac{7}{10},\ P\left(\frac{D}{\overline{\overline{E}}}\right) = \frac{80}{100} = \frac{4}{5}$$

By Baye's theorem,

Required probability $= P\left(\frac{E}{D}\right)$

$$= \frac{P(E)P\left(\frac{D}{E}\right)}{P(E)\,P\left(\frac{D}{E}\right) + P(\overline{E})P\left(\frac{D}{\overline{E}}\right)}$$

$$= \frac{\frac{3}{5}\times\frac{7}{10}}{\frac{3}{5}\times\frac{7}{10}+\frac{2}{5}\times\frac{4}{5}} = \frac{21}{37}$$

14. Ans: (b)

Sol: Let E_1 bet the event of guessing, E_2 the event of copying and E_3 the event of knowing the answer.

$$\therefore\ P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{6},$$

$$P(E_3) = 1 - \left(\frac{1}{3}+\frac{1}{6}\right) = \frac{1}{2}$$

Let E be the event of writing correct answer.

$$P\left(\frac{E}{E_1}\right) = \frac{1}{4},\ P\left(\frac{E}{E_2}\right) = \frac{1}{8}\ \text{(Given)}$$

$$P\left(\frac{E}{E_3}\right) = 1$$

By Baye's theorem,

Required probability $= P\left(\frac{E_3}{E}\right)$

$$= \frac{P(E_3)P\left(\frac{E}{E_3}\right)}{\sum_{j=1}^{3} P(E_j)\,P\left(\frac{E}{E_j}\right)}$$

$$= \frac{\frac{1}{2}(1)}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1}$$

$$= \frac{24}{29}$$

15. Ans: (c)

Sol: Let E_j be the event that the bag contains j number of red balls (j = 1, 2, 3, 4)

$$\therefore\ P(E_j) = \frac{1}{4}\ (j = 1, 2, 3, 4)$$

Let E be the event of drawing a red ball.

$$P\left(\frac{E}{E_1}\right)=\frac{1}{4},\ P\left(\frac{E}{E_2}\right)=\frac{2}{4},\ P\left(\frac{E}{E_3}\right)=\frac{3}{4}$$

$$P\left(\frac{E}{E_4}\right)=\frac{4}{4}=1$$

∴ By Baye's theorem,

$$P\left(\frac{E_4}{E}\right)=\frac{P(E_4)\,P\left(\frac{E}{E_4}\right)}{\sum_{j=1}^{4}P(E_j)P\left(\frac{E}{E_j}\right)}$$

$$=\frac{\frac{1}{4}\times 1}{\frac{1}{4}\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}\right)}=\frac{2}{5}$$

16. Ans: (d)

Sol: **E_1:** Event of letter coming from LONDON.

E_2: Event of the letter coming from CLIFTON.

E: Event of two consecutive letters ON.

$P(E_1) = P(E_2) = \frac{1}{2}$.

Word LONDON consists of 5 pairs of consecutive letters

(LO, ON, ND, DO, ON) out of which there are 2 ON's.

CLIFTON consists of 6 pairs of consecutive letters

(CL, LI, IF, FT, TO, ON) out of which there is only one 'ON'.

$$\therefore\ P\left(\frac{E_1}{E}\right)=\frac{P(E_1)P\left(\frac{E}{E_1}\right)}{P(E_1)P\left(\frac{E}{E_1}\right)+P(E_2)P\left(\frac{E}{E_2}\right)}$$

$$=\frac{\frac{1}{2}\times\frac{2}{5}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{1}{6}}=\frac{12}{17}$$

17. Ans: (c)

Sol: Total probability = $\sum_{d=1}^{4}C\left(\frac{2^d}{\angle d}\right)=1$

$\Rightarrow C\ (2 + 2 + \frac{4}{3} + \frac{2}{3}) = 1$

$\Rightarrow C = \frac{1}{6}$

Expected demand = E (D)

$=\sum_{d=1}^{4}d.P\,(D=d)$

$=1\left(\frac{2}{6}\right)+2\left(\frac{2}{6}\right)+3\left(\frac{4}{8}\right)+4\left(\frac{2}{18}\right)=\left(\frac{19}{9}\right)$

18. Ans: 5

Sol: Let X = Amount the player wins in rupees

The probability distribution for X is given below

Number of heads	0	1	2
X	x	1	3
P(X)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

For the game to be fair we have to find x, so that E(X) = 0

$\Rightarrow x.\left(\frac{1}{4}\right) + 1.\left(\frac{2}{4}\right) + 3.\left(\frac{1}{4}\right) = 0$

$\Rightarrow x = 5$

$\therefore$ Number of rupees, the player has to lose, if no heads occur = 5.

19.

Sol: P (X is even) = P (X = 2) + P (X = 4) + P (X = 6) + ….∞

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + ...\infty$$

$$= \frac{1}{2^2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ...\infty\right]$$

$$= \frac{1}{4}\left(1 - \frac{1}{4}\right)^{-1} = \frac{1}{3}$$

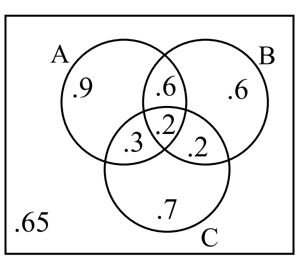
20. Ans: 0.62857 Range(0.62 to 0.63)

Sol: Let

E_1 = The selected reader is reading only one news papers

E_2 = The selected reader is reading atleast one of the newspapers

The Venn diagram for the given data is



Required probability = $P(E_1|E_2)$

$$= \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$= \frac{P(E_1)}{P(E_2)} = \frac{0.22}{0.35}$$

$= 0.62857$

21. Ans: 0 and 0.4

Sol: Here f (x) is an even function

$$E(X) = \int_{-\infty}^{\infty} x.f(x)\,dx = 0$$

($\because$ x f(x) is an odd function)

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-1}^{0} x^2(1+x)\,dx + \int_{0}^{1} x^2(1+x)\,dx = \frac{1}{6}$$

Variance of X = $E(X^2) - (E(X))^2 = \frac{1}{6}$

22. Ans: (a)

Sol: Required probability

$$= C(10,5).\left(\frac{1}{2}\right)^5.\left(\frac{1}{2}\right)^5.1$$

$$= \frac{^{10}C_5}{2^{10}}$$

23. Ans: (b)

Sol: If the person is one step away, then we have two cases:

Case1: 6 forward steps and 5 backward

(or)

Case2: 6 backward steps and 5 forward.

Required Probability

$= C(11,6)(0.4)^6 + C(11,5)(0.6)^6\,(0.4)^5$

$= C(11,5)\,(0.4)^5\,(0.6)^5\,(0.4 + 0.6)$

$= 462 \times (0.24)^5$

24. Ans: (d)

Sol: E_1 = Event of writing good book

E_2 = Event of not writing a good book

E = Probability of publication

$P(E_1) = P(E_2) = \frac{1}{2}$, $P\left(\frac{E}{E_1}\right) = \frac{2}{3}$,

$P\left(\frac{E}{E_2}\right) = \frac{1}{4}$

$P(E) = P(E \cap E_1) + P(E \cap E_2)$

$= \frac{1}{2}.\frac{2}{3} + \frac{1}{2}.\frac{1}{4}$

$= \frac{11}{24}$

X denote the number of books published.

$\therefore$Required probability$= P(X = 1) + P(X = 2)$

$= {}^2C_1 \frac{11}{24} \times \frac{13}{24} + {}^2C_2\left(\frac{11}{24}\right)^2$

$= 2 \times \frac{11}{24} \times \frac{13}{24} + \left(\frac{11}{24}\right)^2$

$= \frac{407}{576}$

25. Ans: (a)

Sol: $P(A) = P(B) = P(C) = \frac{1}{3}$

E = Event of getting 2 heads and 1 tail

$P\left(\frac{E}{A}\right) = {}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$

$P\left(\frac{E}{B}\right) = {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{9}$

$P\left(\frac{E}{C}\right) = {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{2}{9}$

Required probability =

$$P\left(\frac{A}{E}\right) = \frac{P(A)P\left(\frac{E}{A}\right)}{P(A)P\left(\frac{E}{A}\right) + P(B)P\left(\frac{E}{B}\right) + P(C)P\left(\frac{E}{C}\right)}$$

$$= \frac{\frac{1}{3} \times \frac{3}{8}}{\frac{1}{3} \times \frac{3}{8} + \frac{1}{3} \times \frac{4}{9} + \frac{1}{3} \times \frac{2}{9}}$$

$$= \frac{27}{75} = \frac{9}{25}$$

26. Ans: 0.335

Sol: Let X = Number of times we have to toss a pair of dice.

P = probability of getting 7 in one throw

$= \frac{1}{6}$

q = 1 – P

= probability of not getting 7 in one throw

$= \frac{5}{6}$

q^6 = probability of not getting a 6 in 6 throws

P ($X \le 6$) = Probability that it take less than 6 tosses to get a 7

$$= 1 - \left(\frac{5}{6}\right)^6$$

Required probability = P ($X > 6$)

$$= 1 - \left\{1 - \left(\frac{5}{6}\right)^6\right\} = \left(\frac{5}{6}\right)^6$$

$$\approx 0.335$$

27. Ans: 7

Sol: The probability of missing the target is $q = 1 - p = 0.7$. Hence the probability that n missiles miss the target is $(0.7)^n$. Thus, we seek the smallest n for which

$1 - (0.7)^n > 0.90$ or

equivalently $(0.7)^n < 0.10$

Compute

$(0.7)^1 = 0.7$, $(0.7)^2 = 0.49$, $(0.7)^3 = 0.343$,

$(0.7)^4 = 0.240$, $(0.7)^5 = 0.168$,

$(0.7)^6 = 0.118$, $(0.7)^7 = 0.0823$

Thus, atleast 7 missiles should be fired.

28. Ans: 0.0045

Sol: Let X = number of accidents between 5 P.M and 6 P.M.

For Poisson distribution,

$$\lambda = np = (1000)(0.0001) = 0.1$$

$$P(X = x) = \frac{e^{-\lambda}.\lambda^x}{\angle x} \quad (x = 0, 1, 2, \ldots\ldots)$$

Required Probability = $P(X \ge 2)$

$= 1 - P(X < 2)$

$= 1 - \{P(X = 0) + P(X = 1)\}$

$= 1 - e^{-0.1}(1 + 0.1)$

$= 0.0045$

29. Ans: 0.122

Sol: We view the number of misprints on one page as the number of successes in a sequence of Bernoulli trials. Here n = 300 since there are 300 misprints, and $p = \frac{1}{500}$, the probability that a misprint appears on a given page. Since p is small, we use the Poisson approximation to the binomial distribution with $\lambda = np = 0.6$.

We have

P(0misprint) = f(0; 0.6)

$$= \frac{(0.6)^0 e^{-0.6}}{0!} = e^{-0.6} = 0.549$$

P(1 misprint) = f(1; 0.6)

$$= \frac{(0.6)^1 e^{-0.6}}{1!} = (0.6)(0.549)$$

$= 0.329$

Required probability

$= 1 - (0.549 + 0.329)$

$= 0.122$

30. Ans: 0.1353

Sol: Given that λ = 900 vehicles/hour

= 1 vehicle/ 4 sec

= 2 vehicles/8 sec

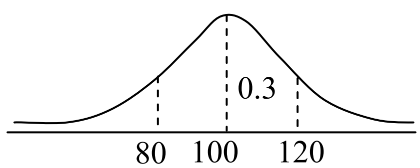
Probability for k vehicles in a time gap of 8 seconds = $P(X = k) = \frac{\lambda^k e^{-\pi}}{k!}$

Required probability = $P(X = 0) = e^{-\lambda} = e^{-2}$

= 0.1353

31. Ans: 0.2

Sol: The area under normal curve is 1 and the curve is symmetric about mean.



$\therefore$ P(100 < X < 120) = P(80 < X < 120) = 0.3

Now, P(X < 80) = 0.5 – P(80 < X < 120)

= 0.5 – 0.3 = 0.2

32. Ans: 0.7939

Sol: This is a binomial experiment B(n, p) with n = 3500, p = 0.04, and q = 1 – p

= 0.96.

Then $\mu = np = (3500)(0.04)$

= 140,

$\sigma^2 = npq = (3500)(0.04)(0.96)$

= 134.4,

$\sigma = \sqrt{134.4} = 11.6$

Let X denote the number of people with Alzheimer's disease.

We seek BP(X < 150) or, approximately, NP (X ≤ 149.5). (BP denote Binomial Probability and NP denote Normal Probability)

We have 149.5 in standard units

$$= \frac{(149.5 - 140)}{11.6} = 0.82$$

Therefore,

Required Probability = NP (X ≤ 149.5)

= NP(Z ≤ 0.82) = 0.5000 + ϕ(0.82)

= 0.5000 + 0.2939

= 0.7939

33. Ans: 2

Sol: $f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$ is the probability density function of Normal Distribution

$$\therefore \int_{-\infty}^{\infty} f(z)\, dz = 1$$

$\therefore$ The value of given integral

$$= 2\int_{-\infty}^{\infty} f(z)\, dz = 2$$

34. Ans: 0.3085

Sol: Let X = diameter of cable in inches

mean = μ = 0.80

Standard deviation = $\sigma = \sqrt{0.0004} = 0.02$

The standard normal variable $Z = \frac{X-\mu}{\sigma}$

When X = 0.81, $Z = \frac{0.81-0.80}{0.02} = \frac{1}{2}$

Required probability = P (X > 0.81)

$= P(Z > \frac{1}{2})$

=1 – (Area under the normal curve to the left of Z = 0.5)

= 1 – 0.6915 = 0.3085

35. Ans: (i) 28 (ii) 28 (iii) 205

Sol: The parameters of normal distribution are $\mu = 68$ and $\sigma = 3$

Let X = weight of student in kgs

Standard normal variable $= Z = \frac{X-\mu}{\sigma}$

(i) When X = 72, we have Z = 1.33

Required probability = P(X > 72)

= Area under the normal curve to the right of Z = 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh greater than 72 kgs = 300 × 0.0918

= 28

(ii) When X = 64, we have Z = –1.33

Required probability = P(X ≤ 64)

= Area under the normal curve to the left of Z = – 1.33

= 0.5 – (Area under the normal curve between Z = 0 and Z = 1.33)

(By symmetry of normal curve)

= 0.5 – 0.4082

= 0.0918

Expected number of students who weigh less than 68 kgs = 300 × 0.0918

= 28

(iii) When X = 65, we have Z = –1

When X = 71, we have Z = +1

Required probability = P(65< X < 71)

= Area under the normal curve to the left of Z = – 1 and Z = +1

= 0.6826

(By Property of normal curve)

Expected number of students who weighs between 65 and 71 kgs = 300 × 0.6826

≈ 205

36. Ans: (b)

Sol: If X has uniform distribution in [a, b] then

$$\text{variance} = \frac{(b-a)^2}{12}$$

$$= \frac{[3a-(-a)]^2}{12} = \frac{16a^2}{12} = \frac{4a^2}{3}$$

37. Ans: (b)

Sol: Let X be a uniformly distributed random variable defined on [a, b].

Mean is $\frac{a+b}{2} = 1 \Rightarrow a + b = 2$ ……. (1)

Variance is $\frac{(b-a)^2}{12} = \frac{1}{3} \Rightarrow b - a = 2$ …..(2)

On solving, we get a = 0, b = 2

$\therefore$ The PDF of f(x) is = $\frac{1}{b-a}$, $a \le x \le b$

$= \frac{1}{2}, 0 \le x \le 2$

$P(X < \frac{1}{2})$ $\int_0^{\frac{1}{2}} f(x)dx = \int_0^{\frac{1}{2}} \frac{1}{2} dx = \frac{1}{4}$

38. Ans: (c)

Sol: $f(x) = \frac{1}{4}, -2 \le X \le 2$

$|X-1| \ge \frac{1}{2}$

$= \left(\frac{-1}{2} \le (X-1) < -2\right) \cup \left(\frac{1}{2} \le (X-1) < 2\right)$

$|X-1| \ge \frac{1}{2}$

$= \left(-1 < X \le \frac{1}{2}\right) + \left(\frac{3}{2} \le X < 3\right)$

$P\left(|X-1| \ge \frac{1}{2}\right)$

$= P\left(-1 < X \le \frac{1}{2}\right) + P\left(\frac{3}{2} \le X < 3\right)$

$= \int_{-1}^{\frac{1}{2}} f(x)dx + \int_{\frac{3}{2}}^{3} f(x)dx$

$= \int_{-1}^{\frac{1}{2}} \frac{1}{4} dx + \int_{\frac{3}{2}}^{3} \frac{1}{4} dx$

$= \frac{1}{4}\left(\frac{1}{2} + 1 + 3 - \frac{3}{2}\right)$

$= \frac{3}{4}$

39. Ans: 0.393

Sol: The probability density function of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & , x \le 0 \end{cases}$$

$P(X<5) = \int_0^5 f(x)\,dx$

$= \int_0^5 \frac{1}{10} e^{\frac{-x}{10}} dx \approx 0.393$

40. Ans: (a)

Sol: Mean $\frac{1}{\theta} = 0.5 \Rightarrow \theta = \frac{1}{0.5} = 2$

The probability function of exponential distribution is $f(x) = 2e^{-2x}$, $x \ge 0$.

$P(X > \frac{1}{2}) = \int_{\frac{1}{2}}^{\infty} 2e^{-2x} dx = \left(-e^{-2x}\right)_{\frac{1}{2}}^{\infty}$

$= (0) - \{-e^{-1}\} = e^{-1}$

41. Ans: (d)

Sol: The density function $f(x) = \frac{1}{5} e^{\frac{-1}{5}x}$

We require $P(x>8) = \int_8^{\infty} f(x)\,dx = e^{-8/5}$

$= 0.2$

42.

Sol: Mean $= \frac{\sum x_i}{n} = 34$

Median is the middle most value of the data by keeping the data points in increasing order or decreasing order.

Mode = 36

S.D = 4.14

43. Ans: (b)

Sol: Mean = $\Sigma x_i p_i = 3$

Variance = $\sum x_i^2 p_i - \mu^2$

$= 10.2 - 9 = 1.2$

44.

Sol: We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 k(x - x^2)\,dx = 1$$

$$\Rightarrow k\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1\right] = 1$$

$$\Rightarrow k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$$

$$\Rightarrow k\left(\frac{3-2}{6}\right) = 1$$

$$\Rightarrow k = 6$$

Mean = $\int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_0^1 6(x^2 - x^3)\,dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{2}$$

Median is that value 'a' for which

$$P(X \le a) = \frac{1}{2}$$

$$\int_0^a 6(x - x^2)\,dx = \frac{1}{2}$$

$$\Rightarrow 6\left(\frac{a^2}{2} - \frac{a^3}{3}\right) = \frac{1}{2}$$

$$\Rightarrow 3a^2 - 2a^3 = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{2}$$

Mode a that value at which f(x) is max/min

$\therefore$ $f(x) = 6x - 6x^2$

$f^1(x) = 6 - 12x$

For max or min $f^1(x) = 0 \Rightarrow 6 - 12x = 0$

$$\Rightarrow \quad x = \frac{1}{2}$$

$f^{11}(x) = -12$

$f^{11}\left(\frac{1}{2}\right) = -12 < 0$

$\therefore$ maximum at $x = \frac{1}{2}$

$\therefore$ mode is $\frac{1}{2}$

S.D = $\sqrt{E(x^2) - (E(x))^2}$

$$= \frac{1}{2\sqrt{5}}$$

45. Ans: 0.95

Sol:

x	y	u= x – 5	v = y – 12	u^2	v^2	uv
1	9	–4	–3	16	9	12
2	8	–3	–4	9	16	12
3	10	–2	–2	4	4	12
4	12	–1	0	1	0	0
5	11	0	–1	0	1	0
6	13	1	1	1	1	1
7	14	2	2	4	4	4
8	16	3	4	9	16	12
9	15	4	3	16	9	12
Total		0	0	60	60	57

correlation coefficient =

$$r_{xy} = r_{uv} = \frac{\sum uv}{\sqrt{\sum u^2 . \sum v^2}}$$

$$= \frac{57}{\sqrt{60.60}}$$

$$= 0.95$$

46. Ans: 0.18

Sol: We have $r = \sqrt{1.6 \times 0.4} = \sqrt{0.64} = 0.8$

$$b_{yx} = r.\frac{\sigma_y}{\sigma_x}$$

$$\Rightarrow \frac{\sigma_y}{\sigma_x} = \frac{b_{yx}}{r} = \frac{1.6}{0.8} = 2$$

$$m_1 = \frac{1}{r}.\frac{\sigma_y}{\sigma_x} = \frac{1}{0.8} \times 2 = \frac{5}{2}$$

$$m_2 = r.\frac{\sigma_x}{\sigma_y} = 0.8 \times 2 = 1.6$$

$$\tan\theta = \left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = \left(\frac{2.5 - 1.6}{1 + 2.5 \times 1.6}\right)$$

$$= \frac{0.9}{5} = 0.18$$

47. Ans: 0.747

Sol: We have, $r(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}}$

$$= \frac{10}{\sqrt{5.5 \times 32.5}}$$

$$= 0.747$$

48. Ans: 0.4

Sol: We have $\bar{x} = \frac{\sum x_i}{n} = \frac{15}{5} = 3,$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{36}{5} = 7.2$$

$$\text{cov}(x, y) = \left(\frac{\sum x_i y_i}{n} - \bar{x}\,\bar{y}\right)$$

$$= \left(\frac{110}{5} - 3 \times 7.2\right) = 0.4$$

49. Ans: i. (a) ii. (c) iii. (d)

Sol: (i) The line of regression of y on x is

$$y = \frac{x}{2} - 2$$

$$\Rightarrow b_{yx} = \frac{1}{2}$$

The line of regression of x on y is

$$x = \frac{y}{2} + 10$$

$\Rightarrow b_{xy} = \frac{1}{2}$

$\therefore r = \sqrt{b_{yx} \times b_{xy}} = \frac{1}{2}$

(ii) $b_{yx} = r.\frac{\sigma_y}{\sigma_x}$

$\Rightarrow \frac{1}{2} = \frac{1}{2}.\frac{\left(\frac{1}{4}\right)}{\sigma_x}$

$\Rightarrow \sigma_x = \frac{1}{4}$

(iii) Let the mean of x = $\bar{x}$

and the mean of y = $\bar{y}$

Then the point $(\bar{x}, \bar{y})$ satisfy the given lines of regression

$\Rightarrow 2\bar{x} - \bar{y} - 20 = 0$

and $2\bar{y} - \bar{x} + 4 = 0$

solving, we get $\bar{x} = 12$ and $\bar{y} = 4$

50. Ans: (b)

Sol: Null Hypothesis H_0: The sample has been drawn from a population whit mean $\mu = 280$ days

Alternate Hypothesis H_1: The sample is not drawn from a population with mean $\mu = 280$

i.e. $\mu \neq 280$

Two-tailed test should be used.

Now the test statistic $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

$\mu = 280$, $\bar{x}$ = mean of the sample = 265

$\sigma = 30$, n = size of the sample = 400

$$Z = \frac{265 - 280}{\frac{30}{\sqrt{400}}} = -10$$

$\Rightarrow$ $|Z| = 10$

$Z_\alpha = 1.96$

Since $|Z| = 10 > 1.96$, we reject null hypothesis

The sample is not drawn from population.

51. Ans: (c)

Sol: $H_0 : P = \frac{1}{5}$, i.e., 20% of the product manufactured is of top quality.

$H_1 : P \neq \frac{1}{5}$.

p = proportion of top quality products in the sample

$= \frac{50}{400} = \frac{1}{8}$

From the alternative hypothesis H_1, we note that two-tailed test is to be used.

Let LOS be 5%. Therefore, $z_\alpha = 1.96$.

$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{\frac{1}{8} - \frac{1}{5}}{\sqrt{\frac{1}{5} \times \frac{4}{5} \times \frac{1}{400}}}$$

Since the size of the sample is equal to 400.

i.e., $z = \frac{3}{40} \times 50 = -3.75$

Now $|z| = 3.75 > 1.96$.

The difference between p and P is significant at 5% level.

Also H_0 is rejected. Hence H_0 is wrong or the production of the particular day chosen is not a representative sample.

95% confidence limits for P are given by

$$\frac{|p-P|}{\sqrt{\frac{pq}{n}}} \le 1.96$$

Note:

We have taken $\sqrt{\frac{pq}{n}}$ in the denominator, because P is assumed to be unknown, for which we are trying to find the confidence limits and P is nearly equal to p.

i.e. $\left(p - \sqrt{\frac{pq}{n}} \times 1.96\right) \le P \le \left(p + \sqrt{\frac{pq}{n}} \times 1.96\right)$

i.e. $\left(0.125 - \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96\right) \le P$

$$\le \left(0.125 + \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96\right)$$

i.e. $0.093 \le P \le 0.157$

Therefore, 95% confidence limits for the percentage of top quality product are 9.3 and 15.7.

52. Ans: (d)

Sol: H_0: p = P, i.e. the hospital is not efficient.

H_1: p < P

One-tailed (left-tailed) test is to be used.

Let LOS be 1%.

Therefore, $z_\alpha = -2.33$.

$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$, where $p = \frac{63}{640} = 0.0984$

P = 0.1726, Q = 0.8274

$$z = \frac{0.0984 - 0.1726}{\sqrt{\frac{0.1726 \times 0.8274}{640}}}$$

$= -4.96$

$\therefore |z| > |z_\alpha|$

Therefore, difference between p and P is significant. i.e., H_0 is rejected and H_1 is accepted.

That is, the hospital is efficient in bringing down the fatality rate of typhoid patients.

Chapter

4 Differential Equations
(With Laplace Transforms)

Leonhard Euler
(1707 – 1783)

01. Ans: (c)

Sol: Given that

(siny–y sinxy)dx+(x cosy – x sinxy) dy = 0

$\Rightarrow$ (sin y) dx + x cos y dy – sin xy (y dx+xdy)= 0

$\Rightarrow$ d (x sin y) – sin(xy) d(xy) = 0

$\Rightarrow \int$ d (x sin y) – $\int$ sin(xy) d(xy) = λ

$\Rightarrow$ x sin y + cos(xy) = λ

where λ is arbitrary constant

02. Ans: (a)

Sol: $\left(xy^2 + 2x^2y^3\right)dx + \left(x^2y - x^3y^2\right)dy = 0$

xy [y dx + x dy] + $2x^2y^3$ dx – x^3y^2 dy =0

$$xy\frac{\left(y\,dx + x\,dy\right)}{x^3\,y^3} + \frac{2x^2y^3\,dx - x^3y^2\,dy}{x^3y^2} = 0$$

$$\frac{y\,dx + x\,dy}{x^2\,y^2} + \frac{2}{x}\,dx - \frac{1}{y}\,dy = 0$$

$$d\left[-\frac{1}{xy}\right] + \frac{2}{x}dx - \frac{1}{y}dy = 0$$

Integrating, we get

$$-\frac{1}{xy} + 2\log x - \log y = C$$

$$\log\left(\frac{x^2}{y}\right) - \frac{1}{xy} = C$$

03. Ans: (a)

Sol: The given equation

$(5x^3 + 3xy + 2y^2)dx + (x^2 + 2xy)dy = 0$

Let M = $5x^3 + 3xy + 2y^2$ and

N = $x^2 + 2xy$

$$\frac{\partial M}{\partial y} = 3x + 4y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x + 2y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{x + 2y}{x^2 + 2xy} = \frac{1}{x}$$

$\therefore$ I.F = $e^{\int \frac{1}{x} + dx} = x$

$\Rightarrow$ k = 1

Multiplying the given Differential Equation by the integrating factor, we get

$(5x^4 + 3x^2y + 2xy^2)dx + (x^3 + 2x^2y)dy = 0$

which is exact

The solution is

$\int(5x^4 + 3x^2y + 2xy^2)dx = C$

$(x^5 + x^3y + x^2y^2) = C$

04. Ans: (c)

Sol: Given that

$$(x + 2y^3)\left(\frac{dy}{dx}\right) = y$$

$y\,dx - x\,dy = 2y^3\,dy$

Leonhard Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He made important discoveries in every branch of mathematic

$$\frac{y\,dx - x\,dy}{y^2} = 2y\,dy$$

$$d\left(\frac{x}{y}\right) = 2y\ dy$$

$$\int d\left(\frac{x}{y}\right) = \int 2y\,dy$$

$$\frac{x}{y} = 2\frac{y^2}{2} + C$$

$x = cy + y^3$

05. Ans: (b)

Sol: Given that

$$\frac{dy}{dx} = \frac{2x}{(x^2 + y^2 - 2y)}$$

$\Rightarrow 2x\ dx = (x^2+y^2 - 2y)\ dy$

$2(x\ dx + y\ dy) = (x^2 + y^2)\ dy$

$$\left(\frac{2x\,dx + 2y\,dy}{x^2 + y^2}\right) = dy$$

$d(\log(x^2+y^2)) = dy$

Integrating both sides

$\log(x^2+y^2) + C = y$

$\Rightarrow y = \log(x^2+y^2) + C$

06. Ans: (c)

Sol: Given that

$(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$

Let $M = 3xy - 2ay^2$ and

$N = x^2 - 2axy$

$$\frac{\partial M}{\partial y} = 3x - 4ay$$

$$\frac{\partial N}{\partial x} = 2x - 2ay$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x}$$

Integrating factor = $e^{\int \frac{1}{x}dx} = x$

By multiplying the given differential equation by the integrating factor, we get

$(3x^2y - 2ay^2x)dx + (x^3 - 2ax^2y)dy = 0$

which is exact

Integrating,

$x^3y - ax^2y^2 = C$

$\Rightarrow x^2y\ (x - ay) = C$

07. Ans: (d)

Sol: Given that

$r \sin\theta\ d\theta + (r^3 - 2r^2 \cos\theta + \cos\theta)dr = 0$

Let $M = r \sin\theta$ and

$N = r^3 - 2r^2 \cos\theta + \cos\theta)$

$$\frac{\partial M}{\partial r} = \sin\theta \quad \text{and} \quad \frac{\partial N}{\partial \theta} = +2r^2 \sin\theta - \sin\theta$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial \theta} - \frac{\partial M}{\partial r}\right) = 2\left(r - \frac{1}{r}\right)$$

Integrating factor (I.F) = $e^{\int 2\left(r - \frac{1}{r}\right)dr}$

$$= \frac{e^{r^2}}{r^2}$$

08.

Sol: Given that

$(x^3y^2 + x)\ dy + (x^2y^3 - y)\ dx = 0$

$(x^2y^2 - 1)\ y\ dx + (x^2y^2 + 1)\ x\ dy = 0$

Let $M = x^3y^2 + x$ and

$N = x^2y^3 - y$

$$I.F = \frac{1}{Mx - Ny}$$

$$= \frac{-1}{2xy}$$

multiplying the given differential equation by I.F, we get

$$\left(-\frac{xy^2}{2}+\frac{1}{2x}\right)dx + \left(\frac{-x^2y}{2}-\frac{1}{2y}\right)dy = 0$$

Integrating

$$\left(\frac{-x^2y^2}{4}\right)+\frac{1}{2}\log x - \frac{1}{2}\log y = k$$

$$\Rightarrow \log\left(\frac{y}{x}\right) + \frac{x^2y^2}{2} = C$$

09.

Sol: Given that

$(4xy + 3y^2 - x)dx + x(x+2y)dy = 0$

Let $M = 4xy + 3y^2 - x$ and

$N = x(x+2y)$

$$\frac{\partial M}{\partial y} = 4x + 6y \quad \frac{\partial N}{\partial x} = 2x + 2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2}{x}$$

Integrating factor = $e^{\int \frac{2}{x}dx} = x^2$

multiplying the given differential equation by I.F, we get

$x^2 (4xy+3y^2 - x)\, dx + x^3(x+2y)dy = 0$

which is exact.

Integrating,

$$4x^4y + 4x^3y^2 - x^4 = C$$

10.

Sol: Given that

$(y^2 + 2x^2y)dx + 2(x^3 - xy)dy = 0$

Let $M = y^2 + 2x^2y$ and

$N = 2(x^3 - xy)$

$$\frac{\partial M}{\partial y} = 4x + 6y \quad \frac{\partial N}{\partial x} = 2x + 2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2}{x}$$

Integrating factor = $e^{\int \frac{-2}{x}dx} = x^{-2}$

multiplying the given differential equation by I.F, we get

$$\frac{1}{x^2}(y^2 + 2x^2y)dx + \frac{2}{x^2}(x^3 - xy)dy = 0$$

which is exact.

Integrating,

$2x^2y - y^2 = Cx$

11.

Sol: Given that

$$\frac{x\,dy}{(x^2+y^2)} = \left(\frac{y}{x^2+y^2} - 1\right)dx$$

$$\frac{x\,dy - y\,dx}{x^2+y^2} = -dx$$

$$\int d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \int -dx + C$$

$$\tan^{-1}\left(\frac{y}{x}\right) = -x + C$$

$y = x\tan(C - x)$

12. Ans: (a)

Sol: Given that

$$x^2\frac{dy}{dx} = \left(3x^2 - 2xy + 1\right)$$

$$x^2\frac{dy}{dx} + 2xy = 3x^2 + 1$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{3x^2+1}{x^2}$$

$$\text{I.F.} = e^{\int \frac{2}{x}dx} = x^2$$

The solution is

$$y.x^2 = \int\left(3 + \frac{1}{x^2}\right).x^2\ dx + C$$

$$y = x + \frac{1}{x} + \frac{C}{x^2}$$

13. Ans: (b)

Sol: $\left(\frac{dy}{dx}\right) + \left(\frac{y}{x}\right) = y^2$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1$$

$$\text{Put } \frac{1}{y} = v$$

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{1}{x}.v = 1$$

$$\frac{dv}{dx} - \frac{1}{x}v = -1$$

$$\text{I.F}= e^{\int \frac{-1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

The solution is

$$v.\frac{1}{x} = \int -1.\frac{1}{x}\ dx + C$$

$$xy(C - \log x) = 1$$

14. Ans: (d)

Sol: Given that

$$\frac{dy}{dx} + \frac{y}{x} = \log x \ \text{ with } y(1) = 1$$

$$\text{I.F} = e^{\int \frac{1}{x}dx} = x$$

The solution is

$$xy = \int \log x.\ x\ dx$$

$$\Rightarrow xy = \log x.\left(\frac{x^2}{2}\right) - \frac{x^2}{4} + C$$

$$y(1) = 1 \quad \Rightarrow \quad C = \frac{5}{4}$$

The solution is

$$y = \frac{x}{2}\log x - \frac{x}{4} + \frac{5}{4x}$$

15.

Sol: Given that

$$\frac{dy}{dx} - x\,\text{Tan}(y - x) = 1$$

$$\text{Put } y - x = t$$

$$\Rightarrow \frac{dy}{dx} - 1 = \frac{dt}{dx}$$

$$\frac{dt}{dx} + 1 - x\ \tan t = 1$$

$$\frac{dt}{dx} = x\ \tan\ t$$

$$\int \cot t\ dt = \int x\ dx$$

$$\log \sin t = \frac{x^2}{2} + C$$

$$\log[\sin(y - x)] = \frac{x^2}{2} + C$$

16.

Sol: Given that $(1 + y^2)\, dx = (\text{Tan}^{-1}y - x)dy$

$$\left(\frac{\tan^{-1} y - x}{1 + y^2}\right)\frac{dy}{dx} = 1$$

Put $\tan^{-1} y = t$

$$\Rightarrow \frac{1}{1 + y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$(t - x)\frac{dt}{dx} = 1$$

$$\frac{dx}{dt} + x = t$$

$$\text{I.F} = e^{\int 1\, dx} = e^x$$

The solution is

$x.\, e^t = \int t.\, e^t\, dt + C$

$x.e^t = e^t[t - 1] + C$

$x = t - 1 + Ce^{-t}$

$$x = (\tan^{-1} y - 1) + C.e^{-\tan^{-1} y}$$

17.

Sol: Given that

$2xy^1 = (10x^3y^5 + y)$

$$\frac{dy}{dx} - \frac{y}{2x} = 5x^2y^5$$

$$\frac{1}{y^5}\frac{dy}{dx} - \frac{y^{-4}}{2x} = 5x^2$$

Put $y^{-4} = t$

$$-4\, y^{-5}\frac{dy}{dx} = \frac{dt}{dx}$$

$$-\frac{1}{4}\frac{dt}{dx} - \frac{t}{2x} = 5x^2$$

$$\frac{dt}{dx} + 2\frac{t}{x} = -20x^2$$

$$\text{I.F} = e^{\int \frac{2}{x} dx} = x^2$$

The solution is

$t.x^2 = \int -20x^2 .x^2\, dx + C$

$$\frac{x^2}{y^4} = -20\frac{x^5}{5} + C$$

$x^2 + (4x^5 - C)y^4 = 0$

18.

Sol: Given that

$$\text{Tan}\, y\frac{dy}{dx} + \text{Tan}\, x = \cos y \cos^2 x$$

$$\sec y \tan y\frac{dy}{dx} + \sec y \tan x = \cos^2 x$$

Put $\sec y = v$

$$\Rightarrow \sec y \tan y\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + (\tan x)v = \cos^2 x$$

$$\text{I.F} = e^{\int \tan x\, dx} = \sec x$$

The solution is

$v.\sec x = \int \cos^2 x .\sec x\, dx + C$

$\sec y = \cos x(\sin x + C)$

19. Ans: (a)

Sol: The auxiliary equation is

$4D^2 - 4D + 1 = 0$

$\Rightarrow D = \frac{1}{2}, \frac{1}{2}$

The solution is

$y = (C_1 + C_2 x)\, e^{\frac{x}{2}}$ .... (i)

$y(0) = 2 \quad \Rightarrow C_1 = 2$

$y^1 = [C_1 + C_2 x]\ e^{\frac{x}{2}} . \frac{1}{2} + C_2\, e^{\frac{x}{2}}$

$y^1(0) = 2 \quad \Rightarrow C_2 = 1$

substituting the values of C_1 & C_2 in (i)

$y = (2 + x)\, e^{\frac{x}{2}}$

20. Ans: (a)

Sol: For the solution $y = C_1 \cos x + C_2 \sin x$ the corresponding roots of the auxiliary equation are $D = \pm i$.

The auxiliary equation is

$(D + i)(D - i) = 0$

$\Rightarrow \quad (D^2 + 1) = 0$

The differential equation is

$$\frac{d^2y}{dx^2} + y = 0$$

Comparing this equation with the given equation

$$\frac{d^2y}{d^2x} + P\frac{dy}{dx} + Q\,y = 0$$

we have P = 0 and Q = 1

Now, the equation

$$\frac{d^2y}{d^2x} + P\frac{dy}{dx} + (Q-1)y = e^x$$

becomes

$$\frac{d^2y}{d^2x} = e^x$$

$$\Rightarrow \frac{dy}{dx} = e^x + C_1$$

$\Rightarrow y = e^x + C_1 x + C_2$

21. Ans: (c)

Sol: The given equation is

$(D^2 - 2D + 5)^2\, y = 0$

The auxiliary equation is

$(D^2 - 2D + 5)^2 = 0$

$\Rightarrow D = 1 \pm 2i, \quad 1 \pm 2i$

The solution is

$y = e^x\,[(C_1 + C_2 x)\cos 2x$

$\qquad + (C_3 + C_4 x)\sin 2x]$

22. Ans: (b)

Sol: The roots of the auxiliary equation are 1, $\pm 2i$.

The differential equation is

$(D - 1)(D + 2i)(D - 2i)\,y = 0$

$\Rightarrow y^{111} - y^{11} + 4y^1 - 4y = 0$

23. Ans: (c)

Sol: The auxiliary equation is

$$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$$

$\Rightarrow D = \frac{-R}{2L}, \ -\frac{R}{2L}$

The solution is

$i = (A + Bt)\ e^{-\frac{Rt}{2L}}$

24. Ans: (c)

Sol: The auxiliary equation is

$D^2 + (3i - 1)D - 3i = 0$

$\Rightarrow D = 1, -3i$

The solution is

$y = C_1 e^x + C_2 e^{-3ix}$

25. Ans: (b)

Sol: If $e^{-x}\ (C_1 \cos \sqrt{3x} + C_2 \sin \sqrt{3x}) + C_3\ e^{2x}$ is the general solution then,

The roots of the auxiliary equation are

$-1 \pm i\sqrt{3}, 2$

The corresponding differential equation is

$(D - 2)\ [D - (-1 + i\sqrt{3}]\ [D - (-1 - i\sqrt{3}]y = 0$

$\Rightarrow (D^3 - 8)y = 0$

$\Rightarrow \frac{d^3y}{dx^3} - 8y = 0$

26. Ans: (d)

Sol: The given equation is

$\frac{d^2y}{dx^2} = e^x$

$\frac{dy}{dx} = e^x + C$

$y = e^x + C_1x + C_2$ .... (i)

$y(0) = 1$

$\Rightarrow C_2 = 0$

$y^1 = e^x + C_1$

$y^1(0) = 2$

$\Rightarrow C_1 = 1$

substituting the values of C_1 & C_2 in (i)

$y = e^x + x$

27. Ans: (d)

Sol: Particular Integral (P.I) = $\frac{1}{D^2+1}\ \cosh 3x$

$= \frac{1}{D^2+1}\left(\frac{e^{3x} + e^{-3x}}{2}\right)$

$= \frac{1}{2}\left[\frac{e^{3x}}{D^2+1} + \frac{e^{-3x}}{D^2-1}\right]$

$= \frac{1}{2}\left[\frac{e^{3x}}{10} + \frac{e^{-3x}}{10}\right]$

28. Ans: (c)

Sol: The auxiliary equation is

$D^2 + 1 = 0$

$\Rightarrow D = \pm i$

Complementary function (C.F)

$= C_1 \cos x + C_2 \sin x$

P.I = $\frac{1}{D^2+1} \sin x$

$= x.\frac{1}{2D} \sin x$

$= \frac{-x}{2} \cos\ x$

The solution is

$$y = C_1 \cos x + C_2 \sin x - \frac{x}{2}\cos x \text{ ...... (i)}$$

$$y\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow C_2 = 0$$

$$y(0) = 1 \quad \Rightarrow C_1 = 1$$

substituting the values of C_1 & C_2 in (i)

$$y = \cos x - \frac{x}{2}\cos x$$

29. Ans: (b)

Sol: The auxiliary equation is

$D^2 + 4 = 0$

$D = \pm 2i$

Complementary function (C.F)

$= C_1 \cos 2t + C_2 \sin 2t$

$$P.I = \frac{1}{D^2+4}\sin 2t$$

$$= t.\frac{1}{2D}\sin 2t$$

$$= \frac{-t}{4}\cos 2t$$

The solution is

$$y = C_1 \cos 2t + C_2 \sin 2t - \frac{t}{4}\cos 2t \text{ ....(i)}$$

$$\frac{dy}{dt} = -2C_1 \sin 2t + 2C_2 \cos 2t - \frac{\cos 2t}{4} + \frac{t\sin 2t}{8}$$

$y(0) = 0$

$\Rightarrow C_1 = 0$

Given $\frac{dy}{dt} = 0$ when $t = 0$

$$\Rightarrow C_2 = \frac{1}{8}$$

substituting the values of C_1 & C_2 in (i)

$$y = \frac{1}{8}(\sin 2t - 2t\cos 2t)$$

30. Ans: (c)

Sol: $P.I = \left(\frac{1}{D^4+1}\right)x^5$

$= (1 + D^4)^{-1} x^5$

$= (1 - D^4 + D^8 - \text{.........})x^5$

$= x^5 - D^4x^4 = x^5 - 120x$

31. Ans: (a)

Sol: $P.I = \left(\frac{1}{4D^2-4D+1}\right)e^{\frac{x}{2}}$

$$= x.\left(\frac{1}{8D-4}\right)e^{\frac{x}{2}}$$

(By Case of failure formula)

$$= \frac{x^2}{8}e^{\frac{x}{2}} \quad \text{(Replacing D with } \frac{1}{2}\text{)}$$

32. Ans: (d)

Sol: $P.I = \left(\frac{1}{D^2+5D+4}\right)(x^2+7x+9)$

$$= \frac{1}{4}\left[1+\left(\frac{D^2+5D}{4}\right)\right]^{-1}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2+5D}{4}\right)+\left(\frac{D^2+5D}{4}\right)^2 - .....\right](x^2+7x+9)$$

$= \frac{1}{4}\left[\left(x^2+7x+9\right)-\frac{1}{2}-\frac{5}{4}\left(2x+7\right)+\frac{25}{16}.2\right]$

$= \frac{1}{4}\left[x^2+\frac{9}{2}x+\frac{23}{8}\right]$

33. Ans: (a)

Sol: Particular Integral (P.I) $= \frac{1}{D^2-4}$ (x sinh x)

$= x\left(\frac{1}{D^2-4}\right)\sin hx - \left[\frac{2D}{\left(D^2-4\right)^2}\right]\sin hx$

$= x\left(\frac{1}{D^2-4}\right)\left(\frac{e^x-e^{-x}}{2}\right)-\left[\frac{2D}{\left(D^2-4\right)^2}\right]\left(\frac{e^x+e^{-x}}{2}\right)$

$= \frac{x}{3}\sin hx - \frac{2}{9}\cos hx$ (applying rule 1)

34. Ans: (b)

Sol: Given that

$x^2y^{11} - xy^1 + y = (\log x)^2$

Let $x = e^t$ and $D_1 = \frac{d}{dt}$

The given equation becomes

$D_1(D_1 - 1)\, y - D_1y + y = t^2$

$(D_1^2 - 2D_1 + 1)y = t^2$

The auxiliary equation

$D_1^2 - 2D_1 + 1 = 0$

$\Rightarrow D_1 = 1, 1$

$C.F = (C_1 + C_2t)e^t$

$P.I = \frac{1}{\left(D_1-1\right)^2}\, t^2$

$= (1 - D_1)^{-2} . t^2$

$= (1 + 2D_1 - 3D_1^2)\, t^2$

$= t^2 + 4t + 6$

The solution is

$y = (C_1 + C_2t)e^t + t^2 + 4t + 6$

$\Rightarrow y = (C_1 + C_2 \log x)\, x + (\log x)^2 + 4\log x + 6$

35. Ans: (c)

Sol: The given equation is

$x^2y^{11} + 6xy^1 + 6y = x$

Let $x = e^t$ and $D_1 = \frac{d}{dt}$

The given equation becomes

$D_1(D_1 - 1)\, y + 6D_1y + 6y = e^t$

$(D_1^2 + 5D_1 + 6)y = e^t$

The auxiliary equation

$D_1^2 + 5D_1 + 6 = 0$

$\Rightarrow D_1 = -2, -3$

$C.F = C_1 e^{-2t} + C_2 e^{-3t}$

$P.I = \frac{1}{\left(D_1^2+5D_1+6\right)}e^t$

$= \frac{e^t}{12}$

The solution is

$y = C_1 e^{-2t} + C_2 e^{-3t} + \frac{e^t}{12}$

$\Rightarrow\ y = \frac{C_1}{x^2}+\frac{C_2}{x^3}+\frac{x}{12}$

36. Ans: (c)

Sol: The differential equation is

$(D-1)(D+1)y = 0$

where $D = \frac{d}{dz}$ and $z = \log x$

$\Rightarrow (D^2 - 1)y = 0$

$\Rightarrow D(D-1)y + Dy - y = 0$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{1}{x}\right)\frac{dy}{dx} - \left(\frac{y}{x^2}\right) = 0$$

37. Ans: (d)

Sol: The given equation is

$x^2y^{11}+2xy^1-12y = 0$

Let $x = e^t$ and $D_1 = \frac{d}{dt}$

The given equation becomes

$D_1(D_1-1)y + 2D_1y - 12y = 0$

$(D_1^2 + D_1 - 12)y = 0$

The auxiliary equation

$D_1^2 + D_1 - 12 = 0$

$\Rightarrow D_1 = -4, 3$

The solution is

$y = C_1 e^{-2t} + C_2 e^{-3t}$

$y = C_1 x^{-4} + C_2 x^3$ ...... (i)

$y(1) = 1$

$\Rightarrow 1 = C_1 + C_2$ ....(ii)

$y^1 = 4C_1 x^{-5} + 3C_2 x^2$

$y^1(1) = 0$

$\Rightarrow -4C_1 + 3C_2$ .... (iii)

solving (ii) & (iii), we get

$C_1 = \frac{3}{7}$ and $C_2 = -\frac{4}{7}$

substituting the values in equation (i), we get

$y = \frac{3}{7}x^{-4} - \frac{4}{7}x^3$

As $x \to \infty$, we have $y \to \infty$

$\therefore$ The solution does not tend to a finite limit as $x \to \infty$.

38. Ans: (a)

Sol: The given equation is

$$x^2\frac{d^3y}{dx^3} - 4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 4$$

Let $x = e^t$ and $D_1 = \frac{d}{dt}$

The given equation becomes

$D_1(D_1-1)(D_1-2)y - 4D_1(D_1-1)y + 6D_1y = 0$

$(D_1^3 - 7D_1^2 + 12D_1)y = 0$

The auxiliary equation

$D_1^3 - 7D_1^2 + 12D_1 = 0$

$D_1 = 0, 4, 3$

$C.F = C_1 + C_2e^{4t} + C_3e^{3t}$

$$P.I = \frac{1}{D_1^3 - 7D_1^2 + 12D_1}.4e^t = \frac{2}{3}e^t$$

The solution is

$$y = C_1 + C_2e^{4t} + C_3e^{3t} + \frac{2}{3}e^t$$

$$\Rightarrow y = \left(C_1 + C_2x^3 + C_3x^4 + \frac{2}{3}x\right)$$

39. Ans: (a)

Sol: The given equation is

$$y^{11} + 4y^{1} + 4y = \frac{e^{-2x}}{x^2} = P \text{ (say)}$$

The auxiliary equation is

$(D+2)^2 = 0 \Rightarrow D = -2, 2$

$C.F = (C_1 + C_2x)e^{-2x}$

$= C_1e^{-2x} + C_2\, xe^{-2x}$

$= C_1\, y_1 + C_2y_2$

where, $C_1 = e^{-2x}$ & $C_2 = xe^{-2x}$

$P.I = A.y_1 + B.y_2$ .... (i)

where, $A = -\int \frac{Py_2}{W} dx$

where, $W = y_1.y_2{}' - y_2.y_1{}' = e^{-4x}$

$$A = -\int \frac{e^{-2x}}{x^2} . \frac{x\,e^{-2x}}{e^{-4x}}\, dx$$

$$= -\log x$$

$$B = \int \frac{Py_1}{W} dx$$

$$= \int \frac{e^{-2x}}{x^2} . \frac{e^{-2x}}{e^{-4x}}\, dx = -\frac{1}{x}$$

substituting the values of A & B in (i)

$P.I = -e^{-2x}[1 + \log x]$

40. Ans: (b)

Sol: The given equation is

$$y'' + 2y' + y = e^{-x} \log x$$

The auxiliary equation is

$(D+1)^2 = 0 \quad \Rightarrow D = -1, -1$

$C.F = (C_1 + C_2x)e^{-x}$

$= C_1e^{-x} + C_2\, xe^{-x}$

$= C_1\, y_1 + C_2y_2$

where, $C_1 = e^{-x}$ & $C_2 = xe^{-x}$

$P.I = A.y_1 + B.y_2$ .... (i)

where,

$A = -\int \frac{Py_2}{W} dx$

where, $W = y_1.y_2{}' - y_2.y_1{}' = e^{-2x}$

$$A = -\int \frac{e^{-x} \log x.\ xe^{-x}}{e^{-2x}} dx$$

$$= \frac{x^2}{4}\left(1 - \log x^2\right)$$

$$B = \int \frac{Py_1}{W} dx$$

$$= \int \frac{e^{-x} \log x.\ e^{-x}}{e^{-2x}} dx$$

$$= x(\log x - 1)$$

41. Ans: (b)

Sol: The given equation is

$$\left(\frac{d^2y}{dx^2}\right) - 4\left(\frac{dy}{dx}\right) + 4y = \frac{e^{2x}}{x}$$

The auxiliary equation is

$(D-2)^2 = 0 \Rightarrow D = 2, 2$

$C.F = (C_1 + C_2x)e^{2x}$

$= C_1e^{2x} + C_2\, xe^{2x}$

$= C_1\, y_1 + C_2y_2$

where, $C_1 = e^{2x}$ & $C_2 = xe^{2x}$

$P.I = A.y_1 + B.y_2$ .... (i)

where, $A = -\int \frac{Py_2}{W} dx$

where, $W = y_1.y_2' - y_2.y_1'$

$= e^{4x}$

$$A = -\int \frac{e^{2x}}{x}.\frac{x e^{2x}}{e^{4x}} dx$$

$= -x$

$$B = \int \frac{Py_1}{W} dx$$

$$= \int \frac{e^{2x}}{x}.\frac{e^{2x}}{e^{4x}} dx$$

$= \log x$

substituting the values of A & B in (i)

$P.I = -xe^{2x} + xe^{2x} \log x$

The solution is

$y = (C_1 + C_2x + x\log x - x)\, e^{2x}$

42.

Sol: The given equation is

$$z = ax + by + a^2 + b^2 \text{ .... (i)}$$

$$\frac{\partial z}{\partial x} = p = a \text{ .... (ii)}$$

$$\frac{\partial z}{\partial y} = q = b \text{ .... (iii)}$$

substituting the values of a & b from (ii) & (iii) in (i)

$z = px + qy + p^2 + q^2$

43.

Sol: The given équation is

$$z = xy + y\sqrt{x^2 - a^2 + b^2}$$

$$p = y + y\frac{2x}{2\sqrt{x^2 - a^2}} \text{ ...... (i)}$$

$$q = x + \sqrt{x^2 - a^2}$$

$$\Rightarrow \sqrt{x^2 - a^2} = q - x \text{ .... (ii)}$$

from (i) & (ii)

$$p = y + \frac{xy}{q - x}$$

$px + qy = pq$

which is the required partial differential equation.

44.

Sol: The given equation is

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

$$p = 2f^1\left(\frac{1}{x} + \log y\right).\left(-\frac{1}{x^2}\right) \text{ .... (i)}$$

$$q = 2y + 2f^1\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right)$$

$$q - 2y = 2f^1\left(\frac{1}{x} + \log y\right).\left(\frac{1}{y}\right) \text{ ....(ii)}$$

dividing (i) by (ii)

$$\frac{p}{q - 2y} = -\frac{y}{x^2}$$

$px^2 + qy = 2y^2$

45.

Sol: The given equation can be written as

$z - xy = \phi(x^2 + y^2)$

Differentiating partially with respect to x

$p - y = \phi^1(x^2 + y^2).2x$ .... (i)

Differentiating partially with respect to y

$q - x = \phi^1(x^2 + y^2).2y$ .... (ii)

Dividing (i) by (ii)

$$\frac{p-y}{q-x} = \frac{x}{y}$$

$qx - py = x^2 - y^2$

46. Ans: (a)

Sol: The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

The general form of 2^{nd} order linear partial differential equation is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, z\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \text{ ....(1)}$$

Equation (1) is said to be

(i) Parabolic if $B^2 - 4AC = 0$

(ii) Elliptic if $B^2 - 4AC < 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Here, A = 1, B = 0 & C = 1

$B^2 - 4AC = -4 < 0$

∴ The given differntial equation is Elliptic

47.

Sol: The given equation is

$$p - q = \log(x + y)$$

∴ The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

$$\frac{dx}{1} = \frac{dy}{-1}$$

$\Rightarrow x + y = C$

$$\frac{dx}{1} = \frac{dz}{\log(x+y)}$$

$$\Rightarrow dx = \frac{1}{\log C} dz$$

$$\Rightarrow x = \frac{z}{\log C} + C_1$$

$$\Rightarrow x - \frac{z}{\log x + y} = C_1$$

The solution is

$$\phi\left[x + y, x - \frac{z}{\log(x+y)}\right] = 0$$

48.

Sol: The auxiliary equations are

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \text{ .... (i)}$$

Using the multipliers 1, 1, 1 each of the fractions in (i)

$$= \frac{dx + dy + dz}{0}$$

$\Rightarrow dx + dy + dz = 0$

$\Rightarrow x + y + z = C_1$ .... (ii)

Using the multipliers x, y, z each of the fractions in (i)

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow x^2 + y^2 + z^2 = C_2$ .... (iii)

The solution is

$f(x + y + z, x^2 + y^2 + z^2) = 0$

49.

Sol: The given equation is

$q = 3p^2$ (Type-I)

Let the solution be

$z = ax + by + c$ .... (i)

$p = a$

$q = b$

substituting in the equation, we have

$b = 3a^2$ ....(ii)

Eliminating b from (i) & (ii)

The solution is $z = ax + 3a^2y + c$

50.

Sol: The given equation is

$q^2 = z^2p^2(1 - p^2)$ (Type-II)

Let $t = x + ay$

$p = \frac{dz}{dt}$ & $q = a.\frac{dz}{dt}$

substituting in the given equation

$$\left(a.\frac{dz}{dt}\right)^2 = z^2\left(\frac{dz}{dt}\right)^2\left(1-\left(\frac{dz}{dt}\right)^2\right)$$

$$\Rightarrow \frac{dz}{dt} = \sqrt{\frac{z^2 - a^2}{z^2}}$$

$$\int d\left(\sqrt{z^2 - a^2}\right) = \int dt + C$$

$$\sqrt{z^2 - a^2} = t + C$$

$$z^2 = (x + ay + C)^2 + a^2$$

51.

Sol: The given equation is

$p^2 + q^2 = x + y$ (Type-III)

$\Rightarrow p^2 - x = y - q^2 = a$ (say)

$\Rightarrow p = \sqrt{a + x}$ and $q = \sqrt{y - a}$

$dz = p\,dx + q\,dy$

$\Rightarrow dz = \sqrt{a + x}\,dx + \sqrt{y - a}\,dy$

Intégrating,

$$z = \left(\frac{2}{3}\right)(a + x)^{3/2} + \left(\frac{2}{3}\right)(y - a)^{3/2} + b$$

52.

Sol: The given equation can be written as

$$z = px + qy + \frac{1}{p - q} \quad \text{(Type-IV)}$$

The solution is

$$z = ax + by + \frac{1}{(a - b)}$$

53.

Sol: The given équation is

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \text{ .... (i)}$$

Let u = X(x).Y(y)

$$\frac{\partial u}{\partial x} = X^1Y \quad \text{and} \quad \frac{\partial u}{\partial y} = XY^1$$

substituting in equation (i)

$X^1Y = 4XY^1$

$$\frac{X'}{X} = \frac{4Y'}{Y} = k$$

$$\frac{X'}{X} = k \quad \text{and} \quad \frac{4Y'}{Y} = k$$

$$\Rightarrow X = C_1\, e^{kx} \quad \text{and} \quad Y = C_2\, e^{\frac{k}{4}y}$$

Now the solution is,

$$u = C_1C_2\, e^{kx}\, e^{\frac{k}{4}y}$$

$$u = C_3\, e^{kx}\, e^{\frac{k}{4}y} \text{ .... (ii)}$$

given $u(0,y) = 8e^{-3y}$

$$\Rightarrow 8e^{-3y} = u(0,1) = C_3\, e^{\frac{k}{4}y}$$

$\Rightarrow C_3 = 8,\ k = -12$

$\therefore\ u = 8\, e^{-12x-3y}$

54.

Sol: The given equation is

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

Let u = X(x).Y(y)

$$\frac{\partial u}{\partial x} = X^1Y \quad \text{and} \quad \frac{\partial u}{\partial y} = XY^1$$

substituting in equation (i)

$3X^1Y + 2XY^1 = 0$

$$\frac{3X'}{X} = \frac{-2Y'}{Y} = k$$

$$\frac{3X'}{X} = k \quad \text{and} \quad \frac{-2Y'}{Y} = k$$

$$\Rightarrow X = C_1\, e^{\frac{k}{3}x} \quad \text{and}$$

$$Y = C_2\, e^{-\frac{k}{2}y}$$

Now the solution is,

$$u = C_1\, e^{\frac{k}{3}x}\, C_2\, e^{-\frac{k}{2}y}$$

$$u = C_3\, e^{\frac{k}{3}x}\, e^{-\frac{k}{2}y}$$

given that $u(x,0) = 4e^{-x}$

$$\Rightarrow 4e^{-x} = C_3\, e^{\frac{k}{3}x}$$

$\Rightarrow C_3 = 4$ and $k = -3$

$$u = 4e^{\frac{1}{2}(-2x+3y)}$$

55. Ans: (b)

Sol: The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2}$$

The general form of 2nd order linear partial differential equation is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}$$

$$+ f\left(x, y, z\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \text{ ....(1)}$$

Equation (1) is said to be

(i) Parabolic if $B^2 - 4AC = 0$

(ii) Elliptic if $B^2 - 4AC < 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Here, $A = C^2, B = 0, C = 0$

$B^2 - 4AC = 0$

$\therefore$ The equation is parabolic

56. Ans: (b)

Sol: The given equation is $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

Let u = X(x).T(t)

$\frac{\partial^2 u}{\partial x^2} = X^{11}T$ and $\frac{\partial u}{\partial t} = XT^1$

substituting in equation (i)

$X^{11}T = \alpha\, XT^1$

$$\frac{X''}{X} = \frac{\alpha T'}{T} = k$$

$$\frac{X''}{X} = \frac{k}{\alpha} \text{ and } \frac{T'}{T} = k$$

$\Rightarrow T = C_1 e^{kt}$

$$X = C_2 e^{x\sqrt{\frac{k}{\alpha}}} + C_3 e^{-x\sqrt{\frac{k}{\alpha}}}$$

The solution is

$$u = C_1 e^{kt}\left[C_2 e^{\left(\sqrt{\frac{k}{\alpha}}\right)x} + C_3 e^{-\left(\sqrt{\frac{k}{\alpha}}\right)x}\right]$$

57. Ans: (d)

Sol: The equation given in option (d) represents one dimensional wave equation.

58.

Sol: The solution of the given equation is with conditions

y(0, t) = 0; y(5,t) = 0; y(x,0) = 0;

is

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{5} \sin\frac{2n\pi t}{5} \quad \text{..... (i)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{5} \cos\frac{2n\pi t}{5}.\left(\frac{2n\pi}{5}\right)$$

given that

$$\frac{\partial y}{\partial t} = 3\sin 2\pi x - 2\sin 5\pi x \quad \text{at } x = 0$$

$$\Rightarrow 3\sin 2\pi x - 2\sin 5\pi x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{5}.\left(\frac{2n\pi}{5}\right)$$

Comparing coefficients of sin 2πx & sin 5πx, we get

$$B_{10} = \frac{3}{4\pi} \quad \text{and} \quad B_{25} = -\frac{1}{5\pi}$$

remaining coefficients are zero.

The solution is

$$y = \frac{3}{4\pi}\sin(2\pi x)\sin(4\pi t) - \frac{1}{5\pi}\sin(5\pi x)\sin(10\pi t)$$

59.

Sol: The given equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The solution of the given equation with conditions

u(0,t) = 0; u(1, t) = 0; u(x,0) = 0 is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin n\pi x \sin n\pi t \quad ..... (i)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n \sin n\pi x \cos n\pi t.(n\pi)$$

given that

$$\frac{\partial u}{\partial t}(x,0) = u_0$$

$$\Rightarrow u_0 = \sum_{n=1}^{\infty} B_n \sin n\pi x (n\pi)$$

By half range sine series, we have

$$B_n (n\pi) = 2\int_0^1 u_0 \sin n\pi x \, dx$$

$$B_n = \frac{2u_0}{n^2 \pi^2}[1 - \cos(n\pi)]$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2u_0}{n^2\pi^2}\left[1-(-1)^n\right] \sin n\pi x \sin n\pi t$$

60.

Sol: The solution of the given equation with conditions

$$y(0,t) = 0; y(l,t) = 0; \frac{\partial y}{\partial t}(x,0) = 0 \quad \text{is}$$

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} \cos\frac{n\pi ct}{\ell} \quad ..... (i)$$

given that

$$y = y_0 \sin^3\left(\frac{x\pi}{\ell}\right)$$

$$\Rightarrow y_0 \sin^3\left(\frac{x\pi}{\ell}\right) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}$$

$$\frac{y_0}{4}\left[3 \sin\frac{\pi x}{4} - \sin\frac{3\pi x}{4}\right] = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}$$

comparing the coefficients both sides

$$A_1 = \frac{3y_0}{4} \quad \text{and} \quad A_2 = \frac{-y_0}{4}$$

$$0 = A_2 = A_4 = A_5 = ...$$

Hence, the solution is

$$y(x,t) = \frac{y_0}{4}\left[3\sin\frac{x\pi}{\ell}\cos\frac{\pi ct}{\ell} - \sin\frac{3x\pi}{\ell}\cos\frac{3\pi ct}{\ell}\right]$$

61.

Sol: The solution of given equation subject to the conditions

$u(0, t) = 0, u(\pi, t) = 0$ is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \sin nx \, e^{-c^2n^2t}$$

given that

$$u(x, 0) = \sin x$$

$$\Rightarrow \sin x = \sum_{n=0}^{\infty} A_n \sin nx \, e^{-c^2n^2t}$$

$$\Rightarrow A_1 = 1 \text{ and } A_2 = A_3 = ... = 0$$

∴ The solution is

$$u(x,t) = \sin x \, e^{-c^2t}$$

62.

Sol: The solution of given equation subject to the conditions

$u(0, t) = 0, u(80, t) = 0$ is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{80} e^{\frac{-c^2n^2\pi^2t}{6400}} \quad ...... (i)$$

given that

$$u(x, 0) = 100 \sin\left(\frac{\pi x}{80}\right)$$

$\Rightarrow 100\sin\left(\frac{\pi x}{80}\right) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{80}$

$\Rightarrow A_1 = 100$ and $A_2 = A_3 = ... = 0$

substituting the values in (i)

$$u(x,t) = 100\sin\left(\frac{\pi x}{80}\right) e^{-\frac{c^2\pi^2 t}{6400}}$$

63.

Sol: The solution of given equation subject to the conditions

$u(0, y) = u(l, y) = u(x, 0) = 0$ is

$$u(x, y) = A\sin\frac{n\pi x}{\ell}\left[e^{\frac{n\pi y}{\ell}} - e^{-\frac{n\pi y}{\ell}}\right]$$

$$u(x, y) = 2A\sin\frac{n\pi x}{\ell}\sinh\frac{n\pi y}{\ell} \quad ...... (i)$$

given that

$$u(x,a) = \sin\frac{n\pi x}{\ell}$$

$$\Rightarrow \sin\frac{n\pi x}{\ell} = 2A\sin\frac{n\pi x}{\ell}\sinh\frac{n\pi y}{\ell}$$

$$\Rightarrow 2A = \frac{1}{\sinh\frac{n\pi a}{\ell}}$$

Now, the solution is

$$u(x,y) = \frac{\sinh\left(\frac{n\pi y}{\ell}\right)}{\sinh\left(\frac{n\pi a}{\ell}\right)}\sin\left(\frac{n\pi x}{\ell}\right)$$

64.

Sol: The solution of given equation subject to the conditions

$u(0, y) = u(l, y) = 0$ is

$$u(x, y) = \sin\frac{n\pi x}{\ell}\left[Ae^{\frac{n\pi y}{\ell}} + Be^{-\frac{n\pi y}{\ell}}\right]$$

given that

$u(x, \infty) = 0$

$\Rightarrow 0 = A = 0$

$$\text{Now, } u(x, y) = \sin\frac{n\pi x}{\ell}\left[Be^{-\frac{n\pi y}{\ell}}\right]$$

The general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}\left[e^{-\frac{n\pi y}{\ell}}\right] \quad ... (i)$$

given that $u(x, 0) = u_0$

$$\Rightarrow u_0 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}$$

$$B_n = \frac{2}{\ell}\int_0^{\ell} u_0 \sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{2u_0}{\ell}\left[\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right]_0^{\ell} = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

The solution is

$$u(x, y) = \sum_{n=0}^{\infty}\frac{2u_0}{n\pi}\left[1-(-1)^n\right]e^{-\frac{n\pi y}{\ell}}\sin\left(\frac{n\pi x}{\ell}\right)$$

65.

Sol: $f(t) = |t-1| + |t+1|, t \geq 0$

$$= \begin{cases} 2 & \text{when } t \leq 1 \\ 2t & \text{when } t > 1 \end{cases}$$

$$L\{f(t)\} = \int_0^{\infty} e^{-st}.f(t)\,dt$$

$$= \int_0^{1} e^{-st}.2\,dt + \int_1^{\infty} e^{-st}.2t\,dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 + 2\left[t.\left[\frac{e^{-st}}{-s}\right] - 1\left[\frac{e^{-st}}{s^2}\right]\right]_1^\infty$$

$$= 2\left[\frac{e^{-st}}{-s} + \frac{1}{s}\right] + 2\left[\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}\right]$$

$$= 2\left[\frac{e^{-s}}{s^2} + \frac{1}{s}\right] = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

66.

Sol: $L\,(1+ t\,e^{-t})^2$

$= L\,(1+2t\,e^{-t} + t^2\,e^{-2t})$

$$= \frac{1}{s} + \frac{2}{(s+1)^2} + \frac{2}{(s+2)^3}$$

(By first shifting property)

67.

Sol: $L(\cos t) = \dfrac{s}{s^2+1}$

By first shifting property

$$L\,(e^{-t}\cos t) = \frac{(s+1)}{(s+1)+1} = \frac{s+1}{s^2+2s+2}$$

By multiplication by t^n property

$$L(t\,e^{-t}\cos t) = (-1)\,\frac{d}{ds}\left(\frac{s+1}{s^2+2s+2}\right)$$

$$= \frac{s^2+2s}{\left(s^2+2s+2\right)^2}$$

68.

Sol: $L(1 - e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

By integral property

$$L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= \left[\log s - \log(s-1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty$$

$$= 0 - \log\left(\frac{s}{s-1}\right)$$

$$= \log\left(\frac{s-1}{s}\right)$$

69.

Sol: $L(\sin t) = \dfrac{1}{s^2+1}$

$$L\{f(t)\} = L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1} ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\{f^1(t)\} = \cot^{-1} s - f(0)$$

$$= \cot^{-1} s - 1$$

70.

Sol: $L(\cos t) = \dfrac{s}{s^2+1}$

By fist shifting property

$$L(e^{-t}.\cos t) = \frac{(s+1)}{(s+1)^2+1}$$

By integral property

$$L\left[\int_0^t e^{-t}\cos\,dt\right] = \frac{1}{s}\left(\frac{s+1}{s^2+2s+2}\right)$$

(By integral property of Laplace Transforms)

71.

Sol: $f(t) = \begin{cases} t, & 0 < t \le 1 \\ 0, & 1 < t < 2 \end{cases}$

$\because$ f(t) is periodic function with period 2

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^1 t.e^{-st}\, dt\, f$$

$$= \frac{1}{1-e^{-2s}} \left[t.\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right) \right]_0^1$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{e^{-s}}{-s}\right) - \left(\frac{e^{-s}}{s^2}\right) + \frac{1}{s^2} \right]$$

$$= \frac{1-e^{-s}-s\,e^{-s}}{s^2\left(1-e^{-2s}\right)}$$

72.

Sol: $L(e^t) = \dfrac{1}{s-1}$

$e^t\, u(t-3) = [e^{t-3}.\, u(t-3)]e^3$

By second shifting property

$L[e^t.u(t-3)] = e^3.\, L[e^{t-3}.\, u(t-3)]$

$$= e^3.\left(\frac{e^{-3s}}{s-1}\right) = \frac{e^{3-3s}}{s-1}$$

73.

Sol: $L(\sin t) = \dfrac{1}{s^2+1}$

$$L(t \sin t) = \int_0^\infty e^{-st}\,(t \sin t)\, dt$$

$$\Rightarrow (-1).\, \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \int_0^\infty e^{-st}(t \sin t)\, dt$$

$$\Rightarrow \frac{2s}{\left(s^2+1\right)^2} = \int_0^\infty e^{-st}\,(t \sin t)\, dt$$

Put s = 3

$$\Rightarrow \frac{2(3)}{\left(3^2+1\right)^2} = \int_0^\infty e^{-st}\; t \sin t\; dt$$

$$\Rightarrow \int_0^\infty e^{-st}\; t.\sin t\; dt = \frac{3}{50}$$

74. Ans: (a)

Sol: $f(t) = L^{-1}\left[\dfrac{1}{s^2(s+1)}\right]$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$\Rightarrow 1 = A\,(s+1) + B(s+1) + cs^2$

C = 1, B = 1, A = –1

$\therefore\ f(t) = L^{-1}\left[\dfrac{-1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s+1}\right]$

$= -1 + t + e^{-t}$

75.

Sol: $L^{-1}\left(\dfrac{1}{s^2}\right) = t$

By first shifting property

$$L^{-1}\left[\frac{1}{(s-2)^2}\right] = e^{2t}.t$$

By second shifting property

$$L^{-1}\left[\frac{e^{-4s}}{(s-2)^2}\right] = e^{2(t-4)}.u(t-4)$$

76.

Sol: $L^{-1}\left[\frac{1}{s(s-1)}\right]$

$= L^{-1}\left[\frac{1}{s-1}-\frac{1}{s}\right]$

$= e^t - 1$

77.

Sol: $L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$

$= L^{-1}\left[\left(\frac{s}{s^2+4}\right)\left(\frac{1}{s^2+4}\right)\right]$

$L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$ and

$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{\sin 2t}{2}$

By convolution theorem,

$L^{-1}\left[\left(\frac{s}{s^2+4}\right).\left(\frac{1}{s^2+4}\right)\right]$

$= \int_0^t \cos 2x.\frac{\sin 2(t-x)}{2}\,dx$

$= \frac{1}{4}\left\{t\sin(2t) + \left[\frac{\cos(2t-4x)}{4}\right]_0^t\right\}$

$= \frac{t\sin 2t}{4}$

78.

Sol: $L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$

$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1}+\frac{Bs+C}{s^2+1}$

$3s + 1 = A(s^2 + 1) + (Bs + C)(s - 1)$

$s = 1 \Rightarrow 4 = 2A \Rightarrow A = 2$

$A + B = 0 \Rightarrow B = -2$

$3 = -B + C \Rightarrow C = 1$

$\therefore L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = L^{-1}\left[\frac{2}{s-1}-2\left(\frac{s}{s^2+1}+\frac{1}{s^2+1}\right)\right]$

$= 2e^t - 2\cos t + \sin t$

79.

Sol: $L^{-1}\left[\frac{1}{(s-1)(s-2)^2}\right]$

$\frac{1}{(s-1)(s-2)^2} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$

$1 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$

$s = 1 \Rightarrow A = 1$

$A + B = 0 \Rightarrow B = -1$

$s = 2 \Rightarrow C = 1$

$L^{-1}\left[\frac{1}{(s-1)(s-2)^2}\right] = L^{-1}\left[\frac{1}{s-1}-\frac{1}{s-2}+\frac{1}{(s-2)^2}\right]$

$= e^t - e^{2t} + t\,e^{2t}$

80.

Sol: $L(y^{11}) + L(y^1) = L(t^2) + 2L(t)$

$\{s^2\,\bar{y} - s\,y(0) - y^1(0)\} + \{s\,\bar{y} - y(0)\}$

$= \frac{2}{s^3+s^2}$

$(s^2 + s)\ \bar{y} - 4s - 2 = \dfrac{2+2s}{s^3}$

$(s^2 + s)\ \bar{y} = 4s + 2 + \dfrac{2s+2}{s^3}$

$$\bar{y} = \frac{4s^4 + 2s^3 + 2s + 2}{s^4(s+1)}$$

$$\frac{4s^4 + 2s^3 + 2s + 2}{s^4(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s+1}$$

$4s^4 + 2s^3 + 2s + 2 = As^3(s+1) + Bs^2(s+1)$

$+ Cs(s+1) + D(s+1) + E.s^4$

$s = 0 \Rightarrow D = 2$

$s = -1 \Rightarrow E = 2$

Comparing s^3 coefficients, $A+B = 2 \Rightarrow B = 0$

Comparing s^4 coefficients, $A + t = 4 \Rightarrow A = 2$

Comparing s^2 coefficients, $B + C = 0 \Rightarrow C = 0$

$$y = L^{-1}\left(\frac{2}{s} + \frac{2}{s^4} + \frac{2}{s+1}\right)$$

$$= 2 + \frac{t^3}{3} + 2e^{-t}$$

Chapter **5**

Complex Variables

Augustin-louis Cauchy
(1789 –1857)

01. Ans: (a)

Sol: Let $u + iv = f(z) = z^2 = (x+iy)^2$

then $u + iv = f(z) = (x^2 - y^2) + i(2xy)$

$\Rightarrow u = x^2 - y^2$ and $v = 2xy$

$\Rightarrow u_x = 2x$ $\quad v_x = 2y$

$\Rightarrow u_y = -2y$ $\quad v_y = 2x$

Here $u_x = v_y$ and $v_x = -u_y$ at every point and also u, v, u_x, u_y, v_x, v_y are continuous at every point.

$\therefore$ f(z) is analytic at every point.

02. Ans: (a)

Sol: Let $u + iv = f(z) = z\,\text{Im}(z) = (x + iy)\,y$

then $u + iv = f(z) = xy + iy^2$

$\Rightarrow u = xy$ and $v = y^2$

$\Rightarrow u_x = y$ $\quad v_x = 0$

$u_y = x$ $\quad v_y = 2y$

Here, $u_x = v_y$ and $v_x = -u_y$ only at one point origin. i.e., C.R equations $u_x = v_y$ and $v_x = -u_y$ are satisfied only at origin. Further v_x, v_y, u_x, u_y are also continuous at origin.

$\therefore$ $f(z) = z\,\text{Im}(z)$ is differentiable only at origin (0,0).

03. Ans: (d)

Sol: sin(z), cos(z) and polynomial $az^2 + bz+c$ are analytic everywhere.

$\therefore$ sin(z), cos(z) and az^2+bz+c are an entire functions.

$\frac{1}{z-1}$ is analytic at every point except at z = 1 because the function $\frac{1}{z-1}$ is not defined at z = 1.

$\Rightarrow \frac{1}{z-1}$ is not analytic at z = 1

$\therefore \frac{1}{z-1}$ is not an entire function

04. Ans: (a)

Sol: Given that z = sin hu.cos v + i coshu. sinv

$\Rightarrow$ z = sinhu.cosh(iv) + coshu.(i sinv)

($\because$ cosh(ix) = cosx & i sin x = i sin hx)

$\Rightarrow$ z = sinhu. cosh(iv) + coshu.sinh(iv)

$\Rightarrow$ z = sinh(u+iv)

($\because$ sinh(A+B) =sinhA coshB+coshA.sinhB)

$\Rightarrow$ z = sinh(w) ($\because$ w = u + iv)

$\Rightarrow w = \sinh^{-1}(z)$

$\Rightarrow w = f(z) = \sinh^{-1}(z)$

$\Rightarrow w^1 = f^1(z) = \frac{1}{\sqrt{1+z^2}}$

Augustin-Louis Cauchy was a French mathematician. "More concepts and theorems have been named for Cauchy than for any other mathematician". Cauchy was a prolific writer; he wrote approximately eight hundred research articles and almost single handedly founded complex analysis.

Here, $f^l(z)$ is defined for all values of z except at $\sqrt{1+z^2} = 0$ (or) $1+z^2 = 0$ (or) $z = i, -i$

$\Rightarrow f^l(z)$ does not exist at $z = i, -i$

$\Rightarrow f(z)$ is not differentiable at $z = i, -i$

$\therefore f(z)$ is not analytic at $z = i, -i$

05. Ans: (a)

Sol: Given that $v = e^x[y \cos y + x \sin y]$

$\Rightarrow v_x = e^x [0 + \sin y] + e^x[y \cos y + x \sin y]$

and $v_y = e^x[-y \sin y + \cos y + x \cos y]$

consider $f^l(z) = u_x - iu_y$

$\Rightarrow f^l(z) = v_y + i v_x$ ($\because u_x = v_y$ & $v_x = -u_x$)

$\Rightarrow f^l(z) = e^x[-y \sin y + \cos y + x \cos y]$

$+ i e^x[\sin y + y\cos y + x\sin y]$

$\Rightarrow \int f^l(z) = ze^z - e^z + e^z + c = z e^z + c$ is a required analytic function.

06. Ans: (c)

Sol: Given that

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\Rightarrow u_x = 3x^2 - 3y^2 + 6x$ and

$u_y = -6xy - 6y$

Consider $f^l(z) = u_x - iu_y$

$\Rightarrow f^l(z) = (3x^2 - 3y^2 + 6x) - i(-6xy - 6y)$

$\Rightarrow f^l(z) = (3z^2 - 0 + 6z) - i(0 - 0)$

(Replacing 'x' by 'z' and 'y' by '0')

$\Rightarrow \int f^l(z)\, dz = \int (3z^2 + 6z)\, z + c$

$$\therefore f(z) = 3\frac{z^3}{3} + 2\frac{z^2}{2} + c$$

$= z^3 + 3z^2 + c$ is a required analytic function where $c = c_1 + ic_2$ is a integral constant & $c = c_1 + ic_2$ because given real point 'u' is containing constant '1'.

07. Ans: (c)

Sol: Given that $u = e^x[x \cos y - y \sin y]$

$\Rightarrow u_x = e^x[\cos y - 0] + e^x[x \cos y - y \sin y]$

and $u_y = e^x[-x \sin y - y \cos y - \sin y]$

Consider $f^l(z) = u_x - iu_y$

$\Rightarrow f^l(z) = e^x[\cos y + x \cos y - y \sin y]$

$- i e^x[-x \sin y - y \cos y - \sin y]$

$\Rightarrow f^l(z) = e^z[1 + z - 0] - i e^z[0 - 0 - 0]$

[Replacing 'x' by 'z' and 'y' by '0']

$\Rightarrow \int f^l(z)\, dz = e^z z\, dz + \int e^z z\, dz + c$

$\therefore f(z) = z e^z - e^z + e^z + c$

$= z e^z + c$

where $c = c_1 + i c_2 = 0 + ic_2$

because the given part 'u' is not containing any constant.

08. Ans: (c)

Sol: Given that $Re\{f^l(z)\} = 2x + 2$, $f(0) = 2$ and $f(1) = 1 + 2i$

Let $f^l(z) = u + iv$

$f^{11}(z) = u_x + i v_x$

$= u_x - i u_y = 2$

$f^l(z) = 2z + c$

$f(z) = z^2 + cz + k$

$f(0) = 2 \Rightarrow k = 2$

$f(i) = 1 + 2i \Rightarrow c = 2$

$\therefore f(z) = z^2 + 2z + 2$

$f^1(z) = 2z + 2$

$= 2(x + i y) + 2$

$= 2(x + 1) + i(2y)$

$\therefore$ Imaginary part of $f^1(z) = 2y$

09. Ans: (c)

Sol: Given $u = (x-1)^3 - 3xy^2 + 3y^2$

$\Rightarrow u_x = 3(x-1)^2 - 3y^2$

and $u_y = -6xy + 6y$

consider $f^1(z) = u_x - i\,u_y$

$\Rightarrow f^1(z) = 3(x-1)^2 - 3y^2 - i(-6xy + 6y)$

$\Rightarrow f^1(z) = 3(z-1)^2 - 0 - i(-0 + 0)$

(Replacing 'x' by 'z' & 'y' by '0')

$\Rightarrow \int f^1(z)\,dz = \int 3(z-1)^2\,dz + c$

$\Rightarrow f(z) = (z-1)^3 + ic$

because the given real part does not contain any constant.

10. Ans: (b)

Sol: Given that $u - v = e^x[\cos y - \sin y]$

Let $f(z) = u + iv$ .... (1) be the required analytic

then $i\,f(z) = iu - v$ .... (2)

Adding (1) & (2), we get

$f(z) + i\,f(z) = (u + iv) + (iu - v)$

$\Rightarrow (1 + i)\,f(z) = (u - v) + i\,(u + v)$

Let $F(z) = U + iV$

where

$F(z) = (1+i)\,f(z)$, $U = u - v$ & $V = u+v$

then $F(z) = U + iV$ is analytic

($\because f(z) = u + iv$ is analytic)

Now $U_x = e^x[\cos y - \sin y]$

& $U_y = i\,e^x[-\sin y - \cos y]$

consider $F^1(z) = U_x - iU_y$

$\Rightarrow F^1(z) = e^x[\cos y - \sin y] - ie^x[-\sin y - \cos y]$

$\Rightarrow F^1(z) = e^z(1 - 0) - i\,e^z(0 - 1)$

$\Rightarrow F^1(z) = e^z(1+i)$

$\Rightarrow \int F^1(z)\,dx = (1+i)\int e^z\,dz + c$

$\Rightarrow \quad F(z) = (1+i)\,e^z + c$

$\Rightarrow (1+i)\,f(z) = (1+i)\,e^z + c$

$\therefore f(z) = e^z + k$ where $k = \dfrac{C}{1+i}$

11. Ans: (b)

Sol: Given $u^2 = x^2 - y^2 - 3x$

$\Rightarrow u_x = 2x - 3$ and $u_y = -2y$

consider $dv = V_x\,dx + V_y\,dy$

$\Rightarrow dv = (-u_y)\,dx + (u_x)\,dy$

($\because u_x = v_y$ & $v_x = -u_y$)

$\Rightarrow dv = (2y)\,dx + (2x - 3)\,dy$

which is exact differential equation

$\Rightarrow \int dv = \int 2y\,dx + \int(-3)\,dy + k$

$\therefore v(x, y) = 2xy - 3y + k$

12. Ans: (d)

Sol: Given that $v = x^3 - 3xy^2$

$\Rightarrow v_x = 3x^2 - 3y^2$ and $v_y = -6xy$

Consider $du = u_x\,dx + u_y\,dy$

$\Rightarrow du = (v_y)\,dx + (-v_x)\,dy$

$(\because u_x = v_y \ \& \ v_x = -u_y)$

$\Rightarrow du = (-6xy)\,dx + (-3x^2 + 3y^2)\,dy$

which is exact differential equation

$\Rightarrow \int du = \int(-6xy)\,dx + \int(3y^2)\,dy + k$

$\therefore u(x, y) = -3x^2y + y^3 + k$

13. Ans: (c)

Sol: Given $u(r, \theta) = e^{-\theta}\cos(\log r)$

$\Rightarrow u_r = -e^{-\theta}\sin(\log r).\dfrac{1}{r}$ and

$u_\theta = -e^{-\theta}\cos(\log r)$

Consider $dv = \left(\dfrac{\partial v}{\partial r}\right)dr + \left(\dfrac{\partial v}{\partial \theta}\right)d\theta$

$\Rightarrow dv = (v_r)\,dr + (v_\theta)\,d\theta$

$= \left(\dfrac{-1}{r}u_\theta\right)dr + (r\,u_r)\,d\theta$

$\Rightarrow dv = \dfrac{1}{r}e^{-\theta}\cos(\log r)\,dr + \left(-e^{-\theta}.\sin(\log r)\right)d\theta$

$\Rightarrow \int dv = \int e^{-\theta}.\dfrac{1}{r}.\cos(\log r)\,dr + \int 0\,d\theta + c$

$\therefore v(r, \theta) = e^{-\theta}\sin(\log r) + c$

14. Ans: (b)

Sol: Given that $u = (x-1)^3 - 3xy^2 + 3y^2$

$\Rightarrow u_x = 3(x-1)^2 - 3y^2$ and

$u_y = -6xy + 6y$

consider $dv = v_x\,dx + v_y\,dy$

$\Rightarrow dv = (-u_y)\,dx + (u_x)\,dy$

$\Rightarrow dv = -(-6xy + 6y)\,dx + (3x^2 - 6x + 3 - 3y^2)$

$\Rightarrow \int dv = -\int(-6xy + 6y)\,dx + \int(3 - 3y^2)\,dy + k$

$\therefore v(x, y) = 3x^2y - 6xy - y^3 + 3y + k$

$= 3(x-1)^2y - y^3 + c$

15. Ans: (b)

Sol: Let $f(z) = e^z + \sin z$ and $z_0 = \pi$

Then Taylor's series expansion of f(z) about a point $z = z_0$ (or) in power of $(z - z_0)$ is given by $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

Here, the coefficient of $(z - z_0)^n$ in the Taylor's series expansion of f(z) about $z = z_0$ is given by $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

$\therefore a_2 = \dfrac{f''(z_0)}{2!} = \dfrac{f''(\pi)}{2!}$

$= \dfrac{(e^z - \sin z)_{z=\pi}}{2} = \dfrac{e^\pi}{2}$

16. Ans: –1

Sol: Given $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ and $|z| > 2$

Now, $|z| > 2$

$\Rightarrow |z| > 2 > 1$

$\Rightarrow |z| > 2$ and $|z| > 1$

$\therefore \left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$\Rightarrow f(z) = \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$\Rightarrow f(z) =$

$$\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+....\right] - \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+......\right]$$

$$\Rightarrow f(z) = (1-1)\frac{1}{z} + (1-2)\frac{1}{z^2} + (1-2^2)\frac{1}{z^3} + ...$$

$\therefore$ The coefficient of $\frac{1}{z^2} = -1$

17. Ans: 1

Sol: Let $f(z) = \log\left(\frac{z}{1-z}\right)$ and $|z| > 1$

(or) $\left|\frac{1}{z}\right| < 1$

then $f(z) = \log\left[\frac{z}{z\left(1-\frac{1}{z}\right)}\right]$

$$= \log\left(\frac{1}{1-\frac{1}{z}}\right)$$

$$\Rightarrow f(z) = \log\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\log\left(1-\frac{1}{z}\right), \quad \left|\frac{1}{z}\right| < 1$$

$\because \log(1-z) =$

$$-\left[-\left\{\frac{1}{z}+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{1}{z}\right)^3+........\right\}\right], \quad \left|\frac{1}{z}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}.\frac{1}{z^3} + ..., \quad \left|\frac{1}{z}\right| < 1$$

$\therefore$ The coefficient of $\frac{1}{z}$ is 1

18. Ans: (a)

Sol: Given $f(z) = \frac{1}{(z-1)(z+3)}$ in $0 < |z+1| < 2$

Let $z + 1 = t$ then $z = t - 1$ and $0 < |t| < 2$

Now, $f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{t(t+2)}$

Instead of expanding $\frac{1}{(z+1)(z+3)}$ in powers of $z + 1$ it is enough to expand $\frac{1}{t(t+2)}$ in powers of t in $0 < |t| < 2$

$f(z) = \frac{1}{t(t+2)}$ in $0 < |t| < 2$ or $\left|\frac{t}{2}\right| < 1$

$$\Rightarrow f(z) = \frac{1}{t}.\frac{1}{2\left(1+\frac{t}{2}\right)} = \frac{1}{t}.\frac{1}{2}.\left[1+\frac{t}{2}\right]^{-1}, \quad \left|\frac{t}{2}\right| < 1$$

$$\Rightarrow f(z) = \frac{1}{2t}\left[1-\left(\frac{t}{2}\right)+\left(\frac{t}{2}\right)^2-.....\right],\ \left|\frac{t}{2}\right|<1$$

$$\Rightarrow f(z) = \frac{1}{2t}-\frac{1}{4}+\frac{1}{8}t-..........$$

$$\therefore\ f(z) = \frac{1}{2(z+1)}-\frac{1}{4}+\frac{(z+1)}{8}-\frac{(z+1)^2}{16}+....$$

19. Ans: (b)

Sol: Given $f(z) = \dfrac{1}{z^2-3z+2}$ in $|z| > 2$

(or) $|z| > 2 > 1$

$$\Rightarrow f(z) = \frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{(z-2)}-\frac{1}{z-1}\ \text{ in } |z|>2\ \ \&\ |z|>1$$

$$\Rightarrow f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}\quad \text{in}\quad \left|\frac{2}{z}\right|<1$$

and $\left|\dfrac{1}{z}\right|<1$

$$\Rightarrow f(z) = \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$\Rightarrow f(z) = \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+.....\right]$$

$$-\frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+.....\right]$$

$$\Rightarrow f(z) = \frac{1}{z}\sum_{n=0}^{\infty}2^n\left(\frac{1}{z}\right)^{n+1}-\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

$$\text{(or) } f(z) = \sum_{n=0}^{\infty}2^n\left(\frac{1}{z}\right)^{n+1}-\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

20. Ans: (a)

Sol: Given that $f(z) = \dfrac{1}{4z-z^2}$ in $0 < |z| < 4$

$$\Rightarrow f(z) = \frac{1}{z(4-z)}\ \text{ in } |z|<4\ \text{ or }\ \left|\frac{z}{4}\right|<1$$

$$\Rightarrow f(z) = \frac{1}{4z\left(1-\frac{z}{4}\right)} = \frac{1}{4z}\left[1-\frac{z}{4}\right]^{-1}\ \text{ in } \left|\frac{z}{4}\right|<1$$

$$\Rightarrow f(z) = \frac{1}{4z}\left[1+\left(\frac{z}{4}\right)+\left(\frac{z}{4}\right)^2+.....\right]\ \text{ in } \left|\frac{z}{4}\right|<1$$

$$\Rightarrow f(z) = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n\ \text{ in } \left|\frac{z}{4}\right|<1$$

$$\therefore\ f(z) = \sum_{n=0}^{\infty}\frac{1}{4^{n+1}}z^{n-1}\ \text{ in } \left|\frac{z}{4}\right|<1$$

21. Ans: (a)

Sol: Given $f(z) = \dfrac{3}{3z-z^2}$ and $z_0 = 1$

The given function is analytic at $z = 1$

$\therefore$ Taylor's series expansion of f(z) is possible at $z = 1$

Now, $f^{1}(z) = \dfrac{-3(3-2z)}{(3z-z^2)^2}$

$$= \frac{6z-9}{(3z-z^2)^2}$$

$$\Rightarrow f^{11}(z) = \frac{(3z-z^2)(6)-(6z-9)2(3z-z^2)(3-2z)}{(3z-z^2)^4}$$

$$\Rightarrow f^{1}(1) = \frac{-3}{4},$$

$$f^{11}(1) = \frac{18}{8}\ \text{ and } f(1) = \frac{3}{2}$$

The Taylor's series of f(z) about $z = z_0$ is given by

$$f(z) = f(z_0) + (z - z_0)\, f^1(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \ldots$$

$$\Rightarrow f(z) = f(1) + (z - 1)\, f^1(1) + \frac{(z-1)^2}{2!} f''(1) + \ldots$$

$$\therefore \frac{3}{(3z - z^2)} = \frac{3}{2} + \left(\frac{-3}{4}\right)(z-1) + \frac{18}{8}\cdot\frac{1}{2!}(z-1)^2 + \ldots$$

22. Ans: (c)

Sol: The given function $f(z) = z^2$ is analytic at every point.

$\therefore$ The value of the given integral is independent of the path joining z = 0 and z = 3 + i

Now, $I = \int_{z=0}^{3+i} z^2 \, dz$

$$\Rightarrow I = \left(\frac{z^3}{3}\right)_0^{3+i} = \frac{(3+i)^3}{3} - \frac{0}{3}$$

$$= \frac{(27 - 27i - 9 - i)}{3}$$

$$\therefore I = 6 + \left(\frac{26}{3}\right) i$$

23. Ans: –1.047

Sol: Let $f(z) = \frac{1}{z^2 + 9}$

$$= \frac{1}{(z+3i)(z-3i)}$$

Then the singular points of f(z) are given by $z^2 + 9 = 0$ (or) $z = 3i, -3i$

But only one singular point $z = -3i$ lies inside the given circle C: $|z + 3i| = 2$

Consider $f(z) = \frac{\phi(z)}{z - z_0}$

$$= \frac{\frac{1}{z-3i}}{[z - (-3i)]}$$

$\therefore$ By Cauchy's Integral Formula, we have

$$\oint_C f(z)\, dz = 2\pi i \left[\frac{1}{z - 3i}\right]_{z=-3i}$$

$$= 2\pi i \left(\frac{1}{-3i - 3i}\right)$$

$$= \frac{-\pi}{3} = -1.04719$$

24. Ans: (c)

Sol: Let $f(z) = \frac{\cos(\pi z)}{z - 1}$

$$= \frac{\phi(z)}{z - z_0}$$

Then the singular point of f(z) is given by $z - 1 = 0$ (or) $z = 1$

Here, the singular point z = 1 lies inside the given circle C: $|z - 1| = 2$.

$\therefore$ By Caychy's Integral Formula, we have

$$\oint_C f(z)\, dx = 2\pi i [\cos(\pi z)]_{z=1}$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$

25. Ans: 0

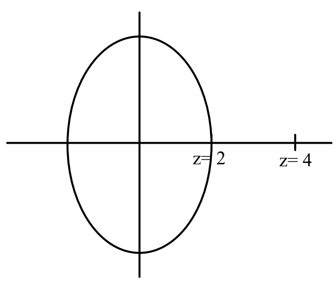
Sol: Let f(z) = $\dfrac{4z^2 + z + 5}{z - 4}$

Then the singular point of f(z) is given by $z - 4 = 0$ (or) $z = 4$

Given that C: $9x^2 + 4y^2 = 36$

$$\Rightarrow \frac{9x^2}{36} + \frac{4y^2}{36} = 1$$

$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$



Here the singular point of the function f(z) lies outside the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

$\therefore$ The given function f(z) has no singular inside and on the curve 'C'

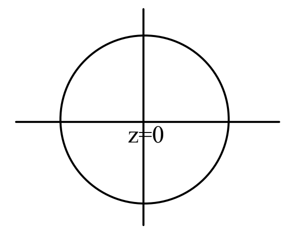
Hence by Cauch's Integral Theorem, we have $\oint_C f(z)\,dz = 0$.

26. Ans: (c)

Sol: Let f(z) = $\dfrac{1}{z^2 e^z} = \dfrac{e^{-z}}{z^2} = \dfrac{e^{-z}}{(z-0)^2}$

Then the singular point of the function f(z) is given by $z^2e^z = 0$ (or) $z = 0$ ($\because$ $e^z \neq 0\ \forall z$)

Here, the singular point z = 0 of the function f(z) lies inside the circle C: |z| = 1.



Let f(z) = $\dfrac{\phi(z)}{[z - z_0]} = \dfrac{e^{-z}}{[z-0]^{1+1}}$

Then by Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \frac{2\pi i}{1!}\left(\frac{d}{dz}e^{-z}\right)_{z=0}$$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i\,(-e^{-z})_{z=0}$$

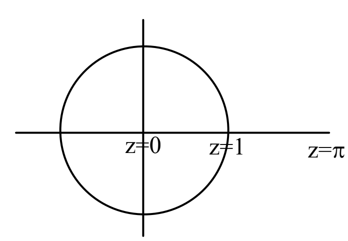
$$\therefore \oint_C f(z)\,dz = -2\pi i$$

27. Ans: (a)

Sol: Let f(z) = $\dfrac{\sin^2(z)}{\left(\dfrac{z-\pi}{6}\right)^3}$

$$= \frac{6^3.\sin^2 z}{(z-\pi)^3}$$

Then the singular point of f(z) is given by $(z-\pi)^3 = 0$ (or) $z = \pi$.



Here the singular $z = \pi$ lies outside the given circle C: $|z| = 1$.

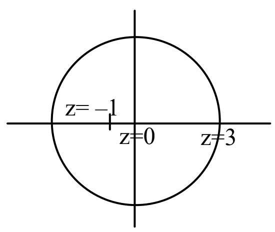
$\therefore$ By Cauchy's Integral Theorem, we have

$$\oint_C f(z)\,dz = 0$$

28. Ans: (d)

Sol: Let $f(z) = \dfrac{e^{2z}}{(z+1)^4}$

Then the singular point of f(z) is given by $(z+1)^4 = 0 \Rightarrow z = -1$.



Here, the singular point $z = -1$ lies inside the given circle C: $|z| = 3$.

Let $f(z) = \dfrac{\phi(z)}{[z - z_0]^{n+1}}$

$$= \frac{e^{2z}}{[z-(-1)]^{3+1}}$$

Then the Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \frac{2\pi i}{3!}\left(\frac{d^3}{dz^3}e^{2z}\right)_{z=-1}$$

$$\Rightarrow \oint_C f(z)\,dz = \frac{2\pi i}{3!}\left(8e^{2z}\right)_{z=-1}$$

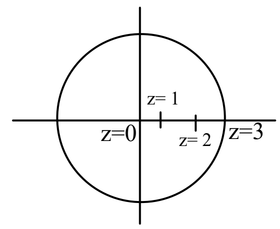
$$\therefore \oint_C f(z)\,dz = \left(\frac{8}{3}\right)\pi i\, e^{-2}$$

29. Ans: (d)

Sol: Let $f(z) = \dfrac{\cos(\pi z^2)}{(z-2)(z-1)}$

Then the singular points of f(z) are given by $(z-2)(z-1) = 0$

$\Rightarrow z = 1$ and $z = 2$



Here, the two singular points $z = 1$ and $z = 2$ lie inside the circle C: $|z| = 3$

Now, $f(z) = \cos(\pi z^2).\left[\dfrac{1}{(z-2)(z-1)}\right]$

$$= \cos(\pi z^2)\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$\Rightarrow f(z) = \frac{\cos(\pi z^2)}{z-2} - \frac{\cos(\pi z^2)}{z-1}$$

$\therefore$ By Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \oint_C \frac{\cos(\pi z^2)}{z-2}dz - \oint_C \frac{\cos(\pi z^2)}{z-1}dz$$

$$= 2\pi i\,[\cos(\pi z^2]_{z=2} - 2\pi i[\cos(\pi z^2)]_{z=1}$$

$$= 2\pi i\,(1) - 2\pi i\,(-1)$$

$$= 4\pi i$$

30. Ans: (c)

Sol: Let $f(z) = z^2 e^{\frac{1}{z}}$

Then

$$f(z) = z^2\left[1+\frac{\left(\frac{1}{z}\right)}{1}+\frac{\left(\frac{1}{z}\right)^2}{2!}+\frac{\left(\frac{1}{z}\right)^3}{3!}+\frac{\left(\frac{1}{z}\right)^4}{4!}+....\right]$$

$$\Rightarrow f(z) = z^2 + z + \frac{1}{2!}+\frac{1}{3!}\frac{1}{z}+\frac{1}{4!}\frac{1}{z^2}+......$$

$$\Rightarrow f(z) = (z-0)^2 + (z-0) + \frac{1}{2!}+\frac{1}{3!}\frac{1}{(z-0)}+\frac{1}{4!}\frac{1}{(z-0)^2}+........$$

$\Rightarrow$ f(z) has a singular point at z = 0.

Here, the singular point z = 0 lies inside the circle |z| = 1.

R_1 = Res(f(z) : z = 0) = The coefficient of $\frac{1}{(z-0)}$ in Laurent series

$$\Rightarrow R_1 = \text{Res}\,(f(z) : z = 0) = \frac{1}{3!} = \frac{1}{6}.$$

$\therefore$ By Cauchy's Residue Theorem, we have

$$\oint_C f(z)\,dz = 2\pi i(R_1)$$

$$= 2\pi i\left(\frac{1}{6}\right)$$

$$= \frac{\pi i}{3}$$

31. Ans: (b)

Sol: Given $f(a) = \int_C \frac{5z^2-4z+3}{z-a}\,dz$

where 'C' is $16x^2 + 9y^2 = 144$ (or)

$$\frac{x^2}{3^2}+\frac{y^2}{4^2}=1$$

Let a= i for finding the value of $f^1(i)$.

Then the singular point z = a = i of the function $\frac{5z^2-4z+3}{z-a}$ lies inside the ellipse

$\therefore$ By Cauchy's integral formula, we have

$$f(a) = \int_C \frac{5z^2-4z+3}{z-a}\,dz$$

$$= 2\pi i\,(5z^2-4z+3)_{z=a}$$

$$\Rightarrow f(a) = 2\pi i\,(5a^2-4a+3)$$

$$\Rightarrow f^1(a) = 2\pi i\,(10a-4)$$

$$\therefore f^1(i) = 2\pi i\,(10i-4)$$

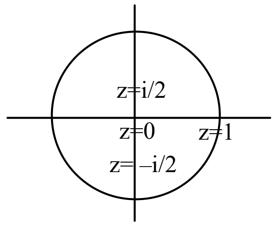
$$= -4\pi\,(5+2i)$$

32. Ans: 0

Sol: Let $f(z) = \frac{\cosh(z)}{4z^2+1}$

$$= \frac{\cosh(z)}{4\left[z^2+\frac{1}{4}\right]} = \frac{\left(\frac{\cosh(z)}{4}\right)}{\left(z-\frac{i}{2}\right)\left(z+\frac{i}{2}\right)}$$

Then the singular points of f(z) are $z = \frac{i}{2}$, $\frac{-i}{2}$.



Here, the two singular point $z = \frac{i}{2}$ and $z = \frac{-i}{2}$ lie inside the circle |z| = 1.

Now, $f(z) = \frac{\cos(z)}{4}\left[\frac{1}{\left(z-\frac{i}{2}\right)\left[z-\left(-\frac{i}{2}\right)\right]}\right]$

$\Rightarrow f(z) = \left[\frac{\cos(z)}{4}\left[\frac{1}{\left(z-\frac{i}{2}\right)\left[\left(\frac{i}{2}+\frac{i}{2}\right)\right]}\right.\right.$

$\left.+\frac{1}{\left(z+\frac{i}{2}\right)\left(\frac{-i}{2}-\frac{i}{2}\right)}\right]$

$$\Rightarrow f(z) = \frac{\left(\frac{\cos h(z)}{4i}\right)}{\left[z-\frac{i}{2}\right]}+\frac{\left(\frac{\cos h(z)}{-4i}\right)}{\left[z-\left(\frac{-i}{2}\right)\right]}$$

By Caychy's Integral Formula, we have

$\oint_C f(z)\,dz$

$$= \frac{1}{4i}\oint_C \frac{(\cos h(z))}{\left[z-\frac{i}{2}\right]}dz+\left(\frac{1}{-4i}\right)\int_C \frac{\cos h(z)}{\left[z-\left(\frac{-i}{2}\right)\right]}\,dz$$

$$= \left(\frac{1}{4i}\right)\left[2\pi i\,(\cos z)_{z=\frac{i}{2}}\right]+\left(\frac{-1}{4i}\right)\left[2\pi i(\cos hz)_{z=\frac{-i}{2}}\right]$$

$$= \frac{\pi}{2}\left[\cos h\left(\frac{i}{2}\right)\right]+\left(\frac{-\pi}{2}\right)\left[\cos h\left(-\frac{i}{2}\right)\right]$$

$$= \frac{\pi}{2}\cos h\left(\frac{i}{2}\right)-\frac{\pi}{2}\cos h\left(\frac{-i}{2}\right)$$

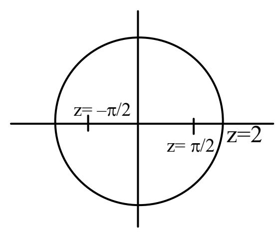
$= 0$ $\quad (\because \cosh(-z) = \cosh(z))$

33. Ans: 0

Sol: The singular points of $f(z) = \frac{\sin z}{z.\cos(z)}$ are given by z.cos(z) = 0

$\Rightarrow z = 0$ and $z = (2n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow z = 0$ and $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, ...



$\Rightarrow z = 0,\ z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ lie inside the circle |z| = 2.

Here, $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ are simple poles of

$f(z) = \frac{\sin z}{z\cos(z)} = \frac{\phi(z)}{\psi(z)}$

where $\psi^1(z) = \cos(z) - z\sin z$

$$R_1 = \text{Res}(f(z): z = \frac{\pi}{2}) = \frac{\phi\left(\frac{\pi}{2}\right)}{\psi'\left(\frac{\pi}{2}\right)} = \frac{1}{0-\frac{\pi}{2}}$$

$$= \frac{-2}{\pi}$$

$$R_2 = \text{Res}(f(z): z = -\frac{\pi}{2}) = \frac{\phi\left(-\frac{\pi}{2}\right)}{\psi'\left(-\frac{\pi}{2}\right)}$$

$$= \frac{-1}{0-\frac{\pi}{2}} = \frac{2}{\pi}$$

Hence, $R_1 + R_2 = \left(\frac{-2}{\pi}\right) + \left(\frac{2}{\pi}\right) = 0$

34. Ans: 0.33

Sol: $f(z) = \frac{1}{z^3} - \frac{1}{z^5}\left[\sin^2(z)\right]$

$$= \frac{1}{z^3} - \frac{1}{z^5}\left[\frac{1-\cos(2z)}{2}\right]$$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{z^5}\left[\frac{1}{2} - \frac{1}{2}\cos(2z)\right]$$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{2z^5} + \frac{1}{2z}\left[1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - -\frac{(2z)^6}{6!} + \ldots\ldots\right]$$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{2z^5} + \left[\frac{1}{2z^5} - \frac{1}{z^3} + \frac{2^4}{2.4!}.\frac{1}{z} - \frac{2^6}{2.6!}z + \ldots\ldots\right]$$

$\therefore$ Res(f(z) : z = 0) = The coefficient of $\frac{1}{z}$ in above series

$$= \frac{2^4}{2.4!} = \frac{1}{3} = 0.333\ldots..$$

35. Ans: 1

Sol: The given singular point z = 0 is a simple pole

(or) 1st order pole of $f(z) = \frac{1+e^z}{z\cos(z)+\sin(z)}$

Now R_1 = Res (f(z) : z = 0) = $\underset{z\to 0}{\text{Lt}}(z-0)\, f(z)$

$$\Rightarrow R_1 = \underset{z\to 0}{\text{Lt}}(z-0).\frac{1+e^z}{z\cos(z)+\sin(z)}$$

$$\left(\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow R_1 = \underset{z\to 0}{\text{Lt}}\ \frac{z(0+e^z)+(1+e^z)}{-z\sin(z)+\cos(z)+\cos(z)}$$

$$\therefore R_1 = \frac{0+1+1}{0+1+1} = 1$$

36. Ans: 0

Sol: Let $f(z) = \frac{z^2+z}{(z-1)^{10}}$

Then the singular point of f(z) is z = 1 and the singular z = 1 lies inside the circle |z| = 2.

Now,

$$f(z) = \frac{\phi(z)}{[z-z_0]^{n+1}} = \frac{z^2+z}{[z-1]^{10}}$$

∴ By Cauchy's Integral formula, we have

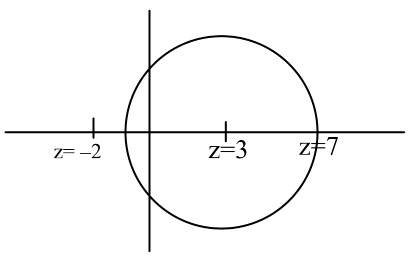
$$\oint_C f(z)\,dz = \frac{2\pi i}{9!}\left[\frac{d^9}{dz^9}\left(z^2+z\right)\right]_{z=1}$$

$$= \frac{2\pi i}{9}(0) = 0$$

37. Ans: (d)

Sol: Let $f(z) = \dfrac{e^z}{(z+2)(z-3)^2}$

Then the singular points of f(z) are z = –2 & z = 3 of these two singular points z = –2 and z = 3 only z = 3 lies inside the circle |z –3| = 4.



Let $f(z) = \dfrac{\phi(z)}{[z-z_0]^{n+1}}$

$$= \frac{\left(\dfrac{e^z}{z+2}\right)}{[z-3]^{1+1}}$$

Then by Cauchy's Integral Formula, we have

$$\oint_C f(z)\,dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\left(\frac{e^z}{z+2}\right)\right]_{z=3}$$

$$= 2\pi i\left[\frac{(z+2)e^z - e^z(1)}{(z+2)^2}\right]_{z=3}$$

$$= 2\pi i\left[\frac{(3+2)e^3 - e^3}{(3+2)^2}\right]$$

$$= \frac{8\pi i\, e^3}{25}$$

Chapter 6 Numerical Methods

Carl David Tolme Runge (1856 – 1927)

Martin Wilhelm Kutta (1867-1944)

01. Ans: (c)

Sol: $f(x) = x^3 - 4x - 9 = 0$

$f(2) = -9 < 0$, $f(3) = 6 > 0$

Let $x_1 = \frac{2+3}{2} = 2.5$ is first approximation to the root

$\therefore f(x_1) = f(2.5) = -3.375 < 0$

Now, Root lies in [2.5, 3]

Let $x_2 = \frac{2.5+3}{2} = 2.75$ is second approximation root.

02. Ans: 0.67

Sol: $f(x) = x^3 + x - 1 = 0$

Let $x_0 = 0.5$, $x_1 = 1$

$f(x_0) = f(0.5) = -0.375$

$f(x_1) = f(1) = 1$

$$\therefore x_2 = \frac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$$

is first approximation root

$$= \frac{1(0.5) - (-0.375)(1)}{1-(-0.375)}$$

$$= \frac{0.5+0.375}{1.375} = \frac{0.875}{1.375}$$

$= 0.6363$

$f(x_2) = f(0.6363)$

$= (0.6363)^3 + (0.6363) - 1$

$= 0.2576 + 0.6363 - 1$

$= -0.1061 < 0$

Root lies in (0.6363, 1)

$$x_3 = \frac{f(x_1)x_2 - f(x_2).x_1}{f(x_1) - f(x_2)}$$

$$= \frac{1(0.6363) - (-0.1061)1}{1+0.1061}$$

$= 0.6711$

03. Ans: (b)

Sol: $f(x) = xe^x - x = 0$

$f(0) = -2 < 0$, $f(1) = 2.7183 - 2 > 0$

Let $x_0 = 0$, $x_1 = 1$

$$x_2 = \frac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$$

$$= \frac{0.7183(0) - (-2).1}{0.7183 - (-2)}$$

$$= \frac{2}{2.7183}$$

$= 0.7357$

$f(x_2) = f(0.7357)$

$= 0.7357\ .e^{0.7357} - 2 = -0.4644$

Take $x_0 = 0.7357$ & $x_1 = 1$

$$\therefore x_3 = \frac{f(x_2).x_1 - f(x_1).x_2}{f(x_2) - f(x_1)}$$

$$= \frac{0.9929}{1.1827} = 0.8395$$

***C. Runge** and **M. W. Kutta** (German mathematicians) developed an important family of implicit and explicit iterative methods, which are used in **temporal discretization** for the approximation of solutions of **ordinary differential equations**. In **numerical analysis**, these techniques are known as Runge–Kutta methods.*

04. Ans: (b)

Sol: $f(x) = x^4 - x - 10 = 0; f^1(x) = 4x^3 - 1$

$f(1) = -10 < 0, f(2) = 4 > 0$

Let $x_0 = 2$ is initial approximation

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$= 2 - \frac{4}{31} = 1.871$$

05. Ans: (c)

Sol: $f(x) = 3x - \cos x - 1$

$f(x_0) = f(0) = -2$

$f^1(x_0) = f^1(0) = 3$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = -\frac{(-2)}{3} = \frac{2}{3}$$

06. Ans: (a)

Sol: Let $x = \sqrt{N}$

$f(x) = x^2 - N = 0$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \text{......... (i)}$$

07. Ans: (b)

Sol: Taking N = 18 & $x_0 = 4$ in equation (i) of previous examples(06), we get

$$x_1 = \frac{4^2 + 18}{8} = 4.25$$

08. Ans: (a)

Sol: $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$$

Let $x_{n+1} = x_n = x$

$$x = \frac{1}{2}\left(x + \frac{3}{x}\right)$$

$x^2 = 3$

09. Ans: (b)

Sol: $$\text{Trap.rule} = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.01}{2}\left[(0.2474 + 0.2860) + 2(0.2571 + 0.2667 + 0.2764)\right]$$

$$= \frac{0.01}{2}[0.5334 + 1.6004]$$

$= 0.005[2.1338]$

$= 0.0106$

10. Ans: (a)

Sol:

x	–1	0	1
f(x) = 5x³ – 3x² + 2x + 1	–9	1	5

$$\int_{-1}^{1} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_2) + 2(0) + 4(y_1)\right]$$

$$= \frac{1}{3}\left[(-4) + 4(1)\right] = 0$$

11. Ans: (a)

Sol: Error = Exact value of the integral – The value of the integral by the simpson's rule

$= 0 - 0 = 0$

12. Ans: (b)

Sol: The area

$$= \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{0.5}{3}[(2 + 2.1) + 2(2.7 + 3) + 4(2.4 + 2.8 + 2.6)]$$

$$= 7.783$$

13. Ans: (c)

Sol:

x	0	1	2	3	4	5	6
f(x) = $\frac{1}{1+x^2}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}\left[\left(1 + \frac{1}{37}\right) + 2\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26}\right)\right]$$

$$= 1.4107$$

14. Ans: (a)

Sol: The volume of cylinder = $\pi\int_0^1 y^2 \, dy$

$$= \pi\frac{h}{2}\left[\left(y_0^2 + y_4^2\right) + 2y_2^2 + 4\left(y_1^2 + y_3^2\right)\right]$$

$$= \pi\frac{0.25}{3}[(1+1) + 2(9) + 4(4+1)]$$

$$= \pi\frac{0.25}{3}[40]$$

$$= \frac{10\pi}{3}$$

15. Ans: (a)

Sol: Error = $\text{Max}\left|\frac{b-a}{12} \times h^2 \times f''(x)\right|$

$$= \frac{1}{12} \times \frac{1}{100} \times 6(2.718)$$

$$= 0.0136$$

Here,

$f(x) = e^{x^2}$

Max $|f^{11}(x)|_{[0,1]} = 6e$

$\therefore h = \frac{b-a}{n}$

$$= \frac{1}{10}$$

16. Ans: (c)

Sol: $\left|\frac{b-a}{180}\times h^4\times \max f^{iv}(x)\right|\le 10^{-5}$

Let $h = \frac{b-a}{n} = \frac{1}{n}$

$f(x) = \frac{1}{x}$

Max $|f^{iv}(x)|_{at\ x=1} = 24$

$\left(\frac{1}{180}\times\frac{1}{n^4}\times 24\right)\le 10^{-5}$

$\Rightarrow n \ge 10.738$

$\therefore n \ge 10.738$

17. Ans: x = 0.9, y = 1 & z = 1

Sol: Let

$10x + y + z = 12$

$2x + 10y + z = 13$

$2x+2y + 10z = 14$ and

$x_0 = 0, y_0 = 0, z_0 = 0$

Then first iteration will be

$x_1 = \frac{1}{10}(12 - y_0 - z_0)$

$= 1.2$

$y_1 = \frac{1}{10}(13 - 2x_1 + 10y_0)$

$= \frac{1}{10}(13 - 2(1.2) - 0) = 1.06$

$z_1 = \frac{1}{10}(14 - 2x_1 - 2y_1)$

$= \frac{1}{10}(14 - 2(1.2) - 2.(1.06)) = 0.95$

Second iteration will be

$x_2 = \frac{1}{10}(12 - y_1 - z_1)$

$= 0.90$

$y_2 = \frac{1}{10}(13 - 2x_2 + 10y_1)$

$= 1.00$

$z_2 = \frac{1}{10}(14 - 2x_2 - 2y_2)$

$= 1.00$

The required solution after second iteration is x = 0.9, y = 1 & z = 1

18. Ans: 0.6

Sol: $y^1 = f(x, y) = 4 - 2xy$

$x_0 = , y_0 = 0.2, h = 0.1$

By Taylor's theorem,

$y(x) = y(x_0 + h)$

$= y(x_0) + h\, y^1(x_0) + \frac{h^2}{2\,!}y^{11}(x_0)$

$= 0.2 + 0.1(4) + \frac{(0.1)^2}{2!}(-0.4)$

$= 0.598 = 0.6$

19. Ans: 0.6

Sol: $f(x, y) = 4 - 2xy$

$x_0 = 0, y_0 = 0.2, f_1 = 0.1$

By Euler's formula

$y_1 = y_0 + h\, f(x_0, y_0) = 0.2 + 0.1(4 - 0)$

$= 0.6$

20. Ans: 0.04

Sol: By Euler's formula,

$y_1 = y_0 + h\, f(x_0, y_0)$

$y_1 = 0 + (0.2)\,(0 + 0) = 0$

$y_2 = y_1 + h\, f(x_1, y_1)$

$y_2 = 0 + 0.2(0.2 + 0)$

$y_2 = 0.04$

21. Ans: 0.095

Sol: $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$k_1 = hf(x_0, y_0) = 0.1\,(1 - 0) = 0.1$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.1\,(1 - 0.1) = 0.09$

$y_1 = 0 + \frac{1}{2}(0.1 + 0.09)$

$= 0.095$

22. Ans: 1.1961

Sol: $f(x, y) = x + \sin y$

$x_0 = 0,\ y_0 = 1,\ h = 0.2$

$k_1 = h(f_0, y_0)$

$= 0.2(0 + \sin 1)$

$= 0.2(0.8414) = 0.1682$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.2(0.2 + \sin(1.1682))$

$= 0.2(0.2 + 0.9200)$

$= 0.2(1.1200)$

$= 0.2240$

$y_1 = 1 + \frac{1}{2}(0.1682 + 0.2240) = 1.1961$

23. Ans: 1.1165

Sol: $f(x, y) = x + y^2$,

$x_0 = 0,\ y_0 = 1,\ f_1 = 0.1$

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1\left[\left(x_0 + \frac{h}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right]$

$= 0.1168$

$k_3 = hf\left(x_0 + h, y_0 + \frac{k_2}{2}\right)$

$= 0.1[0.05 + 1.1185]$

$= 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1347$

$y_1 = y_0 = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= 1 + 0.1164$

$y_1 = 1.1164$

24. Ans: 2.6 – 1.3x, 2.3

Sol: The various summations are given as follows:

	x_i	y_i	x_i^2	x_iy_i
	–2	6	4	–12
	–1	3	1	–3
	0	2	0	0
	1	2	1	2
Σ	–2	13	06	–13

Thus, $\Sigma y_i = na + b\,\Sigma x_i$

$\Sigma x_i y_i = a\,\Sigma x_i + b\,\Sigma x_i^2$

These are called normal equations. Solving for a and b, we get

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \overline{y} - b\overline{x}$$

$$b = \frac{4\times(-13)-(-2)\times 13}{4\times 6 - 6}$$

$= -1.3$

$$a = \frac{13}{4} - 1.3\times\frac{(-2)}{4} = 2.6$$

Therefore, the linear equation is

$y = 2.6 - 1.3x$

The least squares error $= \sum_{i=1}^{4}\{y_i - (a + bx_i)\}^2$

$= (6 - 5.2)^2 + (3 - 3.9)^2 + (2 - 2.6)^2 + (2 - 1.3)^2$

$= 2.3$

25. Ans: i. $8x^2 - 19x + 12$ ii. 6 iii. 13

Sol: $$f(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$$

$f(x) = 8x^2 - 19x + 12$

$f(2) = 6$

$f^1(2) = 13$

$f(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$

$= 1 + (x-1)\,13 + (x-1)(x-3)\,8$

$= 8x^2 - 19x + 12$

$p(2) = 6$

$p^1(2) = 13$

26. Ans: $8x^2 - 19x + 12$, 6, 13

Sol:

x	P(x)	Δp	$\Delta^2 p$
1	1	$\frac{27-1}{3-1} = 13$	
3	27		$\frac{37-13}{4-1} = 8$
		$\frac{64-27}{4-3} = 37$	
4	64		

By Newton's divided difference formula

$P(x) = P(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2)$

$= 1 + (x - 1)13 + (x-1)(x-3).8$

$= 8x^2 - 19x + 12$

$P^1(x) = 16x - 19$

$P(2) = 6$

$P^1(2) = 13$

27. Ans: $x^2 + 2x + 3$, 4.25, 3

Sol: Since the given observations are at equal interval of width unity.

Construct the following difference table.

x	f(x)	Δ f(x)	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

Therefore f(x)

$$f(x) = f(0) + C(x,1)\,\Delta f(0) + C(x, 2)\,f(0)$$

$$= 3 + (x \times 3) + \left(\frac{x(x-1)}{2!}\times 2\right)$$

$$f(x) = x^2 + 2x + 3$$

$$f^1(x) = 2x + 2$$

$$f(0.5) = 4.25$$

$$f^1(0.5) = 3$$

28. Ans: $x^3 + 6x^2 + 11x + 6$, 990, 299

Sol: Let us apply Newton's forward formula

$$\text{Let } u = \frac{x-a}{h} = \frac{x-1}{2}$$

To calculate forward differences

x	f(x)	Δf(x)	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Now by Newton's forward interpolation formula, we have

$$f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$y(x) = 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$$= x^3 + 6x^2 + 11x + 6$$

$$y^1(x) = 3x^2 + 12x + 11$$

$$y(8) = 990$$

$$y^1(8) = 299$$